ANNALES
POLONICI MATHEMATICI
117.1 (2016)

Bergman–Shilov boundary for subfamilies of q-plurisubharmonic functions

Thomas Patrick Pawlaschyk (Wuppertal)

Abstract. We introduce the notion of the Shilov boundary for some subfamilies of upper semicontinuous functions on a compact Hausdorff space. It is by definition the smallest closed subset of the given space on which all functions of that subclass attain their maximum. For certain subfamilies with simple structure we show the existence and uniqueness of the Shilov boundary. We provide its relation to the set of peak points and establish Bishop-type theorems. As an application we obtain a generalization of Bychkov's theorem which gives a geometric characterization of the Shilov boundary for q-plurisubharmonic functions on convex bounded domains.

1. Introduction. In his 1981 article, S. N. Bychkov [Byc81] gave a geometric characterization of the Shilov boundary for bounded convex domains in $\mathbb{C}^n$. Our aim is to generalize his result to the Shilov boundary with respect to q-plurisubharmonic and q-holomorphic functions on bounded convex domains. These families of functions were already studied by different authors, e.g., R. Basener [Bas76], R. L. Hunt and J. J. Murray [HM78] and Z. Słodkowski [Sło84, Sło86]. H. J. Bremermann [Bre59] used a characterization of the Bergman–Shilov boundary (or, for short, the Shilov boundary) based on plurisubharmonic functions without showing its existence. This gap was filled by, e.g., J. Siciak [Sic62]. Given a compact Hausdorff space K and a subfamily $\mathcal{A}$ of upper semicontinuous functions on K, the Shilov boundary for $\mathcal{A}$ is the smallest closed subset of K on which all functions from $\mathcal{A}$ attain their maximum. Existence and uniqueness of such a subset is guaranteed if $\mathcal{A}$ has a simple structure, e.g., if $\mathcal{A}$ forms a convex cone and if sublevel sets of finitely many functions from $\mathcal{A}$ generate the topology of K (see [Sic62, Theorem 1′]). For q-plurisubharmonic functions the condi-

2010 *Mathematics Subject Classification*: Primary 32U05; Secondary 32F10.
Key words and phrases: q-plurisubharmonic, q-convex, convex, Shilov boundary, Bergman boundary.
Received 20 April 2015; revised 29 January 2016.
Published online 17 June 2016.

tion on $\mathcal{A}$ to be a convex cone is too strong, since q-plurisubharmonicity is not stable under summation. It turns out that the above mentioned condition can be relaxed so that the existence of the Shilov boundary for a wide family of upper semicontinuous functions can be guaranteed. Furthermore, the Shilov boundary is strongly connected to peak points. E. Bishop [Bis59] proved that if the compact Hausdorff space is additionally assumed to be metrizable, then for uniform subalgebras of continuous functions, the closure of the set of peak points and the Shilov boundary coincide. This is also true for any Banach subalgebra of continuous functions due to the results of H. G. Dales [Dal71] (see also [Hon88]). Note that similar identities were obtained in [Sic62, Wit83] for upper semicontinuous functions. We partially apply these results to unions of uniform algebras and special families of upper semicontinuous functions in order to establish Bishop-type theorems. This will be the main topic of Section 2.

In Sections 3 and 4, we recall the definitions of q-plurisubharmonic and q-holomorphic functions (see [HM78, Bas76]) and give those of their properties which we will use (for more properties we refer to [Die06], [Fuj90], [Sło86] and [PZ13]). We show the existence of the Shilov boundary for certain subfamilies of q-plurisubharmonic and q-holomorphic functions and investigate their relations. At the same time, we establish Bishop-type peak point properties for these families. In particular, at the end of Section 4, we compare the Shilov boundary for q-holomorphic functions with the classical Shilov boundary of lower-dimensional slices of the given set. As an application, we give estimates on the Hausdorff dimension of the Shilov boundary for q-holomorphic functions of convex sets.

It still seems to be an open question whether the Hausdorff dimension of the classical Shilov boundary for holomorphic functions on convex bodies in $\mathbb{C}^n$ is greater than or equal to n. Bychkov [Byc81] gave a positive answer to this question in the special case $n = 2$. Anyway, using this estimate, we show that the Shilov boundary of a convex body for $(n-2)$-holomorphic functions is not less than $2n-2$ if $n \geq 2$.

In the last Section 5, we study the Shilov boundary of a convex bounded domain D in $\mathbb{C}^n$ for q-plurisubharmonic functions. We generalize Bychkov's theorem in the following way: a boundary point of D does not lie in the Shilov boundary for q-plurisubharmonic or q-holomorphic functions if and only if it is contained in an open part of a complex plane of dimension at least $q+1$ which is fully contained in the boundary of D. We finish the paper with some observations on the local foliation of the boundary of D by parts of complex planes.

2. Bergman–Shilov boundary for upper semicontinuous functions. We will define the Bergman–Shilov boundary for subfamilies of upper semicontinuous functions and show its existence and uniqueness in certain cases. For brevity, we will simply talk about the Shilov boundary instead of the Bergman–Shilov boundary. Anyway, we have to point out that the concept of a distinguished boundary of certain domains in $\mathbb{C}^2$ was already introduced by S. Bergman [Ber31] in 1931.

We start by recalling some basic definitions of upper semicontinuous functions on a compact Hausdorff space X just to fix the notation.

Definition 2.1. A function $f : X \to [-\infty, \infty)$ is called *upper semicontinuous* on X if the sublevel set $\{x \in X : f(x) < c\}$ is open in X for every $c \in \mathbb{R}$. We denote by $\mathcal{USC}(X)$ the set of all upper semicontinuous functions on X and by $\mathcal{C}(X) = \mathcal{C}(X, \mathbb{C})$ the set of all complex-valued continuous functions on X.

From now on, $\mathcal{A}$ is always a subset of $\mathcal{USC}(X)$. Our main objective in this section is to study the Shilov boundary of X with respect to $\mathcal{A}$. Recall that every upper semicontinuous function on a compact Hausdorff space attains its maximum.

Definition 2.2. For a given function $f \in \mathcal{USC}(X)$ we set

$$S(f) = S_X(f) := \Big\{x \in X : f(x) = \max_X f\Big\}.$$

A subset S of X is called a *boundary* (*of* X) *for* $\mathcal{A}$ or an $\mathcal{A}$-*boundary* (*of* X) if $S \cap S(f) \neq \emptyset$ for every $f \in \mathcal{A}$. We denote by $b_{\mathcal{A}} = b_{\mathcal{A}}(X)$ the set of all closed boundaries for $\mathcal{A}$. The intersection

$$\check{S}_{\mathcal{A}} = \check{S}_{\mathcal{A}}(X) := \bigcap_{S \in b_{\mathcal{A}}} S$$

is called the (*Bergman–*) *Shilov boundary* (*of* X) *for* $\mathcal{A}$. A point $x \in X$ is called a *peak point* (*of* X) *for* $\mathcal{A}$ if there is a *peak function* $f \in \mathcal{A}$ such that $S(f) = \{x\}$. We then also say that f *peaks* (*on* X) *at* x, and denote by $P_{\mathcal{A}} = P_{\mathcal{A}}(X)$ the set of all peak points (of X) for $\mathcal{A}$.

Notice that the set $\check{S}_{\mathcal{A}}$ is closed and possibly empty, whereas $b_{\mathcal{A}}$ is never empty, for it contains X. But if $\check{S}_{\mathcal{A}}$ is not empty, it does not mean in general that it is an $\mathcal{A}$-boundary. Anyway, we have the following properties of the Shilov boundary.

Proposition 2.3.

(i) *If* $\mathcal{A}_1 \subset \mathcal{A}_2 \subset \mathcal{USC}(X)$, *then* $b_{\mathcal{A}_2} \subset b_{\mathcal{A}_1}$, $\check{S}_{\mathcal{A}_1} \subset \check{S}_{\mathcal{A}_2}$ *and* $P_{\mathcal{A}_1} \subset P_{\mathcal{A}_2}$.

(ii) *Let* $\mathcal{A} = \bigcup_{j \in J} \mathcal{A}_j$, *where each* $\mathcal{A}_j$ *lies in* $\mathcal{USC}(X)$. *Then*

$$P_{\mathcal{A}} = \bigcup_{j \in J} P_{\mathcal{A}_j}.$$

(iii) *If $\check{S}_{\mathcal{A}_j}$ are $\mathcal{A}_j$-boundaries of X, then $\check{S}_{\mathcal{A}}$ is an $\mathcal{A}$-boundary of X and*

$$\check{S}_{\mathcal{A}} = \overline{\bigcup_{j\in J} \check{S}_{\mathcal{A}_j}}.$$

(iv) *The set $P_{\mathcal{A}}$ lies in $\check{S}_{\mathcal{A}}$, and if $\overline{P_{\mathcal{A}}}$ is an $\mathcal{A}$-boundary, then*

$$\check{S}_{\mathcal{A}} = \overline{P_{\mathcal{A}}} \in b_{\mathcal{A}}.$$

(v) *Let $\overline{\mathcal{A}}^{\downarrow} = \overline{\mathcal{A}}^{\downarrow X}$ denote the set of pointwise limits on X of all decreasing sequences of functions in $\mathcal{A}$. Then $b_{\mathcal{A}} = b_{\overline{\mathcal{A}}^{\downarrow}}$, and thus $\check{S}_{\mathcal{A}} = \check{S}_{\overline{\mathcal{A}}^{\downarrow}}$.*

Proof. Properties (i) and (ii) follow directly from the definition, so we prove (iii). Property (i) implies that $S := \overline{\bigcup_{j\in J} \check{S}_{\mathcal{A}_j}}$ is in $\check{S}_{\mathcal{A}}$. By assumption, the set S, and therefore $\check{S}_{\mathcal{A}}$, is non-empty. Since every $f \in \mathcal{A}$ is in $\mathcal{A}_j$ for some $j \in J$ and since $\check{S}_{\mathcal{A}_j}$ is an $\mathcal{A}_j$-boundary of X by assumption, we obtain

$$\emptyset \neq S(f) \cap \check{S}_{\mathcal{A}_j} \subset S(f) \cap S.$$

This means that S is an $\mathcal{A}$-boundary of X, and so $\check{S}_{\mathcal{A}} \subset S$. Altogether, $S = \check{S}_{\mathcal{A}}$ is an $\mathcal{A}$-boundary of X.

