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On the Class of Perfectly Null Sets and Its Transitive Version

by

Michał KORCH and Tomasz WEISS

Presented by Stanisław KWAPIEŃ

Summary. We introduce two new classes of special subsets of the real line: the class of perfectly null sets and the class of sets which are perfectly null in the transitive sense. These classes may play the role of duals to the corresponding classes on the category side. We investigate their properties and, in particular, we prove that every strongly null set is perfectly null in the transitive sense, and that it is consistent with ZFC that there exists a universally null set which is not perfectly null in the transitive sense. Finally, we state some open questions concerning the above classes. Although the main problem of whether the classes of perfectly null sets and universally null sets are consistently different remains open, we prove some results related to this question.

1. Motivation and preliminaries. Among classes of special subsets of the real line, the classes of *perfectly meager sets* (sets which are meager relative to any perfect set, here denoted by PM) and *universally null sets* (sets which are null with respect to any possible finite diffuse Borel measure, denoted by UN) were considered to be dual (see [13]), though some differences between them have been observed. For example, the class of universally null sets is closed under taking products (see [13]), but it is consistent with ZFC that this is not the case for perfectly meager sets (see [19] and [20]).

Table 1. Classes of special subsets of the real line

category	PM	$\supseteq$	UM	$\supseteq$	PM′	$\supseteq$	SM
measure	$\boxed{?}$		UN		$\boxed{?}$		SN

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In [24], P. Zakrzewski proved that two other earlier defined classes of sets (see [8] and [7]), smaller than PM, coincide and are dual to UN. Therefore, he proposed to call this class the universally meager sets (denoted by UM). A set $A \subseteq 2^\omega$ is *universally meager* if every Borel isomorphic image of A in 2^ω is meager. The class PM was left without a counterpart (see Table 1), and in this paper (answering an oral question of P. Zakrzewski) we try to define a class of sets which may play the role of a dual class to PM.

In [15], the authors introduced a notion of perfectly meager sets in the transitive sense (denoted here by PM′), which turned out to be stronger than the classical notion of perfectly meager sets. In this article, we define an analogous class, which will be called the class of perfectly null sets in the transitive sense PN′, and we investigate its properties.

We assume that the reader is familiar with the standard terminology of special subsets of the reals, and we recall some definitions that are less common (see also [4], [2] and [13]).

Throughout this paper we will work generally in the Cantor space 2^ω. The basic clopen set in 2^ω determined by a finite sequence $w \in 2^{<\omega}$ will be denoted by $[w]$. If F is a set of partial functions $\omega \to 2$, the expression $[F]$ denotes $\bigcup_{f \in F}\{x \in 2^\omega : x{\restriction}_{\operatorname{ran} f} = f\}$. The Cantor space will also be considered as a vector space over $\mathbb{Z}_2$. In particular, for $A, B \subseteq 2^\omega$, we let $A + B = \{t + s : t \in A,\, s \in B\}$.

Recall that a set A is *strongly null* (strongly of measure zero) if for any sequence of $\delta_n > 0$, there exists a sequence $\langle A_n \rangle_{n\in\omega}$ of open sets with $\operatorname{diam} A_n < \varepsilon_n$ for $n \in \omega$, where $\operatorname{diam} X$ denotes the diameter of a set X, and such that $A \subseteq \bigcup_{n\in\omega} A_n$. We denote the class of such sets by SN. Galvin, Mycielski and Solovay [6] proved that $A \in \mathrm{SN}$ (in 2^ω) if and only if for any meager set B, there exists $t \in 2^\omega$ such that $A \cap (B + t) = \emptyset$. Therefore, one can consider a dual class of sets. A set A is called *strongly meager* (strongly first category, denoted by SM) if for any null set B, there exists $t \in 2^\omega$ such that $A \cap (B + t) = \emptyset$.

Finally, we shall say that an uncountable set $L \subseteq 2^\omega$ is a *Lusin* (respectively, *Sierpiński*) *set* if for any meager (respectively, null) set X, $L \cap X$ is countable.

2. Perfectly null sets

2.1. Canonical measure on a perfect set. If P is a closed set in 2^ω, there is a pruned tree $T_P \subseteq 2^{<\omega}$ such that the set of all infinite branches of T_P (usually denoted by $[T_P]$) equals P. If T is a pruned tree, then $[T]$ is perfect if and only if for any $w \in T$, there exist $w', w'' \in T$ such that $w \subseteq w', w \subseteq w''$, but $w' \nsubseteq w''$ and $w'' \nsubseteq w'$. Such a tree is called a *perfect tree*.

If $w \in 2^n$, and $a, b \in \omega$ with $a \le b$, then $w[a,b] \in 2^{b-a+1}$ denotes the finite sequence such that $w[a,b](i) = w(a+i)$ for $i \le b-a$. If $\langle s_0, s_1, \dots s_k\rangle$ is a finite sequence of natural numbers less than n, then $w\,\langle s_0, s_1, \dots s_k\rangle \in 2^{k+1}$ denotes a sequence such that $w\,\langle s_0, s_1, \dots s_k\rangle\,(i) = w(s_i)$ for any $i \le k$.

A finite sequence $w \in T_P$ will be called a *branching point* of a perfect set P if $w^\frown 0, w^\frown 1 \in T_P$. A branching point is *on level* $i \in \omega$ if there exist i branching points below it. The set of all branching points of P on level i will be denoted by $\mathrm{Split}_i(P)$, and $\mathrm{Split}(P) = \bigcup_{i\in\omega} \mathrm{Split}_i(P)$. Let $s_i(P) = \min\{|w| : w \in \mathrm{Split}_i(P)\}$ and $S_i(P) = \max\{|w| : w \in \mathrm{Split}_i(P)\}$. For $i > 0$, we say that $w \in T_P$ is *on level* i in P (written $l_P(w) = i$) if there exist $v, t \in T_P$ such that $v \subsetneq w \subseteq t$, $v \in \mathrm{Split}_{i-1}(P), t \in \mathrm{Split}_i(P)$. We say that $w \in T_P$ is *on level* 0 if $w \subseteq t$ where $t \in \mathrm{Split}_0(P)$.

Let P be a perfect set in 2^ω and $h_P \colon 2^\omega \to P$ be the homeomorphism given by the order isomorphism of $2^{<\omega}$ and $\mathrm{Split}(P)$. We call this homeomorphism the *canonical homeomorphism*. Let m denote the Lebesgue measure (the standard product measure) on 2^ω.

Definition 2.1. Let $A \subseteq P$ be such that $h_P^{-1}[A]$ is measurable in 2^ω. We define $\mu_P(A) = m(h_P^{-1}[A])$. The measure μ_P will be called *the canonical measure* on P. A set $A \subseteq P$ such that $\mu_P(A) = 0$ will be called *P-null*, and a set measurable with respect to μ_P will be called *P-measurable*.

The same idea of the canonical measure on a perfect set was used in [5].

Remark. Sometimes the measure μ_P will be considered as a measure on the whole 2^ω by setting $\mu_P(A) = \mu_P(A \cap P)$ for $A \subseteq 2^\omega$ such that $A \cap P$ is P-measurable.

For $w \in T_P$, we set $[w]_P = [w] \cap P$. Notice that if $w \in T_P$ is on level i in P, then $\mu_P([w]_P) = 1/2^i$. If $Q \subseteq P$ is perfect, then $T_Q \subseteq T_P$, and therefore if $w \in T_Q$, then $l_Q(w) \le l_P(w)$, so $\mu_Q([w]_Q) \ge \mu_P([w]_P)$. By defining the outer measure $\mu_P^*(A) = m^*(h_P^{-1}[A])$, where m^* is the Lebesgue outer measure, we obtain the following proposition.