To verify (iv), let $x \in P_{\mathcal{A}}$ and $f \in \mathcal{A}$ be such that f peaks at x. Given an $\mathcal{A}$-boundary S, it is clear that $S \cap S(f) = \{x\}$. In particular, x lies in S. This yields $P_{\mathcal{A}} \subset S$ and $P_{\mathcal{A}} \subset \check{S}_{\mathcal{A}}$. By the definition of $\check{S}_{\mathcal{A}}$ and since this set is closed, we conclude that $\overline{P_{\mathcal{A}}} \subset \check{S}_{\mathcal{A}}$. Now, if we additionally suppose that $P_{\mathcal{A}} \in b_{\mathcal{A}}$, then by the previous discussion, and again by the definition of $\check{S}_{\mathcal{A}}$, we derive that $\check{S}_{\mathcal{A}} = \overline{P_{\mathcal{A}}}$ is indeed an $\mathcal{A}$-boundary.

The last property (v) follows from the fact that for each decreasing sequence $f_1 \geq f_2 \geq \cdots$ of functions in $\mathcal{A}$ with limit function f, the sequence $(\max_Y f_n)_{n\geq 1}$ of maximums decreases to $\max_Y f$ for each compact subset Y of X. This means that $b_{\mathcal{A}} = b_{\overline{\mathcal{A}}^{\downarrow}}$. ∎

Notice that the very last property (v) also implies that the Shilov boundary for $\mathcal{A}$ exists if and only if it exists for $\overline{\mathcal{A}}^{\downarrow}$.

We can easily relate our concept of Shilov boundary to the classical Shilov boundary for uniform subalgebras of $\mathcal{C}(K)$.

REMARK 2.4. Let $\mathcal{B}$ be a subset of $\mathcal{C}(K)$. The classical Shilov boundary for $\mathcal{B}$ is the smallest closed subset S of K fulfilling $\max_S |f| = \max_K |f|$ for every $f \in \mathcal{B}$. Clearly, it corresponds to the Shilov boundary for the family $\log|\mathcal{B}| := \{\log|f| : f \in \mathcal{B}\}$. It then makes sense to simply write $b_{\mathcal{B}}$, $\check{S}_{\mathcal{B}}$ and $P_{\mathcal{B}}$ instead of $b_{\log|\mathcal{B}|}$, $\check{S}_{\log|\mathcal{B}|}$ and $P_{\log|\mathcal{B}|}$. It is clear that for the uniform closure $\overline{\mathcal{B}}$ of $\mathcal{B}$ in $\mathcal{C}(K)$ we have $b_{\overline{\mathcal{B}}} = b_{\mathcal{B}}$ and $\check{S}_{\overline{\mathcal{B}}} = \check{S}_{\mathcal{B}}$. This means that the Shilov boundary for $\mathcal{B}$ exists if and only if it exists for $\overline{\mathcal{B}}$.

We recall the classical result of Shilov.

Theorem (Shilov). *Let X be a compact Hausdorff space and $\mathcal{B}$ a Banach subalgebra of $\mathcal{C}(X)$. Then $\check{S}_{\mathcal{B}}$ is non-empty and it is a boundary for $\mathcal{B}$.*

Shilov's theorem generalizes to subfamilies of upper semicontinuous functions which satisfy simpler conditions than having a Banach algebra structure.

Definition 2.5. Let $\mathcal{A}$, $\mathcal{A}_1$ and $\mathcal{A}_2$ be subfamilies of upper semicontinuous functions on a Hausdorff space X.

(i) We set $\mathcal{A}_1 + \mathcal{A}_2 := \{f + g : f \in \mathcal{A}_1,\ g \in \mathcal{A}_2\}$.
(ii) The family $\mathcal{A}$ is a *scalar cone* if $nf + b$ lies in $\mathcal{A}$ for every $n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, $f \in \mathcal{A}$ and $b \in \mathbb{R}$. Here we use the convention $-\infty \cdot 0 = 0$.
(iii) An open set V in X is an *$\mathcal{A}$-polyhedron* if there exist finitely many functions $f_1, \ldots, f_n$ in $\mathcal{A}$ and real numbers $C_1, \ldots, C_n$ such that
$$V = V(f_1, \ldots, f_n) = \{x \in X : f_1(x) < C_1, \ldots, f_n(x) < C_n\}.$$
(iv) The family $\mathcal{A}$ *generates the topology of* X if for each $x \in X$ and every neighborhood U of x in X there is an $\mathcal{A}$-polyhedron V such that $x \in V \subset U$.

Now we show that the Shilov boundary for $\mathcal{A}$ is a non-empty boundary for $\mathcal{A}$ if $\mathcal{A}$ has a certain simple structure. The next two statements are based on standard arguments used in the case of Banach subalgebras of continuous functions (see e.g. [AW98, Theorem 9.1]). First, we need the following lemma.

Lemma 2.6. *Let $\mathcal{A}$ be a scalar cone. Assume that there exist an $\mathcal{A}$-boundary $S \in b_{\mathcal{A}}$ and an $\mathcal{A}$-polyhedron $V = V(f_1, \ldots, f_n)$ such that $S \cap V = \emptyset$ and $\mathcal{A} + \{f_j\} \subset \mathcal{A}$ for $j = 1, \ldots, n$. Given another $\mathcal{A}$-boundary $E \in b_{\mathcal{A}}$, it follows that $E \setminus V \in b_{\mathcal{A}}$.*

Proof. Since $\mathcal{A}$ is a scalar cone and $\mathcal{A} + \{f_j\} \subset \mathcal{A}$ for $j = 1, \ldots, n$, the constant function 0 and thus $f_1, \ldots, f_n$ lie in $\mathcal{A}$. Hence, we can assume that V is of the form $\{x \in X : f_1(x) < 0, \ldots, f_n(x) < 0\}$.

Notice first that $E \setminus V$ is non-empty. Indeed, otherwise, $E \subset V$, so $\max_E f_j < 0$ for every $j = 1, \ldots, n$. Since S does not meet V, there is $j_0 \in \{1, \ldots, n\}$ such that $\max_S f_{j_0} \geq 0$. We obtain the contradiction $0 \leq \max_S f_{j_0} = \max_E f_{j_0} < 0$.

Suppose that the statement of the lemma is false, i.e., there are $y \in X$ and $f \in \mathcal{A}$ such that $\max_{E\setminus V} f < \max_X f = f(y)$. Since $\mathcal{A}$ is a scalar cone and $S \in b_{\mathcal{A}}$, we can assume that $f(y) = 0$ and $y \in S$. For $m \in \mathbb{N}$ set $g_j := mf + f_j \in \mathcal{A}$, $j = 1, \ldots, n$. If m is large enough, then $\max_{E\setminus V} g_j < 0$ for each $j = 1, \ldots, n$. Since $\max_X f = 0$, it follows from the definition of V that for every $j = 1, \ldots, n$ we have $g_j(x) < 0$ for all $x \in V$. Hence, $\max_X g_j = \max_E g_j < 0$ for every $j = 1, \ldots, n$.

We conclude that $y \in V$: if not, there is a $j_1 \in \{1,\dots,n\}$ with $f_{j_1}(y) \geq 0$ and thus $g_{j_1}(y) \geq 0$, which is impossible. Consequently, $y \in V \cap S = \emptyset$, a contradiction. ∎

We point out that L. Aizenberg already showed the next result in the case where X is a compact subset of $\mathbb{C}^n$ and $\mathcal{A}$ is a family of upper semicontinuous functions on X satisfying $f+\log\|z-c\| \in \mathcal{A}$ for every $f \in \mathcal{A}$ and $c \in \mathbb{C}^n$ (see [Aiz93, Chapter III, Theorem 14.1 and Corollary 14.2]). Our version extends Aizenberg's result to a more general situation in which X is an arbitrary compact Hausdorff space.

THEOREM 2.7. *If $\mathcal{A}$ contains a subset $\mathcal{A}_0$ which generates the topology of X such that $\mathcal{A}+\mathcal{A}_0 \subset \mathcal{A}$, then the Shilov $\mathcal{A}$-boundary is an $\mathcal{A}$-boundary, i.e., $\check{S}_{\mathcal{A}} \in b_{\mathcal{A}}$.*

Proof. First, assume that $\mathcal{A}$ is a scalar cone. If $\check{S}_{\mathcal{A}} = X$, then there is nothing to show. So we can assume that $\check{S}_{\mathcal{A}} \neq X$. We first treat the case $\check{S}_{\mathcal{A}} \neq \emptyset$.

Suppose $\check{S}_{\mathcal{A}} \notin b_{\mathcal{A}}$. Then there is a function $f \in \mathcal{A}$ such that $\max_{\check{S}_{\mathcal{A}}} f < \max_X f$. Since f is upper semicontinuous on X, there is a neighborhood U of $\check{S}_{\mathcal{A}}$ such that $f(x) < \max_X f$ for every $x \in U$. Then, since $\mathcal{A}_0$ generates the topology of X, we conclude that for every $y \in L := X \setminus U$ there are an $\mathcal{A}_0$-polyhedron V_y and an $\mathcal{A}$-boundary $S_y \in b_{\mathcal{A}}$ such that $y \in V_y$ and $V_y \cap S_y = \emptyset$. The family $\{V_y\}_{y\in L}$ covers L. Hence, by the compactness of L, there are finitely many points $y_1,\dots,y_l \in L$ such that the subfamily $\{V_{y_j}\}_{j=1,\dots,l}$ covers L. Since $\mathcal{A}+\mathcal{A}_0 \subset \mathcal{A}$, we can iteratively apply Lemma 2.6 to obtain

$$E := \big(((X \setminus V_{y_1}) \setminus V_{y_2}) \setminus \cdots \setminus V_{y_l}\big) = X \setminus \bigcup_{j=1}^{l} V_{y_j} \in b_{\mathcal{A}}.$$

Notice that, by construction, the set $\check{S}_{\mathcal{A}}$ lies in E, and hence E is non-empty. Moreover, $E \subset U$, and thus $\max_E f < \max_X f$. But this contradicts $E \in b_{\mathcal{A}}$. Hence, $\check{S}_{\mathcal{A}} \in b_{\mathcal{A}}$.

Suppose $\check{S}_{\mathcal{A}} = \emptyset$. We pick $p \in X$ and a neighborhood U of p in X which is an $\mathcal{A}_0$-polyhedron of the form $U = \{x \in X : f_1(x) < 0,\dots,f_k(x) < 0\}$ such that $U \neq X$. Observe that for every $y \in X \setminus U$ there exists an $\mathcal{A}$-boundary S_y with $y \notin S_y$, since otherwise $y \in \check{S}_{\mathcal{A}}$. Then we can choose an $\mathcal{A}_0$-polyhedron V_y such that $y \in V_y$, $p \notin V_y$ and $S_y \cap V_y$ is empty. By the same argument as above we can construct an $\mathcal{A}$-boundary E such that $p \in E \subset U$. But since $U \neq X$, there exist $x_0 \in X \setminus U$ and $k_0 \in \{1,\dots,k\}$ such that $f_{k_0}(x_0) \geq 0$. This leads to the contradiction

$$0 \leq f_{k_0}(x_0) \leq \max_X f_{k_0} = \max_E f_{k_0} < 0.$$

Thus, $\check{S}_{\mathcal{A}}$ cannot be empty.

If $\mathcal{A}$ is not necessarily a scalar cone, consider the scalar cone generated by $\mathcal{A}$,

$$\mathcal{A}^* := \{nf + c : n \in \mathbb{N}_0,\ f \in \mathcal{A},\ c \in \mathbb{R}\}.$$

Since $\mathcal{A}$ lies in $\mathcal{A}^*$, we have $b_{\mathcal{A}^*} \subset b_{\mathcal{A}}$ and $\check{S}_{\mathcal{A}} \subset \check{S}_{\mathcal{A}^*}$. Pick an $\mathcal{A}$-boundary S in X and a function $nf + c \in \mathcal{A}^*$, where $f \in \mathcal{A}$, $n \in \mathbb{N}$ and $c \in \mathbb{R}$. Since f and $nf + c$ attain their maximum at the same points, we have

$$S \cap S(nf + c) = S \cap S(f) \neq \emptyset.$$

This means that S is also an $\mathcal{A}^*$-boundary in X, so $b_{\mathcal{A}} = b_{\mathcal{A}^*}$ and $\check{S}_{\mathcal{A}} = \check{S}_{\mathcal{A}^*}$. Now observe that the family $\mathcal{A}_0^* := \{nf + c : n \in \mathbb{N}_0,\ f \in \mathcal{A}_0,\ c \in \mathbb{R}\}$ generates the topology of X, since it contains $\mathcal{A}_0$. Moreover, $\mathcal{A}^* + \mathcal{A}_0^* \subset \mathcal{A}^*$ and $\mathcal{A}^*$ is a scalar cone. Thus, by the previous discussion, $\check{S}_{\mathcal{A}} = \check{S}_{\mathcal{A}^*}$ belongs to $b_{\mathcal{A}^*} = b_{\mathcal{A}}$. ∎

We now recall Bishop's peak point theorem for uniform algebras of continuous functions (see [Bis59, Theorem 1]). Further generalizations were obtained by H. G. Dales to Banach function algebras (see [Dal71] and [Hon88]) and by J. Siciak to certain additive subfamilies of continuous functions (see [Sic62, Theorem 3]). Recall that a function family $\mathcal{A}$ is *separating* if for any distinct $x, y \in X$ there exists an $f \in \mathcal{A}$ such that $f(x) \neq f(y)$.

THEOREM 2.8 (Bishop, 1959). *Let X be a compact metrizable Hausdorff space and $\mathcal{B}$ a separating uniform subalgebra of $\mathcal{C}(X)$. Then the Shilov boundary of X for $\mathcal{B}$ exists and coincides with the set of all peak points of X for $\mathcal{B}$.*

Bishop's theorem applies to unions of uniform subalgebras by using Proposition 2.3.

COROLLARY 2.9. *Suppose $\mathcal{B}$ is a union of separating uniform subalgebras $\{\mathcal{B}_j\}_{j\in J}$ of $\mathcal{C}(X)$, where X is a metrizable compact Hausdorff space. Then*

$$\check{S}_{\mathcal{B}} = \overline{P_{\mathcal{B}}} \in b_{\mathcal{B}}.$$

We will give another peak point theorem for a family of upper semicontinuous functions having a subfamily which is stable under small perturbations by functions with compact support.

DEFINITION 2.10. Let $\mathcal{A}$ be a subfamily of upper semicontinuous functions on a Hausdorff space X, and let Θ be a subset of non-negative continuous functions on X with the following property: for each $x \in X$ and each closed subset S of X with $x \notin S$ there exists $\vartheta \in \Theta$ that peaks on X at x and vanishes on S. We say that $f \in \mathcal{A}$ is a *strictly $\mathcal{A}$-function with respect to Θ* if for every $\vartheta \in \Theta$ there is an $\varepsilon_0 > 0$ such that $f + \varepsilon\vartheta \in \mathcal{A}$ for every $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$. The subfamily of $\mathcal{A}$ consisting of all strictly $\mathcal{A}$-functions with respect to Θ is denoted by $\mathcal{A}[\Theta]$.

We can now present another version of Bishop's theorem which better incorporates the properties of subfamilies of q-plurisubharmonic functions (see Definition 3.1 in the next section).

THEOREM 2.11. *Let $\mathcal{A}$ be a subfamily of upper semicontinuous functions on a compact Hausdorff space X. Suppose that there exist a subfamily Θ as in Definition* 2.10 *and a positive function $\omega \in \mathcal{A}[\Theta]$ such that $\mathcal{A}+\{\varepsilon\omega\} \in \mathcal{A}[\Theta]$ for every $\varepsilon > 0$. Then*

$$\check{S}_{\mathcal{A}} = \overline{P_{\mathcal{A}}} \in b_{\mathcal{A}}.$$

Proof. First, observe that $P_{\mathcal{A}[\Theta]}$ is non-empty. Indeed, ω attains its maximum on X, say at $x_0 \in X$. Pick $\vartheta \in \Theta$ with $S_X(\vartheta) = \{x_0\}$. Then there is a $\delta > 0$ such that $\omega + \delta\vartheta \in \mathcal{A}$. Therefore, $2\omega + \delta\vartheta \in \mathcal{A}[\Theta]$ by the assumption on ω. Moreover, $S_X(2\omega + \delta\vartheta) = \{x_0\}$, and thus $x_0 \in P_{\mathcal{A}[\Theta]}$.

The non-empty set $\overline{P_{\mathcal{A}[\Theta]}}$ is a subset of $\check{S}_{\mathcal{A}[\Theta]}$ by Proposition 2.3(iv). To get the converse inclusion, we only need to verify that $S := \overline{P_{\mathcal{A}[\Theta]}}$ is a boundary for $\mathcal{A}[\Theta]$. Suppose that this is not the case. Then there exists $f \in \mathcal{A}[\Theta]$ such that $\max_X f > \max_S f$. If $\varepsilon_0 > 0$ is small enough, $g := f+\varepsilon_0\omega$ also fulfills $\max_X g > \max_S g$. Let $x_1 \in X \setminus S$ be such that $g(x_1) = \max_X g$, and let $\theta \in \Theta$ be such that $S_X(\theta) = \{x_1\}$ and θ vanishes on S. In particular, we have $\theta(x_1) > 0$. Then for $\varepsilon_1 > 0$ small enough the function $f+\varepsilon_1\theta$ is in $\mathcal{A}$. Hence, $h := g + \varepsilon_1\theta = f + \varepsilon_1\theta + \varepsilon_0\omega$ lies in $\mathcal{A}[\Theta]$ and fulfills $S_X(h) = \{x_1\}$. Thus, $x_1 \in P_{\mathcal{A}[\Theta]} \subset S$. But this contradicts the choice of $x_1 \in X \setminus S$. Therefore, S has to be an $\mathcal{A}[\Theta]$-boundary. Hence, Proposition 2.3(iv) yields $\overline{P_{\mathcal{A}[\Theta]}} = \check{S}_{\mathcal{A}[\Theta]} \in b_{\mathcal{A}[\Theta]}$.

Now let $f \in \mathcal{A}$. It follows that for all $n \in \mathbb{N}$ the function $f_n := f+(1/n)\omega$ lies in $\mathcal{A}[\Theta]$, so the sequence $(f_n)_n$ decreases to f. This implies that $\mathcal{A}$ lies in $\overline{\mathcal{A}[\Theta]}^{\downarrow}$. Since $\mathcal{A}[\Theta] \subset \mathcal{A}$, in view of Proposition 2.3(v) we have $b_{\mathcal{A}} = b_{\mathcal{A}[\Theta]}$ and $\check{S}_{\mathcal{A}[\Theta]} = \check{S}_{\mathcal{A}}$. Finally, the proof is finished due to the inclusions

$$\check{S}_{\mathcal{A}} = \check{S}_{\mathcal{A}[\Theta]} = \overline{P_{\mathcal{A}[\Theta]}} \subset \overline{P_{\mathcal{A}}} \subset \check{S}_{\mathcal{A}}. \blacksquare$$

3. Shilov boundary for q-plurisubharmonic functions. Hunt and Murray [HM78] studied q-plurisubharmonic functions, which generalize q-convex functions in the sense of Grauert and Andreotti (see [HL88]) to the upper semicontinuous setting. More precisely, a function is q-convex if and only if it is $\mathcal{C}^2$-smooth and strictly $(q-1)$-plurisubharmonic.

DEFINITION 3.1. Let U be an open set in $\mathbb{C}^n$ and $u : U \to [-\infty, \infty)$ be an upper semicontinuous function on U. Fix $q \in \{0, 1, \dots, n-1\}$.

(i) The function u is called *subpluriharmonic* on U if for every ball $B \Subset U$ and every function h which is pluriharmonic in a neighborhood of the closure of B and fulfills $u \le h$ on bB one has $u \le h$ on B.

(ii) The function u is called *q-plurisubharmonic* in U if it is subpluriharmonic in $U \cap \pi$ for every $(q+1)$-dimensional complex affine plane $\pi \subset \mathbb{C}^n$.

(iii) We denote by $\mathcal{PSH}_q(U)$ the set of all q-plurisubharmonic functions on U. If q is an integer with $q \geq n$, we simply define $\mathcal{PSH}_q(U) := \mathcal{USC}(U)$.

(iv) Given a compact set K in $\mathbb{C}^n$, $\mathcal{PSH}_q(K)$ denotes the set of all functions $u \in \mathcal{USC}(K)$ which have a q-plurisubharmonic extension $\widehat{u}$ into an open neighborhood of K, i.e., there exists an open neighborhood U of K and a function $\widehat{u} \in \mathcal{PSH}_q(U)$ such that $\widehat{u}|K = u$.

According to this definition, the 0-plurisubharmonic functions are the classical plurisubharmonic functions. Moreover, a q-plurisubharmonic function is also $(q+1)$-plurisubharmonic. We now overview the basic properties of q-plurisubharmonic functions.