Proposition 2.2. *If Q, P are perfect sets such that $Q \subseteq P$, and $A \subseteq Q$, then $\mu_P^*(A) \le \mu_Q^*(A)$. In particular, every Q-null set $A \subseteq Q$ is also P-null.* ∎

Proposition 2.3. *If Q, P are perfect sets such that $Q \subseteq P$, and A is a Q-measurable subset of Q, then it is P-measurable.*

Proof. If A is Q-measurable, there exists a Borel set $B \subseteq 2^\omega$ such that $B \cap Q \subseteq A$ and $\mu_Q(A \setminus B) = 0$, so $\mu_P(A \setminus B) = 0$. Let $B' = B \cap Q$. Then B' is Borel, $\mu_P(A \setminus B') = \mu_P(A \setminus B) = 0$ and $B' \subseteq A$. ∎

Corollary 2.4. *If P is perfect, and $Q_n \subseteq P$ for $n \in \omega$ are perfect sets such that $\mu_P(\bigcup_n Q_n) = 1$ and $A \subseteq P$ is such that $A \cap Q_n$ is Q_n-measurable* *for any $n \in \omega$, then A is P-measurable and $\mu_P(A) \leq \sum_{n\in\omega} \mu_{Q_n}(A \cap Q_n)$. In particular, if $A \cap Q_n$ is Q_n-null for all $n \in \omega$, then A is P-null.* ■

We will need the following lemma.

LEMMA 2.5. *Let $P \subseteq 2^\omega$ be a perfect set, $k \in \omega$ and $X \subseteq 2^\omega$ be such that for all $t \in P$, there exist infinitely many $n \in \omega$ such that there exists $w \in 2^k$ with $[t{\restriction}_n {}^\frown w]_P \subseteq P \setminus X$. Then $\mu_P(X) = 0$.*

Proof. Notice that if $k = 0$, then $X \cap P = \emptyset$, so we can assume that $k > 0$. We prove by induction that for any $m \in \omega$, there exists a finite set $S_m \subseteq T_P$ such that $X \cap P \subseteq \bigcup_{s\in S_m} [s]_P$, and

$$\sum_{s\in S_m} \frac{1}{2^{l_P(s)}} \leq \left(\frac{2^k - 1}{2^k}\right)^m.$$

Let $S_0 = \{\emptyset\}$. Given S_m, for each $s \in S_m$ and each $t \in P$ such that $s \subseteq t$, we can find $s_{s,t} \in T_P$ such that $s \subseteq s_{s,t} \subseteq t$ and $w_{s,t} \in 2^k$ with $[s_{s,t} {}^\frown w_{s,t}]_P \subseteq P \setminus X$. Therefore, since $[s]_P$ is compact, we can find a finite set $A_s \subseteq P$ such that $[s]_P = \bigcup_{t\in A_s} [s_{s,t}]_P$ and $[s_{s,t}]_P \cap [s_{s,t'}]_P = \emptyset$ if $t, t' \in A_s$ and $t \neq t'$. Let

$$S_{m+1} = \left\{ s_{s,t} {}^\frown w \colon s \in S_m \wedge t \in A_s \wedge w \in 2^k \setminus \{w_{s,t}\} \right\} \cap T_P.$$

We have $X \cap P \subseteq \bigcup_{s\in S_{m+1}} [s]_P$. Notice also that for $s \in S_m$,

$$\sum_{t\in A_s} \frac{1}{2^{l_P(s_{s,t})}} = \frac{1}{2^{l_P(s)}}.$$

Moreover, if $t \in A_s$, then

$$\sum_{w\in 2^k\setminus\{w_{s,t}\}} \frac{1}{2^{l_P(s_{s,t} {}^\frown w)}} \leq \frac{2^k - 1}{2^k} \cdot \frac{1}{2^{l_P(s_{s,t})}}.$$

Therefore,

$$\sum_{s\in S_{m+1}} \frac{1}{2^{l_P(s)}} \leq \frac{2^k - 1}{2^k} \cdot \sum_{s\in S_m} \frac{1}{2^{l_P(s)}} \leq \left(\frac{2^k - 1}{2^k}\right)^{m+1},$$

which concludes the induction argument.

Thus, $\mu_P(X) \leq (1 - 1/2^k)^m$ for any $m \in \omega$, and so $\mu_P(X) = 0$. ■

Now, we define a possible measure analogue of the class of perfectly meager sets.

DEFINITION 2.6. We shall say that $A \subseteq 2^\omega$ is *perfectly null* if it is null in any perfect set $P \subseteq 2^\omega$ with respect to the measure μ_P. The class of perfectly null sets will be denoted by PN.

Proposition 2.7. *The following conditions are equivalent for a set* $A \subseteq 2^\omega$*:*

(i) *A is perfectly null,*
(ii) *for every perfect $P \subseteq 2^\omega$, $A \cap P$ is P-measurable, but $P \setminus A \neq \emptyset$,*
(iii) *there exists $n \in \omega$ such that for every $w \in 2^n$ and every perfect $P \subseteq [w]$, $A \cap P$ is P-null.*

Proof. Notice that if $A \cap P$ is P-measurable with $\mu_P(A \cap P) > 0$, then we can find a closed uncountable set F such that $F \subseteq A \cap P$. Therefore, there is a perfect set $Q \subseteq F$; hence $Q \subseteq A$, so $Q \setminus A = \emptyset$. Moreover, given any perfect set P we have $P = \bigcup_{w \in 2^n, w \in T_P} [w]_P$, and for any $w \in 2^n$ such that $w \in T_P$, the set $[w]_P$ is perfect. ∎

2.2. The main open problem

Proposition 2.8. UN $\subseteq$ PN.

Proof. Let $A \subseteq 2^\omega$ be universally null, and let P be perfect. Let λ be a measure on 2^ω such that $\lambda(B) = \mu_P(B \cap P)$ for any Borel set $B \subseteq 2^\omega$. Then $\lambda(A) = 0$, so A is P-null. ∎

Unfortunately, we do not know the answer to the following question.

Problem 2.9. *Is it consistent with ZFC that* UN $\neq$ PN*?*

Remark. On the category side every proof of the consistency of the fact that UM $\neq$ PM known to the authors uses the idea of a Lusin function or similar arguments. A Lusin function $\mathcal{L}\colon \omega^\omega \to 2^\omega$ was defined in [11], and extensively described in [23]. To get a Lusin function we construct a system $\langle P_s \colon s \in \omega^{<\omega}\rangle$ of perfect sets such that for $s \in \omega^{<\omega}$ and $n, m \in \omega$, $\operatorname{diam} P_s \leq 1/2^{|s|}$, $P_{s^\frown n} \subseteq P_s$ is nowhere dense in P_s, $\bigcup_{k\in\omega} P_{s^\frown k}$ is dense in P_s, and if $n \neq m$, then $P_{s^\frown n} \cap P_{s^\frown m} = \emptyset$. Next, we set $\mathcal{L}(x)$ to be the only point of $\bigcap_{n\in\omega} P_{x|_n}$. One can prove that $\mathcal{L}$ is a continuous and one-to-one function. Furthermore, if $Q \subseteq 2^\omega$ is a perfect set, then $\mathcal{L}^{-1}[\bigcup\{P_s \colon P_s \text{ is nowhere dense in } Q\}]$ contains an open dense set. Therefore, if L is a Lusin set, then $\mathcal{L}[L]$ is perfectly meager (see also [13]). Moreover, $\mathcal{L}^{-1}$ is a function of the first Baire class. Given such a function it is easy to see that if there exists a Lusin set L, then UM $\neq$ PM. This should be clear since UM is closed under taking Borel isomorphic images, so $\mathcal{L}[L] \in$ PM$\setminus$UM.