Proposition 3.2. *Let U be an open set in $\mathbb{C}^n$.*

(i) [Sło84] *Given $c \geq 0$ and functions $u \in \mathcal{PSH}_q(U)$ and $v \in \mathcal{PSH}_r(U)$, we have*

$$\begin{aligned} &cu \in \mathcal{PSH}_q(U), &&\max\{u,v\} \in \mathcal{PSH}_{\max\{q,r\}}(U),\\ &u+v \in \mathcal{PSH}_{q+r}(U), &&\min\{u,v\} \in \mathcal{PSH}_{q+r+1}(U).\end{aligned}$$

(ii) [HM78] *A $\mathcal{C}^2$-smooth function u lies in $\mathcal{PSH}_q(U)$ if and only if its complex Hessian $(\partial^2 u/\partial z_k \partial \overline{z}_l)_{k,l=1}^n$ has at least $n-q$ non-negative eigenvalues at each point in U.*

(iii) [Fuj90, Die06] *If $\psi \in \mathcal{PSH}_q(U)$, then $\psi \circ h \in \mathcal{PSH}_q(W)$ for every holomorphic mapping $h : W \to U$, where W is an open set in $\mathbb{C}^n$.*

(iv) [HM78] *If $(u_n)_{n\in\mathbb{N}}$ is a decreasing sequence of functions in $\mathcal{PSH}_q(U)$, then $\lim_{n\to\infty} u_n$ lies in $\mathcal{PSH}_q(U)$.*

(v) [HM78] *Let V be an open subset of U. Let $v \in \mathcal{PSH}_q(V)$ and $u \in \mathcal{PSH}_q(U)$ be such that $\limsup_{\zeta\to z,\zeta\in V} v(\zeta) \leq u(z)$ for every $z \in U \cap bV$. Then*

$$\varphi = \begin{Bmatrix} u & \text{on } U \setminus V \\ \max\{u,v\} & \text{on } V \end{Bmatrix} \in \mathcal{PSH}_q(U).$$

(vi) [HM78, Sło84] *Let A be an analytic subset of an open relatively compact set U in $\mathbb{C}^n$. Assume that the dimension of A at each of its points is at least $q+1$. Then every q-plurisubharmonic function u on U which is upper semicontinuous up to the boundary of U satisfies the maximum principle on A, i.e.,*

$$\max_{\overline{A}} u = \max_{\overline{A}\cap bU} u.$$

Interesting examples of q-plurisubharmonic functions come from characteristic functions of analytic sets.

Example 3.3. Let A be an analytic subset of an open set U in $\mathbb{C}^n$. Assume that the codimension of A is less than or equal to q at each of its points. Fix a plurisubharmonic function ψ on U. Then the function

$$\Psi(z) = \begin{cases} \psi(z), & z \in A, \\ -\infty, & z \in U \setminus A, \end{cases}$$

is q-plurisubharmonic on U. Indeed, it was shown in [Sło84] that the characteristic function $\chi_A \equiv 0$ on A and $\chi_A \equiv -\infty$ on $U \setminus A$ is q-plurisubharmonic. Hence, we see from Proposition 3.2(i) that $\Psi = \psi + \chi_A$ is q-plurisubharmonic on U.

We now present a useful regularization technique derived from [Dem12, Chapter 5, Lemma (5.18)].

Definition 3.4. Let θ be a non-negative $\mathcal{C}^\infty$-smooth function on $\mathbb{R}$ with compact support in $(-1,1)$ such that $\int_{\mathbb{R}} \theta(s)\,ds = 1$ and $\theta(-s) = \theta(s)$ for all $s \in \mathbb{R}$. Given $\varepsilon_1, \ldots, \varepsilon_l \in (0,\infty)$ and $t = (t_1, \ldots, t_l) \in \mathbb{R}^l$, we define the *regularized maximum* by

$$\widetilde{\max}_{(\varepsilon_1,\ldots,\varepsilon_l)}(t) := \int_{\mathbb{R}^l} \max\{t_1 + \varepsilon_1 s_1, \ldots, t_l + \varepsilon_l s_l\}\theta(s_1)\cdot\ldots\cdot\theta(s_l)\,d(s_1,\ldots,s_l).$$

For a single positive number $\varepsilon > 0$ we set $\widetilde{\max}_{\varepsilon}(t) := \widetilde{\max}_{(\varepsilon,\ldots,\varepsilon)}(t)$.

The regularized maximum has the following properties.

Lemma 3.5.

(i) *The function* $t = (t_1, \ldots, t_l) \mapsto \widetilde{\max}_{(\varepsilon_1,\ldots,\varepsilon_l)}(t)$ *is a* $\mathcal{C}^\infty$*-smooth convex function on* $\mathbb{R}^l$ *which is non-decreasing in each variable* $t_1, \ldots, t_l$.

(ii) $\max\{t_1, \ldots, t_l\} \leq \widetilde{\max}_{(\varepsilon_1,\ldots,\varepsilon_l)}(t) \leq \max\{t_1 + \varepsilon_1, \ldots, t_l + \varepsilon_l\}$.

(iii) *If* $t_j + \varepsilon_j < \max_{i\neq j}\{t_i - \varepsilon_i\}$, *then*

$$\widetilde{\max}_{(\varepsilon_1,\ldots,\varepsilon_l)}(t_1, \ldots, t_l) = \widetilde{\max}_{(\varepsilon_1,\ldots,\varepsilon_{j-1},\varepsilon_{j+1},\ldots,\varepsilon_l)}(t_1, \ldots, t_{j-1}, t_{j+1}, \ldots, t_l).$$

As a consequence of Lemma 3.5 and [PZ13, Proposition 2.11], we can apply the regularized maximum to q-plurisubharmonic functions.

Lemma 3.6. *Let* $u_1, \ldots, u_k$ *be finitely many* $\mathcal{C}^\infty$*-smooth functions on an open set* U *in* $\mathbb{C}^n$ *such that for each* $j \in \{1, \ldots, k\}$ *the function* u_j *is* q_j*-plurisubharmonic on* U*. Then for every tuple* $(\varepsilon_1, \ldots, \varepsilon_k)$ *of positive numbers the regularized maximum* $\widetilde{\max}_{(\varepsilon_1,\ldots,\varepsilon_k)}\{u_1, \ldots, u_k\}$ *is* $\mathcal{C}^\infty$*-smooth and* q*-plurisubharmonic on* U*, where* $q = q_1 + \cdots + q_k$.

Let K be a compact set in $\mathbb{C}^n$, and let Θ be the set of all non-negative $\mathcal{C}^\infty$-smooth functions with compact support on $\mathbb{C}^n$. Define $\omega(z) := \|z\|_2^2 + 1$ for $z \in \mathbb{C}^n$. Since $\omega + \varepsilon f$ belongs to $\mathcal{PSH}_q(K)$ for all $\varepsilon > 0$ and $f \in \mathcal{PSH}_q(K)$, we can apply Theorem 2.11 to Θ, $\mathcal{A} = \mathcal{PSH}_q(K)$ and ω to get the following Bishop-type peak point property for q-plurisubharmonic functions. For brevity, we will write $b_{\mathcal{PSH}_q(K)}$, $\check{S}_{\mathcal{PSH}_q(K)}$ and $P_{\mathcal{PSH}_q(K)}$ rather than $b_{\mathcal{PSH}_q(K)}(K)$, $\check{S}_{\mathcal{PSH}_q(K)}(K)$ and $P_{\mathcal{PSH}_q(K)}(K)$.

PROPOSITION 3.7. $\check{S}_{\mathcal{PSH}_q(K)} = \overline{P_{\mathcal{PSH}_q(K)}} \in b_{\mathcal{PSH}_q(K)}$.

A similar result may be obtained for many different subfamilies of q-plurisubharmonic functions using the same argument. For instance, one can take the continuous or $\mathcal{C}^m$-smooth functions in $\mathcal{PSH}_q(K)$, $m \geq 1$; or the q-plurisubharmonic functions in $\mathcal{PSH}_q(K)$ with corners, that is, continuous functions which are locally the maximum of finitely many $\mathcal{C}^\infty$-smooth q-plurisubharmonic functions; or the continuous functions on K which are q-plurisubharmonic in the interior of K. But we will restrict only to the family $\mathcal{PSH}_q(K)$ for the following two reasons: Firstly, in the last section we are only interested in the Shilov boundaries of convex sets X with non-empty interior. Then it is easy to verify that the Shilov boundaries are the same for the families of continuous functions in $\mathcal{PSH}_q(X)$ and continuous functions on X which are q-plurisubharmonic in the interior of X. We simply have to use the retraction $z \mapsto \lambda z$ for small enough numbers $\lambda > 0$, since we can always assume that X contains the origin. Secondly, we can show that the Shilov boundaries for (nearly) all the subfamilies of q-plurisubharmonic functions described above coincide. In view of the peak point property, this means that the Shilov boundary of K for q-plurisubharmonic functions equals the closure of the peak points of K for $\mathcal{C}^\infty$-smooth q-plurisubharmonic functions. This observation is interesting since, in general, it is not possible to (uniformly) approximate a q-plurisubharmonic function by a sequence of smooth ones (see [DF85]).

PROPOSITION 3.8. *Let K be a compact set in $\mathbb{C}^n$. Then*

$$\check{S}_{\mathcal{PSH}_q(K)} = \check{S}_{\mathcal{PSH}_q^\infty(K)}.$$

Here, $\mathcal{PSH}_q^\infty(K)$ is the set of $\mathcal{C}^\infty$-smooth q-plurisubharmonic functions v defined on some open neighborhood V of K, where V depends on v.

Proof. The inclusion $\check{S}_{\mathcal{PSH}_q^\infty(K)} \subset \check{S}_{\mathcal{PSH}_q(K)}$ is trivial, so it suffices to show the converse. Denote by $\mathcal{PSH}_q^c(K)$ the set of all q-plurisubharmonic functions ψ defined in some neighborhood W of K which have corners on W. As before, we mean that W depends on ψ. Then it follows from Słodkowski's and Bungart's approximation techniques [Sło86], [Bun90] and from Proposition 2.3(v) that $\check{S}_{\mathcal{PSH}_q(K)} = \check{S}_{\mathcal{PSH}_q^c(K)}$.

Thus, it remains to show that $\check{S}_{\mathcal{PSH}_q^c(K)} \subset \check{S}_{\mathcal{PSH}_q^\infty(K)}$. To see this, assume that $\psi \in \mathcal{PSH}_q^c(K)$ peaks at some $p \in bK$. Then there exist a bounded open neighborhood U of p and finitely many $\mathcal{C}^\infty$-smooth q-plurisubharmonic functions $\psi_1, \dots, \psi_k$ on U such that $\psi = \max_{j=1,\dots,k} \psi_j$. By picking a slightly smaller neighborhood of p and denoting it again by U, we can arrange that each ψ_j is defined on a neighborhood of $\overline{U}$. Take $j_0 \in \{1, \dots, k\}$ such that $\psi(p) = \psi_{j_0}(p)$. Since ψ peaks at p, we have

$$\psi_{j_0}(p) = \psi(p) > \psi(z) \geq \psi_{j_0}(z) \quad \text{ for every } z \in (U \cap K) \setminus \{p\}.$$

Hence, ψ_{j_0} peaks at p in $K \cap U$. Since ψ_{j_0} is continuous on $\overline{U}$, we can choose $c \in \mathbb{R}$ such that

$$\psi_{j_0}(p) > c > \max_{bU \cap K} \psi_{j_0}.$$

By Lemma 3.6, we can find $\varepsilon > 0$ such that the function $\varphi := \widetilde{\max}_\varepsilon\{\psi_{j_0}, c\}$ is $\mathcal{C}^\infty$-smooth and q-plurisubharmonic on a neighborhood of $\overline{U} \cap K$. In view of Lemma 3.5(iii), we can choose $\varepsilon > 0$ so small that φ peaks at p in $K \cap \overline{U}$ and fulfills $\varphi \equiv c$ on a neighborhood of $bU \cap K$. Thus, we can extend φ by the constant c into a neighborhood of $K \setminus U$ to a function from $\mathcal{PSH}_q^\infty(K)$ which peaks at p. Since p was an arbitrary peak point for $\mathcal{PSH}_q^c(K)$, we obtain $P_{\mathcal{PSH}_q^c(K)} \subset P_{\mathcal{PSH}_q^\infty(K)}$. Since the converse inclusion is obvious, we conclude that $P_{\mathcal{PSH}_q^c(K)} = P_{\mathcal{PSH}_q^\infty(K)}$, so Proposition 3.7 (and the preceding remark) yields $\check{S}_{\mathcal{PSH}_q^\infty(K)} = \check{S}_{\mathcal{PSH}_q^c(K)}$. ∎

We close this section by presenting a subfamily of q-plurisubharmonic functions which arises naturally, but which has a trivial Shilov boundary.

Remark 3.9. Let $q \in \{0, \dots, n-1\}$, let K be a compact set in $\mathbb{C}^n$, and let $\mathcal{A}$ be the family of upper semicontinuous functions on K which are q-plurisubharmonic on the interior of K. In view of Theorem 2.7, the Shilov boundary $\check{S}_\mathcal{A}$ for $\mathcal{A}$ exists. By the maximum principle in Proposition 3.2(vi), the Shilov boundary for $\mathcal{A}$ is contained in the boundary of K. On the other hand, pick a point x in the boundary of K. Then the characteristic function $\chi_{\{x\}}$ lies in $\mathcal{A}$ and peaks at x. Hence, the whole boundary of K coincides with $P_\mathcal{A}$. Since the set of all peak points for $\mathcal{A}$ lies in the Shilov boundary for $\mathcal{A}$, we conclude that $\check{S}_\mathcal{A} = bK$.

4. Shilov boundary for q-holomorphic functions. A generalization of holomorphic functions is given by the so called q-holomorphic functions which were already studied by, e.g., Basener [Bas76], [Bas78] and Hunt and Murray [HM78].

Definition 4.1. Let U be an open set in $\mathbb{C}^n$. Given an integer $q \geq 0$, the set of *q-holomorphic* functions on U is defined by

$$\mathcal{O}_q(U) := \{f \in \mathcal{C}^2(U) : \overline{\partial} f \wedge (\partial\overline{\partial} f)^q = 0\}.$$

Clearly, if $q \geq n$, then $\mathcal{O}_q(U) = \mathcal{C}^2(U)$. Furthermore, each q-holomorphic function is $(q+1)$-holomorphic, and the 0-holomorphic functions are the usual holomorphic functions. Moreover, q-holomorphic functions have the following properties which can be found in [Bas76], and the very last property in [HM78].

PROPOSITION 4.2. *Let U be an open set in $\mathbb{C}^n$.*

(i) *A function $f \in \mathcal{C}^2(U)$ lies in $\mathcal{O}_q(U)$ if and only if*

$$\operatorname{rank}\begin{pmatrix} f_{\bar z_1} & \cdots & f_{\bar z_n} \\ f_{z_1\bar z_1} & \cdots & f_{z_1\bar z_n} \\ \vdots & \ddots & \vdots \\ f_{z_n\bar z_1} & \cdots & f_{z_n\bar z_n}\end{pmatrix} \leq q \qquad \text{on } U.$$

(ii) *If $f \in \mathcal{O}_q(U)$, $g \in \mathcal{O}_r(U)$, $\lambda \in \mathbb{C}$ and $m \in \mathbb{N}$, then*

$$f^m, \lambda f \in \mathcal{O}_q(U) \quad \text{and} \quad fg, f+g \in \mathcal{O}_{q+r}(U).$$

(iii) *If W is an open set in $\mathbb{C}^n$, $f \in \mathcal{O}_q(W)$ and $h : U \to W$ a holomorphic mapping, then $f \circ h \in \mathcal{O}_q(U)$.*

(iv) *If $f \in \mathcal{O}_q(U)$ and h is a complex-valued holomorphic function defined in a neighborhood of the image of f, then $h \circ f \in \mathcal{O}_q(U)$.*

(v) *If $q < n$, then every $f \in \mathcal{O}_q(U)$ satisfies the local maximum modulus principle, i.e., for every relatively compact set $D \Subset U$ we have $\max_{\overline{D}} |f| = \max_{bD} |f|$.*

(vi) *If $f \in \mathcal{O}_q(U)$, then $\operatorname{Re} f$ and $\log|f|$ lie in $\mathcal{PSH}_q(U)$.*

We give some examples of q-holomorphic functions, which can be found in [Bas76] as well.

EXAMPLE 4.3. (i) If there are local coordinates $z_1, \ldots, z_n$ such that a $\mathcal{C}^2$-smooth function g depends holomorphically on $n-q$ variables $z_1, \ldots, z_{n-q}$, then g is q-holomorphic.

(ii) Let $h = (h_1, \ldots, h_q)$ be a holomorphic mapping from U into $\mathbb{C}^q$ and $V := \{z \in U : h(z) = 0\}$. Then

$$\chi_{m,V} = \frac{1}{1 + m(|h_1|^2 + \cdots + |h_q|^2)}$$

lies in $\mathcal{O}_q(U)$ for every $m \in \mathbb{N}$ by (i). Moreover, the sequence $(\chi_{m,V})_{m\in\mathbb{N}}$ decreases to the characteristic function χ_V of V in U.

The last example motivates introducing another subfamily of q-holomorphic functions. It will later serve for the characterization of the Shilov boundary of bounded convex domains using a simple subfamily of q-holomorphic functions.

Definition & Remark 4.4. Let U be an open set in $\mathbb{C}^n$.

(i) Let L be an affine complex plane of codimension $q \in \{0, \dots, n-1\}$. We denote by $\mathcal{O}(L, U)$ the set of all functions $h \in \mathcal{C}^2(U)$ which are holomorphic on $U \cap L'$ for every parallel copy L' of the plane L.
(ii) The family $\mathcal{O}_q^\pi(U)$ is the union of all sets $\mathcal{O}(L, U)$, where L varies over all affine complex planes in $\mathbb{C}^n$ of codimension q.
(iii) We easily derive from Example 4.3(i) that $\mathcal{O}_q^\pi(U) \subset \mathcal{O}_q(U)$.

We will examine the Shilov boundaries for the following sets of functions.

Definition 4.5. Let K be a compact set in $\mathbb{C}^n$.

(i) $\mathcal{O}_q(K)$ denotes the set of all continuous functions f on K which have a q-holomorphic extension into an open neighborhood of K, i.e., there exist an open neighborhood U of K and a function $F \in \mathcal{O}_q(U)$ such that $F|K = f$. For $q = 0$ we write $\mathcal{O}(K)$ instead of $\mathcal{O}_0(K)$.
(ii) We define $A_q(K) := \mathcal{C}(K) \cap \mathcal{O}_q(\mathrm{int} K)$ and $A(K) := A_0(K)$.
(iii) $\mathcal{O}(L, K)$ will mean the set of all continuous functions f on K such that there exist a neighborhood U of K and a function $F \in \mathcal{O}(L, U)$ with $F|K = f$. The family $\mathcal{O}_q^\pi(K)$ is then the union of all sets $\mathcal{O}(L, K)$, where L varies over all affine complex planes of codimension q.
(iv) For the respective Shilov boundaries and peak sets we will prefer the shorter notations $b_\mathcal{B}$, $\check{S}_\mathcal{B}$ and $P_\mathcal{B}$ rather than $b_\mathcal{B}(K)$, $\check{S}_\mathcal{B}(K)$ and $P_\mathcal{B}(K)$ if $\mathcal{B}$ is any of the families introduced in this definition.

Note that for historical reasons [Ber31], the Shilov boundary for the family $\mathcal{O}(K)$ is also called the *Bergman boundary* of K. However, for simplicity we will keep the notation introduced at the beginning of this paper.

We have the following properties of the Shilov boundaries for the above subfamilies of q-holomorphic functions.

Proposition 4.6. *Let $q \in \{0, \dots, n-1\}$ and let K be a compact set in $\mathbb{C}^n$. If $\mathcal{B}$ is any of the families from Definition* 4.5, *then*

$$\check{S}_\mathcal{B} = \overline{P_{\overline{\mathcal{B}}}} \in b_\mathcal{B}.$$

Here, $\overline{\mathcal{B}}$ means the uniform closure of $\mathcal{B}$ in $\mathcal{C}(K)$.

Proof. Pick $f \in \mathcal{B}$ and denote by $\mathcal{B}_f$ the uniform closure (in $\mathcal{C}(K)$) of the algebra generated by f and the family $\mathcal{O}(\mathbb{C}^n)$ of holomorphic functions. Then Proposition 4.2(ii) shows that $\mathcal{B}_f \subset \overline{\mathcal{B}}$ for every $f \in \mathcal{B}$. Therefore, if we set $\mathcal{B}_\bullet := \bigcup_{f \in \mathcal{B}} \mathcal{B}_f$, we get the inclusions

$$\mathcal{B} \subset \mathcal{B}_\bullet \subset \overline{\mathcal{B}} \quad \text{and} \quad b_\mathcal{B} = b_{\overline{\mathcal{B}}} \subset b_{\mathcal{B}_\bullet} \subset b_\mathcal{B}. \tag{4.1}$$

Hence, all the Shilov boundaries for $\mathcal{B}$, $\mathcal{B}_\bullet$ and $\overline{\mathcal{B}}$ are the same. Since we can apply Bishop's Theorem 2.8 to each $\mathcal{B}_f$, Corollary 2.9 and (4.1) yield

$$\check{S}_{\mathcal{B}} = \check{S}_{\mathcal{B}_\bullet} = \overline{P_{\mathcal{B}_\bullet}} \in b_{\mathcal{B}_\bullet} = b_{\mathcal{B}}.$$

Finally, the remaining peak point property $\check{S}_{\mathcal{B}} = \overline{P_{\overline{\mathcal{B}}}}$ follows from

$$\check{S}_{\mathcal{B}} = \check{S}_{\mathcal{B}_\bullet} = \overline{P_{\mathcal{B}_\bullet}} \subset \overline{P_{\overline{\mathcal{B}}}} \subset \check{S}_{\overline{\mathcal{B}}} = \check{S}_{\mathcal{B}}. \ \blacksquare$$

In the next statement, we compare the Shilov boundary for subfamilies of q-holomorphic functions defined on subspaces of lower dimensions. Some ideas of its proof are similar to those in [Bas78, proof of Theorem 3].