Therefore, to prove PN $\neq$ UN, we possibly need some analogue of a Lusin function.

Problem 2.10. *Is there an analogue of a Lusin function for perfectly null sets?*

But even if such an analogue exists, it cannot be constructed in a similar way to Lusin's argument. Indeed, if we equip ω^ω with the natural measure

m defined by

$$m([w]) = \prod_{i=0}^{|w|-1} \frac{1}{2^{w(i)+1}},$$

where $w \in \omega^{<\omega}$ and $[w] = \{f \in \omega^\omega : w \subseteq f\}$, we get the following proposition.

PROPOSITION 2.11. *Let $\mathcal{S}: \omega^\omega \to 2^\omega$ be a function such that there exists a sequence $\langle P_s : s \in \omega^{<\omega} \rangle$ such that each $P_s \subseteq 2^\omega$ is a perfect set, and for $n, m \in \omega$, $n \neq m \to P_{s^\frown n} \cap P_{s^\frown m} = \emptyset$, $P_{s^\frown n} \subseteq P_s$, $\operatorname{diam} P_s \leq 1/2^{|s|}$, and $\mathcal{S}(x)$ is the only element of $\bigcap_{n\in\omega} P_{x\restriction_n}$. Then there exists a perfect set $Q \subseteq 2^\omega$ such that $m(\mathcal{S}^{-1}[\bigcup\{P_s : \mu_Q(P_s) = 0\}]) < 1$.*

Proof. We define $T \subseteq \omega^{<\omega}$ inductively as follows: in the nth step we construct $T_n = T \cap \omega^n$ such that $|T_n| < \omega$ for all $n \in \omega$. Let $T_0 = \{\emptyset\}$. Assume that T_n is constructed and $w \in T_n$. Let $M_w \geq 2$ be such that $2^{M_w} \geq 2^{n+2} \cdot |T_n| \cdot m([w])$ and $T_{n+1} = \{w^\frown k : w \in T_n \wedge k \in \omega \wedge k < M_w\}$.

Therefore, if $w \in T_n$, then

$$\begin{aligned} m\Big([w] \setminus \bigcup\{w^\frown k : k < M_w\}\Big) &= m\Big(\bigcup\{w^\frown k : k \geq M_w\}\Big) \\ &= m([w]) \cdot \sum_{i=M_w}^{\infty} \frac{1}{2^{i+1}} = \frac{m([w])}{2^{M_w}} \leq \frac{1}{2^{n+2}|T_n|}. \end{aligned}$$

Thus, for all $n \in \omega$,

$$m\Big(\bigcup\{[s] : s \in T_n\} \setminus \bigcup\{[s] : s \in T_{n+1}\}\Big) \leq \frac{1}{2^{n+2}},$$

so

$$m\Big(\bigcup\{[s] : s \notin T\}\Big) = m\Big(\bigcup_{n\in\omega}\Big(\bigcup\{[s] : s \in T_n\} \setminus \bigcup\{[s] : s \in T_{n+1}\}\Big)\Big) \leq \frac{1}{2}.$$

Let $Q = \bigcap_{n\in\omega} \bigcup_{s\in T_n} P_s$. Obviously, Q is a closed set. Moreover, if $s \in T$, there exists $w \in 2^{<\omega}$ with $[w]_Q \subseteq P_s$. This should be clear since for all $n \in \omega$, $\{P_s : s \in T_n\}$ is a finite collection of disjoint perfect sets, and $Q \subseteq \bigcup_{s\in T_n} P_s$. Therefore, Q is perfect and $\mu_Q(P_s) > 0$. On the other hand, if $s \notin T$, then $P_s \cap Q = \emptyset$, so $\mu_Q(P_s) = 0$. Therefore, if $\mathcal{S}(x) \in P_s$ and $\mu_Q(P_s) = 0$, then $s \notin T$ and $x \in [s]$, so

$$m\Big(\mathcal{S}^{-1}\Big[\bigcup\{P_s : \mu_Q(P_s) = 0\}\Big]\Big) = m\Big(\bigcup\{[s] : s \notin T\}\Big) \leq 1/2. \ \blacksquare$$

Obviously, since for every diffuse Borel measure μ, there exists a Borel isomorphism of 2^ω mapping μ to the Lebesgue measure (see e.g. [12, Theorem 4.1(ii)]), if the class PN is closed under Borel automorphisms of 2^ω, then UN = PN, which motivates the following question, which was asked by the reviewer.

Problem 2.12. *Is the class* PN *closed under homeomorphisms of* 2^ω *onto itself?*

2.3. Simple perfect sets. To understand what may happen when one attempts to solve the main open problem mentioned above, we restrict our attention to some special subfamilies of perfect sets. This will lead to an important result in Theorem 2.25.

Definition 2.13. A perfect set P will be called *balanced* if $s_{i+1}(P) > S_i(P)$ for all $i \in \omega$. This generalizes the notion of uniformly perfect set (see [3]). A perfect set P is *uniformly* perfect if for any $i \in \omega$, either $2^i \cap T_P \subseteq \mathrm{Split}(P)$ or $2^i \cap \mathrm{Split}(P) = \emptyset$. If additionally, in a uniformly perfect set P,

$$\forall_{w,v \in T_P,\, |w|=|v|}\ \forall_{j \in \{0,1\}}\ (w^\frown j \in T_P \to v^\frown j \in T_P),$$

then P is called a *Silver* perfect set (see for example [10]).

A set that is null in any balanced (respectively, uniformly, Silver) perfect set will be called balanced perfectly null (respectively, uniformly perfectly null, Silver perfectly null). The class of such sets will be denoted by bPN (respectively, uPN, vPN). Obviously, PN $\subseteq$ bPN $\subseteq$ uPN $\subseteq$ vPN.

Lemma 2.14. *There exists a perfect set E such that for every balanced perfect set B, we have either $\mu_B(E) = 0$ or $\mu_E(B) = 0$.*

Proof. Consider $K = \{000, 001, 011, 111\} \subseteq 2^3$ and a perfect set $E \in 2^\omega$ such that $x \in E$ if and only if $x[3k, 3k+2] \in K$ for every $k \in \omega$ (see Figure 1). Let B be a balanced perfect set. Imagine now how T_B looks like in a K-block of T_E (see Figure 1, where T_B is shown as dotted lines). Let $k \in \omega$ and $w \in T_E \cap 2^{3k}$. The following two situations are possible: either $\{w^\frown s\colon s \in K\} \subseteq T_B$ (possibility (a)), or $\{w^\frown s\colon s \in K\} \setminus T_B \neq \emptyset$ (possibility (b)).

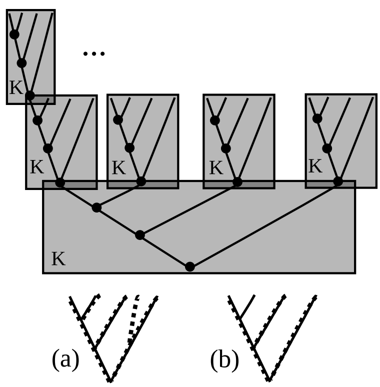

Fig. 1. Proof of Lemma 2.14

Assume that for all $t \in E$, there exist infinitely many $k \in \omega$ such that $\{t{\restriction}_{3k}{}^\frown s \colon s \in K\} \setminus T_B \neq \emptyset$ (case (b)). Then $\mu_E(B) = 0$ by Lemma 2.5.

On the other hand, assume that there exists $t \in E$ such for all but finite $k \in \omega$, we have $\{t{\restriction}_{3k}{}^\frown s \colon s \in K\} \subseteq T_B$ (case (a)). Then there exists $i \in \omega$ such that B has a branching point of length j for all $j \geq i$, so $s_{j+1}(B) \leq S_j(B) + 1$ for any $j \geq i$. And since B is a balanced perfect set, this implies that $s_j(B) = S_j(B)$ and $s_{j+1}(B) = s_j(B) + 1$ for any $j > i$. In other words, for $w \in T_B \cap 2^i$, $B \cap [w] = [w]$, and therefore for any $v \in T_B \cap 2^{3k}$ with $3k > i$, there exists $w \in 2^3$ such that $v^\frown w \in T_B \setminus T_E$. It follows that $\mu_B(E) = 0$ by Lemma 2.5. ∎

Proposition 2.15. *Suppose that there exists a Sierpiński set. Then* $\mathrm{PN} \subsetneq \mathrm{bPN}$.

Proof. Let E be the perfect set defined in Lemma 2.14, and let $S \subseteq E$ be a Sierpiński set with respect to μ_E. Obviously, S is not perfectly null. But if B is a balanced perfect set, then either $\mu_B(E) = 0$, so $\mu_B(S) = 0$, or $\mu_E(B) = 0$, so $S \cap B$ is countable. Thus, $\mu_B(S) = 0$. So $S \in \mathrm{bPN} \setminus \mathrm{PN}$. ∎

Proposition 2.16. $\mathrm{bPN} \subsetneq \mathrm{uPN} \subsetneq \mathrm{vPN}$.

Proof. The first inclusion is proper, because if we take any balanced perfect set B such that $|\mathrm{Split}(B) \cap 2^i| = 1$ for each $i \in \omega$, and any uniformly perfect set U, then $\mu_U(B) \leq (n+1)/2^n$ for any $n \in \omega$, so B is U-null. Thus, $B \in \mathrm{uPN} \setminus \mathrm{bPN}$.

To see that the second inclusion is proper, notice that the uniformly perfect set $U = \{\alpha \in 2^\omega \colon \forall_{i\in\omega}\ \alpha(2i+1) = \alpha(2i)\}$ is null in every Silver perfect set. Indeed, let S be a Silver perfect set. Let $i \in \omega$ be such that for every $w \in 2^{2i} \cap S$, $w \in \mathrm{Split}(S)$, or for every $w \in 2^{2i+1} \cap S$, $w \in \mathrm{Split}(S)$. The following two cases are possible:

- For every $w \in 2^{2i} \cap S$, we have $w \in \mathrm{Split}(S)$, so $w^\frown 0, w^\frown 1 \in T_S$. Then $w^\frown 0^\frown 1 \in T_S$ or $w^\frown 0^\frown 0 \in T_S$. In the first case $w^\frown 0^\frown 1 \in T_S \setminus T_U$, while in the second $w^\frown 1^\frown 0 \in T_S$, but $w^\frown 1^\frown 0 \notin T_U$.
- For every $w \in 2^{2i} \cap S$, we have $w \notin \mathrm{Split}(S)$. Without loss of generality, assume that $w^\frown 0 \in T_S$. Then $w^\frown 0 \in \mathrm{Split}(S)$ and $w^\frown 0^\frown 1 \in T_S \setminus T_U$.

Since there exist infinitely many $i \in \omega$ such that $2^{2i} \cap S \subseteq \mathrm{Split}(S)$ or $2^{2i+1} \cap S \subseteq \mathrm{Split}(S)$, Lemma 2.5 can be applied to conclude that $\mu_S(U) = 0$. ∎

Proposition 2.17. *The following conditions are equivalent for* $A \subseteq 2^\omega$*:*

(i) *A is perfectly null,*
(ii) *for every perfect set $P \subseteq 2^\omega$, $A \cap P$ is P-measurable, but for every balanced perfect set $Q \subseteq 2^\omega$, $Q \setminus A \neq \emptyset$,*
(iii) *for every perfect set $P \subseteq 2^\omega$, $A \cap P$ is P-measurable and $A \in \mathrm{bPN}$.*

Proof. Notice that there exists a balanced perfect set in every perfect set. Therefore, in the proof of Proposition 2.7 we can require that the perfect set Q is balanced. ∎

Remark. Notice that even if a set is P-measurable for any perfect set and does not contain any uniformly perfect set, it need not be perfectly null. An example is the set B from the proof of Proposition 2.16.

Proposition 2.18.

(i) $A \in \mathrm{bPN}$ *if and only if for every balanced perfect* $P \subseteq 2^\omega$, $A \cap P$ *is* P*-measurable, but* $P \setminus A \neq \emptyset$.
(ii) $A \in \mathrm{uPN}$ *if and only if for every uniformly perfect* $P \subseteq 2^\omega$, $A \cap P$ *is* P*-measurable, but* $P \setminus A \neq \emptyset$.
(iii) $A \in \mathrm{vPN}$ *if and only if for every Silver perfect* $P \subseteq 2^\omega$, $A \cap P$ *is* P*-measurable, but* $P \setminus A \neq \emptyset$.

Proof. We proceed as in the proof of Proposition 2.7. For uniformly perfect and Silver perfect sets we use [10, Lemma 2.4], which states that there exists a Silver perfect set in every set of positive Lebesgue measure, and we notice that if P is a uniformly (respectively, Silver) perfect set, and $h_P \colon 2^\omega \to P$ is the canonical homeomorphism, then the image of any Silver perfect set is uniformly (respectively, Silver) perfect. ∎

2.4. Perfectly null sets and the s_0 and v_0 ideals. Recall that a set A is a *Marczewski s_0-set* if for any perfect set P, there exists a perfect set $Q \subseteq P$ such that $Q \cap A = \emptyset$.

Proposition 2.19. $\mathrm{PN} \subseteq \mathrm{bPN} \subseteq s_0$.

Proof. Indeed, if P is perfect and $X \in \mathrm{bPN}$, let $B \subseteq P$ be a balanced perfect set. Then $\mu_B(B \setminus X) = 1$, so there exists a closed set $F \subseteq B \setminus X$ of positive measure. Therefore, it is uncountable, and there exists a perfect set $Q \subseteq F \subseteq P \setminus X$. ∎

Remark. Obviously, $\mathrm{uPN} \not\subseteq s_0$ (see the proof of Proposition 2.16).

We say that a set X has the v_0 *property* if for every Silver perfect set P, there exists a Silver perfect set $Q \subseteq P \setminus X$ (see [10]).

Proposition 2.20. $\mathrm{PN} \subseteq \mathrm{vPN} \subseteq v_0$.

Proof. Let $P \subseteq 2^\omega$ be a Silver perfect set, and let $X \in \mathrm{vPN}$. Notice that the image of any Silver perfect set under the canonical homeomorphism $h_P \colon 2^\omega \to P$ is a Silver perfect set. Since $m(2^\omega \setminus h_P^{-1}[X]) = 1$, there exists a Silver perfect set $Q \subseteq 2^\omega \setminus h_P^{-1}[X]$ (see [10, Lemma 2.4]). So, $h_P[Q] \subseteq P \setminus X$ is a Silver perfect set. ∎

M. Scheepers [22] proved that if X is a measure zero set with the s_0 property, and S is a Sierpiński set, then $X + S$ is also an s_0-set. Therefore, we easily obtain the following proposition.