PROPOSITION 4.7. *Let K be a compact set in $\mathbb{C}^n$ which admits a Stein neighborhood basis. Given a complex plane L in $\mathbb{C}^n$, we have*

$$\check{S}_{\mathcal{O}(K\cap L)} = \check{S}_{\mathcal{O}(L,K)} \cap L.$$

Proof. Observe that $K \cap L$ is non-empty if and only if $\check{S}_{\mathcal{O}(L,K)} \cap L$ is non-empty. Indeed, assume that $K \cap L \neq \emptyset$, but L does not intersect $S_0 := \check{S}_{\mathcal{O}(L,K)}$. For $n \in \mathbb{N}$ let $\chi_n := \chi_{n,L}$ be the functions from Example 4.3. It is obvious that $\chi_n \in \mathcal{O}(L,K)$, since it is constant on each complex plane parallel to L and of the same dimension as L. Recall that χ_n decreases to the characteristic function of L. Then for large enough $n \in \mathbb{N}$, we can arrange that

$$\max_K \chi_n \geq \max_{K\cap L} \chi_n > \max_{S_0} \chi_n,$$

which contradicts the definition of the Shilov boundary for $\mathcal{O}(L,K)$. The other direction is obvious, because $\check{S}_{\mathcal{O}(L,K)}$ is a non-empty subset of K due to Proposition 4.6.

We now prove that $\check{S}_{\mathcal{O}(K\cap L)} \subset \check{S}_{\mathcal{O}(L,K)} \cap L$. Let again $S_0 := \check{S}_{\mathcal{O}(L,K)}$. We have to show that $\max_{K\cap L}|f| = \max_{S_0\cap L}|f|$ for every $f \in \mathcal{O}(K\cap L)$. Fix $f \in \mathcal{O}(K \cap L)$. Then $f \in \mathcal{O}(U \cap L)$ for some open neighborhood U of K. Since K has a Stein neighborhood basis, we can assume that U is pseudoconvex. Let F be a holomorphic extension of f to the whole of U. Then $F_n := F \cdot \chi_{n,L} \in \mathcal{O}(L,K)$ for every $n \in \mathbb{N}$. Furthermore,

$$\max_{K\cap L}|f| = \lim_{n\to\infty} \max_K |F_n| = \lim_{n\to\infty} \max_{S_0} |F_n| = \max_{S_0\cap L}|f|.$$

By definition this means that $\check{S}_{\mathcal{O}(K\cap L)} \subset S_0 \cap L$.

For the other inclusion, take $p \in L \cap P_{\overline{\mathcal{O}(L,K)}}$. Then there is a function f in $\overline{\mathcal{O}(L,K)}$ which peaks on K at p. It is easy to see that $g = f|L$ lies in $\overline{\mathcal{O}(K\cap L)}$ and peaks on $K \cap L$ at p, because $p \in K \cap L$. Thus

$$L \cap P_{\overline{\mathcal{O}(L,K)}} \subset P_{\overline{\mathcal{O}(K\cap L)}} \subset \check{S}_{\overline{\mathcal{O}(K\cap L)}} = \check{S}_{\mathcal{O}(K\cap L)}.$$

Together with Proposition 4.6 we conclude that

$$L \cap \check{S}_{\mathcal{O}(L,K)} = L \cap \overline{P_{\overline{\mathcal{O}(L,K)}}} \subset \check{S}_{\mathcal{O}(K\cap L)}. \ \blacksquare$$

As a consequence, we can compute the Shilov boundary of $\mathcal{O}_q^\pi(K)$ using the classical Shilov boundary.

Corollary 4.8. *Let K be a compact set in $\mathbb{C}^n$ which admits a Stein neighborhood basis, and let Π be the set of all complex planes of codimension q. Then*

$$\check{S}_{\mathcal{O}_q^\pi(K)} = \overline{\bigcup_{L\in\Pi} \check{S}_{\mathcal{O}(K\cap L)}}.$$

In particular, $\check{S}_{\mathcal{O}_{n-1}^\pi(K)} = \check{S}_{\mathcal{O}_{n-1}(K)} = \check{S}_{\mathcal{PSH}_{n-1}(K)} = bK$.

Proof. The first identity follows immediately from Propositions 2.3(iii) and 4.7. The series of identities is derived from the first one by invoking the fact that $\check{S}_{\mathcal{O}(X)} = bX$ for each compact set X in $\mathbb{C}$ together with Proposition 4.2(vi) and the inclusions

$$\log|\mathcal{O}_{n-1}^\pi(K)| \subset \log|\mathcal{O}_{n-1}(K)| \subset \mathcal{PSH}_{n-1}(K).$$

Therefore, $bK = \check{S}_{\mathcal{O}_{n-1}^\pi(K)} \subset \check{S}_{\mathcal{O}_{n-1}(K)} \subset \check{S}_{\mathcal{PSH}_{n-1}(K)} \subset bK$. ∎

Another consequence of Proposition 4.7 is the next observation. It was shown in [Byc81] that the Hausdorff dimension of the Shilov boundary of a convex body K in $\mathbb{C}^2$ for $A(K)$ is no less than 2. It remains an open question whether the Hausdorff dimension of the Shilov boundary for holomorphic functions of a convex body in $\mathbb{C}^n$ is at least n. Anyway, we partially generalize Bychkov's result to the Shilov boundary for q-holomorphic functions using slices of a given convex bounded domain D in $\mathbb{C}^n$. Therefore, we need the next statement which can be found in, e.g., [Fal03, Corollary 7.12]. We also refer to that book for the definition and properties of the Hausdorff dimension $\dim_{\mathrm{H}}$.

Lemma 4.9. *Let I be an m-dimensional cube in $\mathbb{R}^m$, J be an n-dimensional cube in $\mathbb{R}^n$ and F be a subset of $I \times J$. For $x \in I$ consider the slice $F_x := F \cap (\{x\} \times J)$. If $\dim_{\mathrm{H}} F_x \ge \alpha$ for every $x \in I$, then $\dim_{\mathrm{H}} F \ge \alpha + m$.*

The next proposition says that we only need to know a lower estimate for the Shilov boundary for holomorphic functions in order to get a lower estimate for the Shilov boundary for q-holomorphic functions.

Corollary 4.10. *Let D be a convex bounded domain in $\mathbb{C}^n$ and fix $q \in \{0, \dots, n-2\}$. Suppose that there are a constant $\alpha \ge 0$ and a collection $\{L_j\}_{j\in J}$ of disjoint parallel complex affine planes in $\mathbb{C}^n$ with the following properties:*

- *The union $\bigcup_{j\in J} L_j$ is open and intersects D.*
- *For every $j \in J$ the plane L_j has complex codimension q and*

$$\dim_{\mathrm{H}} \check{S}_{\mathcal{O}(\overline{D}\cap L_j)} \ge \alpha.$$

Then

$$\dim_{\mathrm{H}} \check{S}_{\mathcal{O}_q(\overline{D})} \geq \alpha + 2q. \tag{4.2}$$

In particular, $\dim_{\mathrm{H}} \check{S}_{\mathcal{O}_{n-2}(\overline{D})} \geq 2n-2$.

Proof. Propositions 4.7 and 4.6 imply that

$$\bigcup_{j\in J} \check{S}_{\mathcal{O}(\overline{D}\cap L_j)} \subset \bigcup_{j\in J} \check{S}_{\mathcal{O}(L_j,\overline{D})} \subset \check{S}_{\mathcal{O}_q^{\pi}(\overline{D})} \subset \check{S}_{\mathcal{O}_q(\overline{D})}.$$

Hence, it follows from Lemma 4.9 that $\dim_{\mathrm{H}} \check{S}_{\mathcal{O}_q(\overline{D})} \geq \alpha + 2q$.

Now consider the case $q = n-2$. It was shown in [Byc81, Theorem 3.1] that $\dim_{\mathrm{H}} \check{S}_{A(\overline{D}\cap L)} \geq 2$ for every complex two-dimensional affine plane L such that $L \cap D \neq \emptyset$. Since $\check{S}_{A(X)} = \check{S}_{\mathcal{O}(X)}$ for any convex body X in $\mathbb{C}^n$, in view of the inequality (4.2) we conclude that

$$\dim_{\mathrm{H}} \check{S}_{\mathcal{O}_{n-2}(\overline{D})} \geq 2 + 2(n-2) = 2n-2. \ \blacksquare$$

5. Generalization of Bychkov's theorem. S. N. Bychkov [Byc81] gave a characterization of the Shilov boundary for bounded convex domains $D \subset \mathbb{C}^n$. Our goal in this section is to generalize this theorem to Shilov boundaries for subfamilies of q-plurisubharmonic and q-holomorphic functions (see Theorem 5.14 below). First, we recall the main result of Bychkov's article.

THEOREM 5.1 (Bychkov, 1981). *Let D be a bounded convex open set in $\mathbb{C}^n$. A boundary point $p \in bD$ is not in $\check{S}_{A(\overline{D})}$ if and only if there is a neighborhood U of p in bD such that U consists only of complex points* (*see Definition* 5.4). *For $n = 2$, the complement of $\check{S}_{A(\overline{D})}$ in bD admits a local fibration by pieces of complex lines.*

REMARK 5.2. If $D \Subset \mathbb{C}^n$ is a bounded convex domain, it is easy to verify that $\check{S}_{A_q^{\pi}(\overline{D})} = \check{S}_{\mathcal{O}_q^{\pi}(\overline{D})}$ and $\check{S}_{A_q(\overline{D})} = \check{S}_{\mathcal{O}_q(\overline{D})}$ using Proposition 4.2(iii) and the retraction $z \mapsto \lambda(z-p)$ for $\lambda > 0$ and some fixed point $p \in D$.