PROPOSITION 2.21. *The algebraic sum of a Sierpiński set and a perfectly null set is an s_0-set.* ∎

2.5. Products. We consider PN sets in the product $2^\omega \times 2^\omega$ using the natural homeomorphism $h\colon 2^\omega \times 2^\omega \to 2^\omega$ defined as

$$h(x, y) = \langle x(0), y(0), x(1), y(1), \ldots \rangle .$$

It is consistent with ZFC that the product of two perfectly meager sets is not perfectly meager (see [20], [19]). If the answer to Problem 2.9 is positive, then it makes sense to ask the following question.

PROBLEM 2.22. *Is the product of any two perfectly null sets perfectly null?*

This problem remains open, but in the easier case of Silver perfect sets, the answer is affirmative. First, notice the following simple lemma.

LEMMA 2.23. *Let $P, Q \subseteq 2^\omega$ be perfect sets. Then $\mu_{P\times Q} = \mu_P \times \mu_Q$. In particular, if $X \subseteq 2^\omega \times 2^\omega$ is such that $\pi_1[X]$ is P-null, then $\mu_{P\times Q}(X) = 0$.*

Proof. First, we shall prove that for any $n \in \omega$ and $v \in 2^{2n}$,

$$\mu_{P\times Q}([v]_{P\times Q}) = \frac{1}{2^{l_P(w_P)}} \cdot \frac{1}{2^{l_Q(w_Q)}},$$

where $w_P, w_Q \in 2^n$ are such that for any $i < n$, $w_P(i) = v(2i)$ and $w_Q(i) = v(2i+1)$. This can be proved by induction on n. For $n = 0$, we get $v = w_P = w_Q = \emptyset$, and

$$\mu_{P\times Q}([v]_{P\times Q}) = 1 = \frac{1}{2^{l_P(w_P)}} \cdot \frac{1}{2^{l_Q(w_Q)}}.$$

Now consider $v \in 2^{2(n+1)}$.

- If both $w_P{\restriction}_n$ and $w_Q{\restriction}_n$ are branching points in P and Q respectively (so $l_P(w_P) = l_P(w_P{\restriction}_n) + 1$ and $l_Q(w_Q) = l_Q(w_Q{\restriction}_n) + 1$), then $v{\restriction}_{2n} \in \mathrm{Split}(P \times Q)$ and $v{\restriction}_{2n+1} \in \mathrm{Split}(P \times Q)$, and so

$$\begin{aligned}\mu_{P\times Q}([v]_{P\times Q}) &= 1/2 \cdot 1/2 \cdot \mu_{P\times Q}([v{\restriction}_{2n}]_{P\times Q}) \\ &= 1/2 \cdot 1/2 \cdot 1/2^{l_P(w_P{\restriction}_n)} \cdot 1/2^{l_Q(w_Q{\restriction}_n)} \\ &= 1/2^{l_P(w_P)} \cdot 1/2^{l_Q(w_Q)}.\end{aligned}$$

- If $w_P{\restriction}_n$ or $w_Q{\restriction}_n$ (but not both) is a branching point in P or Q respectively, we may assume without loss of generality that $w_P{\restriction}_n \in \mathrm{Split}(P)$ and $w_Q{\restriction}_n \notin \mathrm{Split}(Q)$ (so $l_P(w_P) = l_P(w_P{\restriction}_n) + 1$ and $l_Q(w_Q) =$

$l_Q(w_Q{\restriction}_n))$. Then $v{\restriction}_{2n} \in \mathrm{Split}(P \times Q)$, but $v{\restriction}_{2n+1} \notin \mathrm{Split}(P \times Q)$, and so

$$\begin{aligned}\mu_{P\times Q}([v]_{P\times Q}) &= 1/2 \cdot 1 \cdot \mu_{P\times Q}([v{\restriction}_{2n}]_{P\times Q})\\ &= 1/2 \cdot 1 \cdot 1/2^{l_P(w_P{\restriction}_n)} \cdot 1/2^{l_Q(w_Q{\restriction}_n)}\\ &= 1/2^{l_P(w_P)} \cdot 1/2^{l_Q(w_Q)}.\end{aligned}$$

- If $w_P{\restriction}_n \notin \mathrm{Split}(P)$ and $w_Q{\restriction}_n \notin \mathrm{Split}(Q)$ (so $l_P(w_P) = l_P(w_P{\restriction}_n)$ and $l_Q(w_Q) = l_Q(w_Q{\restriction}_n))$, then $v{\restriction}_{2n}, v{\restriction}_{2n+1} \notin \mathrm{Split}(P \times Q)$, and so

$$\begin{aligned}\mu_{P\times Q}([v]_{P\times Q}) &= 1 \cdot 1 \cdot \mu_{P\times Q}([v{\restriction}_{2n}]_{P\times Q})\\ &= 1/2^{l_P(w_P{\restriction}_n)} \cdot 1/2^{l_Q(w_Q{\restriction}_n)} = 1/2^{l_P(w_P)} \cdot 1/2^{l_Q(w_Q)}.\end{aligned}$$

This concludes the induction argument. Since every open set in $P \times Q$ is a countable union of sets of the form $[v]_{P\times Q}$, with $v \in 2^{2n}$, $n \in \omega$, this concludes the proof. ∎

Proposition 2.24. *If $X, Y \in \mathrm{vPN}$, then $X \times Y \in \mathrm{vPN}$ in $2^\omega \times 2^\omega$.*

Proof. Fix a Silver perfect set P. Recall that it is uniquely determined by a sequence $\langle a_n\rangle_{n\in\omega}$, $a_n \in \{-1, 0, 1\}$, such that $\{n \in \omega : a_n = -1\}$ is infinite, T_P splits on all branches at length $n \in \omega$ if and only if $a_n = -1$, and $t(n) = a_n$ for all $t \in P$ for any other $n \in \omega$. Let T_1 be a tree which splits on all branches at length n if and only if $a_{2n} = -1$, and $t(n) = a_{2n}$ for any $t \in [T_1]$ for any other $n \in \omega$. Finally, let T_2 be a tree which splits on all branches at length n if and only if $a_{2n+1} = -1$, and $t(n) = a_{2n+1}$ for any $t \in [T_2]$ for any other $n \in \omega$. Let $P_1 = [T_1]$ and $P_2 = [T_2]$. If $\{2n \in \omega : a_n = -1\}$ is infinite, then P_1 is a Silver perfect set. On the other hand, if $\{2n \in \omega : a_n = -1\}$ is finite, then P_1 is also finite. Analogously, if $\{2n+1 \in \omega : a_n = -1\}$ is infinite, then P_2 is a Silver perfect set. On the other hand, if $\{2n+1 \in \omega : a_n = -1\}$ is finite, then P_2 is also finite. Moreover, $P = P_1 \times P_2$.

If P_1 and P_2 are Silver perfect, then by Lemma 2.23, $\mu_P(X \times Y) = 0$.

The other case is when P_1 or P_2 (but not both) is finite. Without loss of generality, we may assume that P_2 is finite. Then $P = \bigcup_{t\in P_2} P_1 \times \{t\}$. Obviously, for any $t \in Y$, $\mu_{P_1\times\{t\}}(X \times Y) = \mu_{P_1}(X) = 0$, so by Corollary 2.4, also $\mu_P(X \times Y) = 0$. ∎

On the other hand, it is consistent with ZFC that the classes uPN and bPN are not closed under taking products.