In his article [Byc81], Bychkov divides the boundary points of a convex body into *real* and *complex* points. We recall their construction. For more definitions and details on convex sets we refer to [Roc70].

DEFINITION & REMARK 5.3. Let K be a *convex body* in $\mathbb{R}^m$, i.e., a compact convex set with non-empty interior. A subset of the boundary bK of K which results from an intersection of K with supporting hyperplanes is called a *face* of K. A face is again a lower-dimensional convex set. The empty set and K itself are also considered to be faces. A face of a face of K does not need to be a face of K. An arbitrary intersection of faces of K is again a face of K.

Given a convex body K, there is a unique minimal face $F_1 = F_{\min}(p, K)$ of $F_0 := K$ in the boundary of K containing the point p. It can be defined as the intersection of K and all supporting hyperplanes for K at p. Then there are two options for p: either it is an inner point of the convex body F_1 or it lies on the boundary of F_1. In the second case, p might again lie either in the interior of the minimal face $F_2 = F_{\min}(p, F_1)$ of F_1 or in the boundary of F_2. Inductively, we obtain a finite sequence $(F_j)_{j=0,\dots,j(p)}$ of convex bodies F_j in K of dimension m_j such that $F_{j+1} = F_{\min}(p, F_j) \subset F_j$ for each $j \in \{0, \dots, j(p) - 1\}$, and if $m_{j(p)} > 0$, then p is an interior point of $F_{j(p)}$, while if $m_{j(p)} = 0$, then the minimal face $F_{j(p)}$ consists only of the point p.

The convex body $F_p(K) := F_{j(p)}$ thus obtained will be called the *face essentially containing* p. It is contained in a plane $E_p = E_p(K)$ of dimension $m_{j(p)}$ which satisfies $E_p \cap K = F_{j(p)}$.

In the following, let D be always a bounded convex domain in $\mathbb{C}^n$. Then real and complex points are defined as follows.

Definition 5.4. Let $p \in bD$, and let $E_p^{\mathbb{C}} = E_p^{\mathbb{C}}(\overline{D})$ be the largest complex plane inside E_p passing through p. We define $\nu(p)$ to be the complex dimension of $E_p^{\mathbb{C}}$. If $\nu(p) = 0$, then E_p is totally real and we say that the point p is *real*.

The symbol $\Pi_p = \Pi_p(\overline{D})$ denotes the set of all complex planes π in $\mathbb{C}^n$ such that there exists a domain $G \subset \mathbb{C}^n$ with $p \in G \cap \pi \subset bD$. If $\Pi_p \neq \{p\}$, then p is called *complex*.

We will need the next two statements (see [Byc81, Lemma 2.5 and its corollary]).

Lemma 5.5. *If $I \subset bD$ is an open segment containing $p \in bD$, then $I \subset F_p$.*

Corollary 5.6. *A boundary point $p \in bD$ is either real or complex.*

From this, it easily follows that p is complex if and only if $\nu(p) \geq 1$.

Corollary 5.7. *If $p \in bD$ is complex, then $E_p^{\mathbb{C}} \in \Pi_p$.*

Proof. Since p is complex, it cannot be real due to Corollary 5.6. Thus, $E_p^{\mathbb{C}}$ is not trivial, so the face essentially containing p cannot be a single point. The point p is then an inner point of the convex body F_p in E_p. Hence, there is an open ball B' with center p in F_p, and we can find another open ball B in $\mathbb{C}^n$ with center p such that $B \cap E_p = B'$. Then

$$E_p^{\mathbb{C}} \cap B \ \subset\ E_p \cap B = B' \ \subset F_p \ \subset\ bD.$$

It now follows from the definition of Π_p that $E_p^{\mathbb{C}}$ lies in Π_p. ∎

Another consequence is that $E_p^{\mathbb{C}}$ is maximal in Π_p.

Corollary 5.8. *If $\pi \in \Pi_p$, then π lies in $E_p^{\mathbb{C}}$.*

Proof. Let G be an open neighborhood of p in $\mathbb{C}^n$ such that $p \in G \cap \pi \subset bD$. It follows from Lemma 5.5 that $p \in G \cap \pi$ lies in F_p. Since $G \cap \pi$ is open in π, we see that $\pi \subset E_p$. Since π is a complex plane containing p, and $E_p^{\mathbb{C}}$ is the largest complex plane inside E_p, we conclude that $\pi \subset E_p^{\mathbb{C}}$. ∎

We specify complex points in the following way.

Definition & Remark 5.9. Let p be a complex boundary point of D and let $q \in \{1, \ldots, n-1\}$.

(i) The point p is called *q-complex* if $\nu(p) = \dim_{\mathbb{C}} E_p^{\mathbb{C}} \geq q$.
(ii) Corollaries 5.7 and 5.8 yield the following characterization: A boundary point p in bD is q-complex if and only if there is a domain G in $\mathbb{C}^n$ and a complex plane of dimension at least q such that $p \in G \cap \pi \subset bD$.

The next lemma asserts that a complex point p is a lower-dimensional real point.

Lemma 5.10. *Let p be a complex point in bD. Let π be a complex affine plane of codimension $\nu(p)$ such that $E_p^{\mathbb{C}} \cap \pi = \{p\}$. Then p is a real boundary point of $\overline{D} \cap \pi$.*

Proof. If $\nu(p) = n-1$, the statement is obvious, since every boundary point of $\overline{D} \cap \pi$ is real.

Suppose that $\nu(p) \leq n-2$ and that the statement is false. Then by Corollary 5.6, p is a complex boundary point of $\overline{D} \cap \pi$. By Corollary 5.7 there exist a domain $G \subset \mathbb{C}^n$ and a complex line $\mathbb{L}$ in π such that $p \in G \cap \mathbb{L} \subset bD \cap \pi \subset bD$. Hence, $\mathbb{L} \in \Pi_p$. By Corollary 5.8 the line $\mathbb{L}$ lies in $E_p^{\mathbb{C}}$. But since $E_p^{\mathbb{C}} \cap \pi = \{p\}$ and $\mathbb{L} \subset \pi$, it follows that $\mathbb{L} = \{p\}$, which is absurd. Thus, p has to be a real boundary point of $\overline{D} \cap \pi$. ∎

We now generalize Proposition 2.6 in [Byc81] which states that a real boundary point always lies in the Shilov boundary for $A(\overline{D})$.

Proposition 5.11. *If $p \in bD$, then $p \in \check{S}_{\mathcal{O}^{\pi}_{\nu(p)}(\overline{D})}$.*

Proof. By Corollary 5.6, p is either real or complex. If p is real, then by [Byc81, Proposition 2.6] and Remark 5.2 we have

$$p \in \check{S}_{A(\overline{D})} = \check{S}_{\mathcal{O}(\overline{D})} = \check{S}_{\mathcal{O}^{\pi}_0(\overline{D})}.$$

Recall that it follows from the definition that $\mathcal{O}(\overline{D}) = \mathcal{O}^{\pi}_0(\overline{D})$.

If p is complex, then there are a domain G and a $\nu(p)$-dimensional complex plane L' such that $p \in G \cap L' \subset bD$. Let L be a complex affine plane of codimension $\nu(p)$ such that $L' \cap L = \{p\}$. Then by Lemma 5.10 the point p

is a real boundary point of the convex body $\overline{D}\cap L$ in L. By [Byc81, Proposition 2.6] and Propositions 4.7 and 4.6 we obtain

$$p \in \check{S}_{A(\overline{D}\cap L)} = \check{S}_{\mathcal{O}(\overline{D}\cap L)} \subset \check{S}_{\mathcal{O}(L,\overline{D})} \subset \check{S}_{\mathcal{O}^{\pi}_{\nu(p)}(\overline{D})}. \blacksquare$$

The next lemma follows from the maximum principle for q-plurisubharmonic functions on analytic sets (see Proposition 3.2(vi) or [Sło86, Corollary 5.3]).

LEMMA 5.12. *Let $p \in bD$ and $q \in \{0,\dots,n-2\}$. If there exists an analytic set in bD which contains p and which is at least $(q+1)$-dimensional at each of its points, then p is not a peak point for $\mathcal{PSH}_q(\overline{D})$. In particular, no $(q+1)$-complex point can be contained in $P_{\mathcal{PSH}_q(\overline{D})}$.*

We are now able to generalize Bychkov's theorem.

DEFINITION 5.13. For $q \in \{1,\dots,n-1\}$ denote by $\mathit{\Gamma}_q(\overline{D})$ the relative interior of the set of all q-complex points of $\overline{D}$ in bD.

THEOREM 5.14. *Let $q \in \{0,\dots,n-2\}$. Then*

$$\check{S}_{\mathcal{O}^{\pi}_q(\overline{D})} = \check{S}_{\mathcal{O}_q(\overline{D})} = \check{S}_{\mathcal{PSH}_q(\overline{D})} = bD \setminus \mathit{\Gamma}_{q+1}(\overline{D}).$$

Proof. If $p \in bD \setminus \check{S}_{\mathcal{O}^{\pi}_q(\overline{D})}$, then there is a neighborhood U of p in bD such that $U \cap \check{S}_{\mathcal{O}^{\pi}_q(\overline{D})} = \emptyset$. Thus, if $w \in U$, then $\nu(w) \geq q+1$ due to Proposition 5.11. This means that U consists only of $(q+1)$-complex points. Hence, $p \in \mathit{\Gamma}_{q+1}(\overline{D})$ and we conclude that $bD \setminus \mathit{\Gamma}_{q+1}(\overline{D}) \subset \check{S}_{\mathcal{O}^{\pi}_q(\overline{D})}$.

On the other hand, if there is a neighborhood U of p in bD such that U contains only $(q+1)$-complex points, then $U \cap P_{\mathcal{PSH}_q(\overline{D})} = \emptyset$ by Lemma 5.12. This implies that $p \notin \overline{P}_{\mathcal{PSH}_q(\overline{D})}$. Since, by Proposition 4.6, the latter set coincides with $\check{S}_{\mathcal{PSH}_q(\overline{D})}$, we get the other inclusion

$$\check{S}_{\mathcal{PSH}_q(\overline{D})} \subset bD \setminus \mathit{\Gamma}_{q+1}(\overline{D}).$$

The proof is now complete since $\log|\mathcal{O}^{\pi}_q(\overline{D})|$ lies in $\mathcal{PSH}_q(\overline{D})$. $\blacksquare$

In what follows, we give some interesting consequences of the previous theorem.