Theorem 2.25. *If there exists a Sierpiński set, then there are $X, Y \in$ bPN such that $X \times Y \notin$ uPN.*

Proof. Let $J \subseteq 2^8$ be as shown in Figure 2 ($J = \{$00000000, 00010111, 00101011, 00111111, 01001010, 01011111, 01101011, 01111111, 10000101, 10010111, 10101111, 10111111, 11001111, 11011111, 11101111, 11111111$\}$).

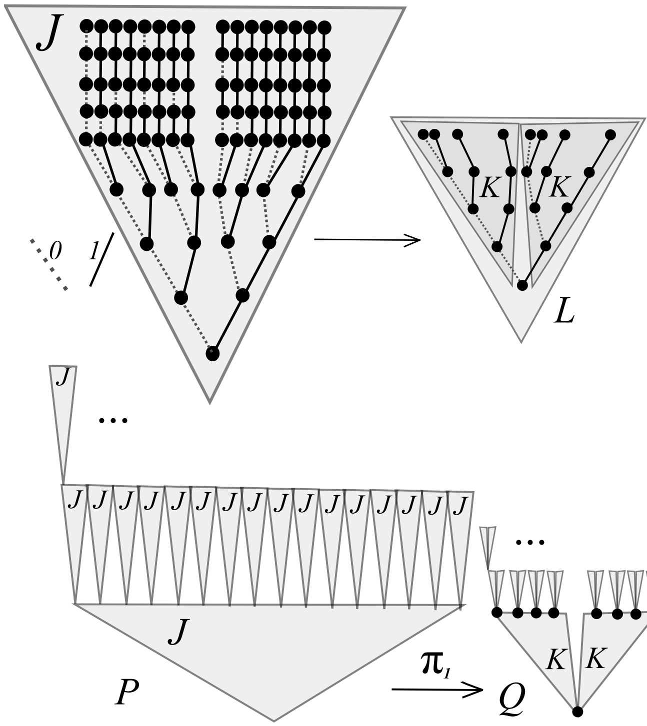

Fig. 2. Proof of Theorem 2.25

Let P be a perfect set such that $x \in P$ if and only if $x[8n, 8n+7] \in J$ for all $n \in \omega$. Obviously, P is a uniformly perfect set. Let $Q = \pi_1[P]$. Notice that $x \in Q$ if and only if for all $n \in \omega$, we have $x[4n, 4n+3] \in L$, where $L = \{0000, 0001, 0011, 0111, 1000, 1001, 1011, 1111\} \subseteq 2^4$ (see Figure 2 and Table 2).

Notice that L consists of two K-blocks (see the proof of Lemma 2.14) joined by an additional root.

Also, if B is a balanced perfect set, then $\mu_B(Q) = 0$ or $\mu_Q(B) = 0$. The argument is the same as in the proof of Lemma 2.14, namely there are two possibilities. If for all $t \in Q$, there exists infinitely many $k \in \omega$ such that $\{t{\restriction}_{4k}{}^\frown s\colon s \in L\} \setminus T_B \neq \emptyset$, then $\mu_Q(B) = 0$ by Lemma 2.5. If it is not the case, there exists $t \in Q$ such that for all but finite $k \in \omega$, we have $\{t{\restriction}_{4k}{}^\frown s\colon s \in L\} \subseteq T_B$. It follows that there exists $i \in \omega$ such that B has a branching point of length j for all $j \geq i$, so $s_{j+1}(B) \leq S_j(B) + 1$ for any $j \geq i$. And since B is a balanced perfect set, this implies that $s_j(B) = S_j(B)$ and $s_{j+1}(B) = s_j(B) + 1$ for any $j > i$. In other words, for $w \in T_B \cap 2^i$, we have $B \cap [w] = [w]$, and therefore for any $v \in T_B \cap 2^{4k}$ with $3k > i$, there exists $w \in 2^4$ such that $v^\frown w \in T_B \setminus T_Q$. It follows that $\mu_B(Q) = 0$ by Lemma 2.5.

Table 2. Proof of Theorem 2.25

$s \in J$	$w = s\langle 0,2,4,6\rangle \in L$	μ_Q $\{x \in Q\colon x[4n,4n+3] = w\}$	μ_P $\pi_1^{-1}[\{x \in Q\colon x[4n,4n+3] = w\}]$
00000000	0000	1/16	1/16
00010111	0001	1/16	1/16
00101011	0111	1/4	4/16
00111111	0111		
01101011	0111		
01111111	0111		
01001010	0011	1/8	2/16
01011111	0011		
10000101	1000	1/16	1/16
10010111	1001	1/16	1/16
10101111	1111	1/4	4/16
10111111	1111		
11101111	1111		
11111111	1111		
11001111	1011	1/8	2/16
11011111	1011		

Moreover, if A is Q-null, then $A \times 2^\omega$ is P-null. Indeed, if $n \in \omega$ and $w \in L$, then

$$\begin{aligned}\mu_Q(\{x \in Q\colon x[4n,4n+3] = w\}) &= \frac{|\{s \in J\colon w = s\langle 0,2,4,6\rangle\}|}{16}\\ &= \mu_P(\pi_1^{-1}[\{x \in Q\colon x[4n,4n+3] = w\}])\end{aligned}$$

(see Table 2). Thus, if $\varepsilon > 0$ and $\langle w_i\rangle_{i\in\omega}$ is a sequence such that $w_i \in T_Q$, $\bigcup_{i\in\omega}[w_i]_Q$ covers A and $\sum_{i\in\omega}\mu_Q([w_i]_Q) \le \varepsilon$, then $\mu_P(\pi_1^{-1}[[w_i]_Q]) = \mu_Q([w_i]_Q)$, so $\bigcup_{i\in\omega}\pi_1^{-1}[[w_i]_Q]$ is a covering of $A \times 2^\omega$ of measure μ_P not greater than ε.

Let $S \subseteq P$ be a Sierpiński set with respect to μ_P, and let $X = \pi_1[S] \subseteq Q$. Suppose that B is a balanced perfect set. Then either $\mu_B(Q) = 0$, so $\mu_B(X) = 0$, or $\mu_Q(B) = 0$, so $\mu_P(\pi_1^{-1}[Q \cap B]) = 0$. In the latter case, $S \cap \pi_1^{-1}[Q \cap B]$ is countable, so $X \cap B$ is countable and $\mu_B(X) = 0$. Hence $X \in \mathrm{bPN}$.

Notice also that $\pi_2[P] = Q$ as well (see Table 3). So analogously, one can check that $Y = \pi_2[S] \in \mathrm{bPN}$.

But $S \subseteq X \times Y$, so $X \times Y$ is not P-null, and therefore $X \times Y \notin \mathrm{uPN}$. ∎

Remark. The above result seems to be interesting as it resembles the argument of Recław [20] that if there exists a Lusin set, then the class of perfectly meager sets is not closed under taking products. In his proof, Recław actually constructs a perfect set $D \subseteq 2^\omega \times 2^\omega$ and shows that given a Lusin set $L \subseteq D$, its projections are perfectly meager. The same happens in the above

Table 3. $\pi_2[P] = Q$

$s \in J$	$w = s\langle 1,3,5,7\rangle \in L$
00000000	0000
00010111	0111
00111111	0111
10010111	0111
10111111	0111
00101011	0001
01101011	1001
01111111	1111
01011111	1111
11011111	1111
11111111	1111
01001010	1000
10000101	0011
10101111	0011
11101111	1011
11001111	1011

proof where we consider a Sierpiński set and the class bPN. Nevertheless, we do not know yet whether this can be done for the class PN.