COROLLARY 5.15. *For $q \in \{1,\dots,n-1\}$ let $\mathit{\Gamma}^A_q(\overline{D})$ be the relative interior in bD of the set of all boundary points p of D having the following property: there is an open set U and an analytic set of U which has dimension at least q and which contains p and lies in bD. Then*

$$\mathit{\Gamma}^A_q(\overline{D}) = \mathit{\Gamma}_q(\overline{D}).$$

Proof. Indeed, the inclusion $\mathit{\Gamma}_q(\overline{D}) \subset \mathit{\Gamma}^A_q(\overline{D})$ follows directly from the definition of these two sets and the definition of q-complex points. Now pick $p \in \mathit{\Gamma}^A_q(\overline{D})$. Then Lemma 5.12 and Proposition 4.6 imply that p is not in $\check{S}_{\mathcal{PSH}_{q-1}(\overline{D})}$. Thus, $p \in \Gamma_q(\overline{D})$ by Theorem 5.14. This shows the other inclusion. ∎

In some cases, the Shilov boundary of convex bodies for q-plurisubharmonic functions admits an analytic structure.

Corollary 5.16. *Let $q \in \{1, \ldots, n-1\}$, and assume the set $\{z \in bD : \nu(z) \geq q+1\}$ is open. If*

$$\mathcal{F}_q(\overline{D}) := \operatorname{int}_{bD}(\check{S}_{\mathcal{PSH}_q(\overline{D})} \setminus \check{S}_{\mathcal{PSH}_{q-1}(\overline{D})})$$

is not empty, then it admits a local fibration by complex q-dimensional planes in the following sense: for every point $p \in \mathcal{F}_q(\overline{D})$ there exists a neighborhood U of p in bD such that for each $z \in U$ there is a domain G_z in $\mathbb{C}^n$ and a unique complex q-dimensional plane π_z with $z \in \pi_z \cap G_z \subset U$. In the special case $q = n-1$, these complex (hyper-)planes are parallel to one another.

Proof. We set $\Gamma_n := \emptyset$. By Theorem 5.14 and since $bD = \check{S}_{\mathcal{PSH}_{n-1}}(\overline{D})$, we have $\mathcal{F}_q(\overline{D}) = \Gamma_q(\overline{D}) \setminus \overline{\Gamma_{q+1}(\overline{D})}$. If the set $\{z \in bD : \nu(z) \geq q+1\}$ is open, then it coincides with $\Gamma_{q+1}(\overline{D})$. Thus,

$$\mathcal{F}_q(\overline{D}) = \Gamma_q(\overline{D}) \setminus \overline{\{z \in bD : \nu(z) \geq q+1\}}.$$

Now if $p \in \mathcal{F}_q(\overline{D})$, then there is a neighborhood W of p in $\mathcal{F}_q(\overline{D})$ such that $\nu(z) = q$ for every $z \in W$. Hence, the open set $\mathcal{F}_q(\overline{D})$ consists only of *exactly q-complex points.* Then Corollaries 5.7 and 5.8 imply existence and uniqueness of an open part of a complex q-dimensional plane $\pi_z = E_z^{\mathbb{C}}(\overline{D})$ containing z and lying in U.

For $q = n-1$ the set $\mathcal{F}_{n-1}(\overline{D}) = \Gamma_{n-1}(\overline{D})$ is a convex hypersurface foliated by complex hyperplanes. By a result of Beloshapka and Bychkov [BB86], they have to be parallell (see also Example 5.17 below). ∎

We close the paper by giving some examples related to the previous statement.

Example 5.17. (i) If the openness assumption in Corollary 5.16 is dropped, then it may happen that $\mathcal{F}_1(\overline{D})$ does not admit a foliation by planes in the above sense. To see this, consider the domain G in $\mathbb{C} \times \mathbb{R}$ given by

$$G = \{(x, y, u) \in \mathbb{C} \times \mathbb{R} : x^2 + (1-y^2)u^2 < (1-y^2),\ |y| < 1\}.$$

It is easy to compute that the function $h(y, u) := \sqrt{(1-y^2)(1-u^2)}$ is concave on $[-1, 1]^2$. Since G is the intersection of the sublevel set $\{x < h(y, u)\}$ of the concave function h and the superlevel set $\{x > -h(y, u)\}$ of the convex function $-h$ over $[-1, 1]^2$, it is convex in $\mathbb{C} \times \mathbb{R}$.

The boundary of G contains the *flat* parts $\{\pm i\} \times (-1, 1)$ and $\{0\} \times [-1, 1] \times \{\pm 1\}$ whereas the rest of the boundary consists of *strictly convex* points. By setting $D := G \times (-1, 1)^3$ and $q = 1$ we obtain a convex domain

D in $\mathbb{C}^3$ such that

$$\{z \in bD : \nu(z) \geq 2\} = \{\pm i\} \times (-1,1)^4,$$

and $\Gamma_1(\overline{D})$ is the whole boundary of D. In particular, $\Gamma_2(\overline{D})$ is empty. Thus, the boundary points z in bD with $\nu(z) \geq 2$ lie in $\Gamma_1(\overline{D})$, but there is no unique foliation by complex one-dimensional planes near these points.

(ii) In general, one cannot expect a foliation by parallel planes due to an example by N. Nikolov and P. J. Thomas [NT12] (except in the case of $q = n-1$ as we have seen in the proposition above). Consider the function $\varrho(z) = \operatorname{Re}(z_2)^2 - \operatorname{Re}(z_1)\operatorname{Re}(z_3)$ for $z \in \mathbb{C}^3$. Then the set

$$D := \{z \in \mathbb{C}^3 : \operatorname{Re}(z_1) > 0,\ \varrho(z) < 0\}$$

is convex, and an open part of its boundary is foliated by a real 3-dimensional parameter family of open parts of non-parallel complex lines of the form

$$\{(a^2\zeta + ib, a\zeta + ic, \zeta) : \zeta \in \mathbb{C}\}, \qquad a, b, c \in \mathbb{R}.$$

Acknowledgements. The author was partially supported by the German Research Foundation (DFG, Deutsche Forschungsgemeinschaft) under the project 'Pluripotential Theory, Hulls and Foliations', grant SH 456/1-1. The author would like to thank Prof. N. V. Shcherbina for his supervision during the writing process.

References

[Aiz93] L. Aizenberg, *Carleman's Formulas in Complex Analysis*, Math. Appl. 244, Kluwer, Dordrecht, 1993.

[AW98] H. Alexander and J. Wermer, *Several Complex Variables and Banach Algebras*, 3rd ed., Grad. Texts in Math. 35, Springer, New York, 1998.

[Bas76] R. F. Basener, *Nonlinear Cauchy–Riemann equations and q-pseudo-convexity*, Duke Math. J. 43 (1976), 203–213.

[Bas78] R. F. Basener, *Several dimensional properties of the spectrum of a uniform algebra*, Pacific J. Math. 74 (1978), 297–306.

[BB86] V. K. Beloshapka and S. N. Bychkov, *Convex hypersurfaces in* $\mathbf{C}^n$, Math. Notes 40 (1986), 854–857.

[Ber31] S. Bergmann, *Über ausgezeichnete Randflächen in der Theorie der Funktionen von zwei komplexen Veränderlichen*, Math. Ann. 104 (1931), 611–636.

[Bis59] E. Bishop, *A minimal boundary for function algebras*, Pacific J. Math. 9 (1959), 629–642.

[Bre59] H. J. Bremermann, *On a generalized Dirichlet problem for plurisubharmonic functions and pseudo-convex domains. Characterization of Šilov boundaries*, Trans. Amer. Math. Soc. 91 (1959), 246–276.

[Bun90] L. Bungart, *Piecewise smooth approximations to q-plurisubharmonic functions*, Pacific J. Math. 142 (1990), 227–244.

[Byc81] S. N. Bychkov, *On geometric properties of the boundary of a domain of holomorphy*, Math. USSR-Izv. 16 (1981), 41–54.

[Dal71] H. G. Dales, *Boundaries and peak points for Banach function algebras*, Proc. London Math. Soc. (3) 22 (1971), 121–136.

[Dem12] J.-P. Demailly, *Complex analytic and differential geometry*, http://www-fourier.ujf-grenoble.fr/∼demailly/manuscripts/agbook.pdf, 2012.

[DF85] K. Diederich and J. E. Fornæss, *Smoothing q-convex functions and vanishing theorems*, Invent. Math. 82 (1985), 291–305.

[Die06] N. Q. Dieu, *q-plurisubharmonicity and q-pseudoconvexity in* $\mathbb{C}^n$, Publ. Mat. 50 (2006), 349–369.

[Fal03] K. Falconer, *Fractal Geometry*, 2nd ed., Wiley, Hoboken, 2003.

[Fuj90] O. Fujita, *Domaines pseudoconvexes d'ordre général et fonctions pseudoconvexes d'ordre général*, J. Math. Kyoto Univ. 30 (1990), 637–649.

[HL88] G. M. Henkin and J. Leiterer, *Andreotti–Grauert Theory by Integral Formulas*, Progr. Math. 74, Birkhäuser Boston, Boston, MA, 1988.

[Hon88] T. G. Honary, *The density of peak points in the Shilov boundary of a Banach function algebra*, Proc. Amer. Math. Soc. 103 (1988), 480–482.

[HM78] L. R. Hunt and J. J. Murray, *q-plurisubharmonic functions and a generalized Dirichlet problem*, Michigan Math. J. 25 (1978), 299–316.

[NT12] N. Nikolov and P. J. Thomas, *Convex domains with locally Levi-flat boundaries*, arXiv:1112.1867v3 (2012).

[PZ13] T. Pawlaschyk and E. S. Zeron, *On convex hulls and pseudoconvex domains generated by q-plurisubharmonic functions, part I*, J. Math. Anal. Appl. 408 (2013), 394–408.

[Roc70] R. T. Rockafellar, *Convex Analysis*, Princeton Math. Ser. 28, Princeton Univ. Press, Princeton, NJ, 1970.

[Sic62] J. Siciak, *On function families with boundary*, Pacific J. Math. 12 (1962), 375–384.

[Sło84] Z. Słodkowski, *The Bremermann–Dirichlet problem for q-plurisubharmonic functions*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) 11 (1984), 303–326.

[Sło86] Z. Słodkowski, *Local maximum property and q-plurisubharmonic functions in uniform algebras*, J. Math. Anal. Appl. 115 (1986), 105–130.

[Wit83] R. Wittmann, *Shilov points and Shilov boundaries*, Math. Ann. 263 (1983), 237–250.

Thomas Patrick Pawlaschyk
Faculty of Mathematics and Natural Sciences
University of Wuppertal
42119 Wuppertal, Germany
E-mail: pawlaschyk@math.uni-wuppertal.de