3. Perfectly null sets in the transitive sense

3.1. The definition. In relation to the algebraic sum of sets belonging to different classes of small subsets of 2^ω, the class of perfectly sets in the transitive sense (PM') has been defined in [15]. The definition was also motivated by the obvious fact that a set X is perfectly meager if and only if for any perfect set P, there exists an F_σ set $F \supseteq X$ such that $F \cap P$ is meager in P. We say that a set X is *perfectly meager in the transitive sense* if for any perfect set P, there exists an F_σ set $F \supseteq X$ such that for any t, the set $(F+t) \cap P$ is a meager set relative to P. Further properties of PM' sets were investigated in [14], [16], [18] and [17], but some open questions remain.

Obviously, a set is perfectly null if and only if for any perfect set P, there exists a G_δ set $G \supseteq X$ such that $\mu_P(G) = 0$. We define the following new class of small sets.

Definition 3.1. We call a set X *perfectly null in the transitive sense* if for any perfect set P, there exists a G_δ set $G \supseteq X$ such that for any t, the set $(G+t) \cap P$ is P-null. The class of such sets will be denoted by PN'.

We do not know whether this class is a σ-ideal.

Similarly we define the ideals $\mathrm{bPN}', \mathrm{uPN}'$ and vPN'.

Proposition 3.2. *The following inclusions hold:*

$$\begin{array}{ccccccc} \mathrm{PN}' & \subseteq & \mathrm{bPN}' & \subsetneq & \mathrm{uPN}' & \subsetneq & \mathrm{vPN}' \\ \cap\!\mid & & \cap\!\mid & & \cap\!\mid & & \cap\!\mid \\ \mathrm{PN} & \subseteq & \mathrm{bPN} & \subsetneq & \mathrm{uPN} & \subsetneq & \mathrm{vPN} \end{array}$$

Proof. The above inclusions follow immediately from the definitions. The sets B and U defined in the proof of Proposition 2.16 are obviously also in $\mathrm{uPN}' \setminus \mathrm{bPN}'$ and $\mathrm{vPN}' \setminus \mathrm{uPN}'$, respectively. ■

3.2. PN′ sets and other classes of special subsets. In [14], [16], [18] and [17] the authors prove that $\mathrm{SM} \subseteq \mathrm{PM}' \subseteq \mathrm{UM}$, and that it is consistent with ZFC that those inclusions are proper. Therefore, we study the relation between the class PN' and the classes of strongly null sets and universally null sets.

Theorem 3.3. *Every strongly null set is perfectly null in the transitive sense.*

Proof. Let X be a strongly null set, and let P be a perfect set. If $w \in T_P$ and $|w| = S_n(P) + 1$, then $\mu_P([w]_P) \le 1/2^{n+1}$. It is well-known that if a set A is strongly null, then we can find a sequence of open sets of any given diameters, the union of which covers X in such a way that every point of A is covered by infinitely many sets from this sequence (see e.g. [4]). So let $\langle A_n : n \in \omega\rangle$ be a sequence of open sets such that $X \subseteq \bigcap_{m\in\omega}\bigcup_{n\ge m} A_n$ and $\operatorname{diam} A_n \le 1/2^{S_n(P)+1}$. Let $t \in 2^\omega$ and $B_n = (A_n + t) \cap P$. Since $\operatorname{diam} B_n \le 1/2^{S_n(P)+1}$, we have $B_n \subseteq [w_n]_P$, where $w_n \in T_P$ and $|w_n| = S_n(P) + 1$. Therefore, $\mu_P(B_n) \le 1/2^{n+1}$. But

$$(X + t) \cap P \subseteq \Big(\bigcap_{m\in\omega}\bigcup_{n\ge m} A_n + t\Big) \cap P \subseteq \bigcap_{m\in\omega}\bigcup_{n\ge m} B_n$$

and $\mu_P(\bigcap_{m\in\omega}\bigcup_{n\ge m} B_n) = 0$, so X is perfectly null in the transitive sense. ■

The following problem remains open.

Problem 3.4. *Does there exist a* PN' *set which is not strongly null?*

In particular, the authors have not been able to answer the following question.

Problem 3.5. *Does there exist an uncountable* PN' *set in every model of ZFC?*

Recall that in every model of ZFC there exists an uncountable PM' set (see [14]).

In [17], it is proved that $\mathrm{PM}' \subseteq \mathrm{UM}$. One can ask the following:

Problem 3.6. *Is it true that* $\mathrm{PN}' \subseteq \mathrm{UN}$*?*

If this inclusion holds in ZFC, then it is consistent with ZFC that it is proper. Motivated by [21, Theorem 1], we get the following theorem.

Theorem 3.7. *If there exists a universally null set of cardinality* $\mathfrak{c}$*, then there exists* $Y \in \mathrm{UN} \setminus \mathrm{bPN}' \subseteq \mathrm{UN} \setminus \mathrm{PN}'$*.*

Proof. As in [17], we apply the ideas presented in [21] in the case of subsets of 2^ω. Notice that there exists a perfect set $P \subseteq 2^\omega$ which is linearly independent over $\mathbb{Z}_2$. Indeed, define $\varphi\colon 2^{<\omega} \to 2^{<\omega}$ by induction. Let $\varphi(\emptyset)=\emptyset$. Given $\varphi(w) = v \in 2^{<\omega}$ for $w \in 2^{<\omega}$ with $n = |w|$, let $\varphi(w^\frown 0) = v^\frown \varepsilon_{2k}^{2^{n+1}}$ and $\varphi(w^\frown 1) = v^\frown \varepsilon_{2k+1}^{2^{n+1}}$, where $\varepsilon_l^m = 0\ldots010\ldots0$ is of length m with 1 on the lth position, and $k \in \omega$ has binary expansion w. For example, $\varphi(0) = 10$, $\varphi(1) = 01$, $\varphi(00) = 101000$, $\varphi(01) = 100100$, $\varphi(10) = 010010$, $\varphi(11) = 010001$, $\varphi(000) = 10100010000000$, and so on. Now, notice that $\langle[\varphi(w)]\rangle_{w\in 2^{<\omega}}$ is a Cantor scheme, so define

$$P = \bigcup_{\alpha\in 2^\omega} \bigcap_{n\in\omega} [\varphi(\alpha\restriction_n)].$$

Let $\alpha_1, \ldots, \alpha_n \in P$ be pairwise distinct. There exists $l \in \omega$ such that $\alpha_i\restriction_{2^l-2} \neq \alpha_j\restriction_{2^l-2}$ for any distinct $i, j \leq n$. Then $\alpha_1, \ldots, \alpha_n$ restricted to $[2^l - 2, 2^{l+1} - 2)$ are basis vectors of 2^l. Thus, P is linearly independent over $\mathbb{Z}_2$. The existence of such a set also follows from the Kuratowski–Mycielski Theorem (see [9, Theorem 19.1]).

Next, we follow the argument from [21]. Let C, D be perfect and disjoint subsets of P. We can require D to be a balanced perfect set. Assume that $X \subseteq C$ is a universally null set and $|X| = \mathfrak{c}$. Let $\langle B_x\colon x \in X\rangle$ enumerate all G_δ sets. For every $x \in X$, let $y_x \in x + D$ be such that $y_x \notin B_x$ if only $(D + x) \setminus B_x \neq \emptyset$. Otherwise, choose any $y_x \in x + D$. Set $Y = \{y_x\colon x \in X\}$.

Notice that $+\colon C \times D \to C + D$ is a homeomorphism. Obviously, $+$ is continuous and open on $C \times D$. Since $(C + C) \cap (D + D) = \{0\}$ (because P is linearly independent), we see that $+$ is one-to-one. Since $\pi_1[+^{-1}[Y]] = \pi_1[\{\langle x, d_x\rangle\colon x + d_x = y_x \wedge x \in X\}] = X$ is universally null, Y is universally null as well.

Now, we prove that Y is not perfectly null in the transitive sense. Indeed, if $B_x \supseteq Y$ is a G_δ set, then $y_x \in B_x$, so $(D+x)\setminus B_x = \emptyset$ and $D\cap(B_x+x) = D$. Therefore, $\mu_D(D \cap (B_x + x)) = 1$. ∎

Recall that non(N) denotes the minimal possible cardinality of a subset of the real line which is not of Lebesgue measure zero (see e.g. [4]).

Corollary 3.8. *If* $\mathrm{non}(\mathrm{N}) = \mathfrak{c}$*, then* $\mathrm{PN}' \neq \mathrm{UN}$*.*

Proof. If $\mathrm{non}(\mathrm{N}) = \mathfrak{c}$, then there exists a universally null set of cardinality $\mathfrak{c}$ (see [4, Theorem 8.8]). ∎

Taking into account Proposition 2.8, we have the following.

COROLLARY 3.9. *If* $\mathrm{non}(\mathrm{N}) = \mathfrak{c}$, *then* $\mathrm{PN}' \neq \mathrm{PN}$. ∎

The class of perfectly meager sets in the transitive sense is closed under taking products (see [17]). We do not know whether this holds for PN' sets.

PROBLEM 3.10. *Let* $X, Y \in \mathrm{PN}'$. *Is it always true that* $X \times Y \in \mathrm{PN}'$*?*

The answer is affirmative for vPN' sets.

PROPOSITION 3.11. *Let* $X, Y \in \mathrm{vPN}'$. *Then* $X \times Y \in \mathrm{vPN}'$.

Proof. Follows easily from the proof of Proposition 2.24. ∎

3.3. Additive properties of PN′ sets. We conclude this paper by investigating some additive properties of the class of sets perfectly null in the transitive sense.

PROPOSITION 3.12. *Let* $A \subseteq 2^\omega$ *be open,* μ *be any Borel diffuse measure on* 2^ω *and* $0 \leq \varepsilon < 1$. *Then the set* $A_\varepsilon = \{t \in 2^\omega \colon \mu(A+t) > \varepsilon\}$ *is also open.*

Proof. Let $A = \bigcup_{n\in\omega}[s_n]$. If $A_\varepsilon = \emptyset$, it is obviously open. Otherwise, let $t_0 \in A_\varepsilon$. There exists $N \in \omega$ such that $\mu(\bigcup_{n\leq N}[s_n] + t_0) > \varepsilon$. Let $M = \max\{|s_n| \colon n \leq N\}$. For any t such that $t{\restriction}_M = t_0{\restriction}_M$,

$$\mu(A+t) \geq \mu\Big(\bigcup_{n\leq N}[s_n] + t\Big) = \mu\Big(\bigcup_{n\leq N}[s_n] + t_0\Big) > \varepsilon.$$

So A_ε is open. ∎

Recall that a set A is called *null-additive* ($A \in \mathrm{N}^*$) if for any null set X, $A + X$ is null. Let $\leq^*$ denote the standard dominating order on ω^ω.

LEMMA 3.13. *Let* μ *be a Borel diffuse measure on* 2^ω *and* $G \subseteq 2^\omega$ *be a* G_δ *set. Let* $Y \in \mathrm{N}^*$ *be such that for every Borel map* $\varphi\colon Y \to \omega^\omega$, *there exists* $\alpha \in \omega^\omega$ *such that* $\varphi(y) \leq^* \alpha$ *for every* $y \in Y$. *Moreover, assume that for* $\mu(G+y) = 0$ *all* $y \in Y$. *Then* $\mu(G+Y) = 0$.

Proof. Let $G = \bigcap_{m\in\omega} G_m$, where for any $m \in \omega$, G_m is open and $G_{m+1} \subseteq G_m$. For $m \in \omega$, let $G_m = \bigcup_{i\in\omega}[w_{i,m}]$ with $w_{i,m} \in 2^{<\omega}$, $|w_{i,m}| > m$, and $[w_{i,m}] \cap [w_{j,m}] = \emptyset$ for $i \neq j$. Let $F_n = \{w_{i,m} \colon i, m \in \omega \wedge |w_{i,m}| = n\} \subseteq 2^n$. Notice that

$$G = \bigcap_{m\in\omega}\bigcup_{n\geq m}[F_n].$$

Let $\varphi\colon Y \to \omega^\omega$ be defined as follows:

$$\varphi(y)(k) = \min\Big\{i \in \omega \colon \mu\Big(\bigcup_{n\geq i}[F_n + y{\restriction}_n]\Big) \leq \frac{1}{2^{k+1}\cdot k!}\Big\}.$$

Notice that φ is well defined, as $\mu(G+y)=0$ for any $y\in Y$. By Proposition 3.12, the set

$$\varphi^{-1}[\{\gamma\in\omega^\omega\colon \gamma(k)>i\}]=\Big\{y\in Y\colon \mu\Big(\bigcup_{n\geq i}[F_n]+y\Big)>\frac{1}{2^{k+1}\cdot k!}\Big\}$$

is open for any $i,k\in\omega$, and therefore φ is Borel, so there exists a strictly increasing $\alpha\in\omega^\omega$ such that $\varphi(y)\leq^*\alpha$ for every $y\in Y$. For $p\in\omega$, set $Y_p=\{y\in Y\colon \forall_{k\geq p}\ \varphi(y)(k)\leq\alpha(k)\}$.

Recall now the characterization of null-additive sets due to S. Shelah (see [2, Theorem 2.7.18(3)]): $A\in\mathrm{N}^*$ if and only if for any increasing $F\colon\omega\to\omega$, there exists a sequence $\langle I_q\rangle_{q\in\omega}$ of sets such that for $q\in\omega$, $I_q\subseteq 2^{[F(q),F(q+1))}$, $|I_q|\leq q$ and $A\subseteq\bigcup_{r\in\omega}\bigcap_{q\geq r}[I_q]$.

Fix $p\in\omega$, and apply the above characterization for Y_p and the function α. There exists a sequence $\langle I^p_q\rangle_{q\in\omega}$ of sets such that for $q\in\omega$, $I^p_q\subseteq 2^{[\alpha(q),\alpha(q+1))}$, $|I^p_q|\leq q$ and $Y_p\subseteq\bigcup_{r\in\omega}\bigcap_{q\geq r}[I^p_q]$. For $r\in\omega$, let $Y_{p,r}=Y\cap\bigcap_{q\geq r}[I^p_q]$.

Then $Y_p=\bigcup_{r\in\omega}Y_{p,r}$. For any $q>r$, set $K_{p,q,r}=\{y{\restriction}_{\alpha(q+1)}\colon y\in Y_{p,r}\}$. Notice that $K_{p,q,r}$ has at most

$$|2^{\alpha(r)}|\prod_{n=r}^{q}|I^p_n|=2^{\alpha(r)}\prod_{n=r}^{q}n\leq 2^{\alpha(r)}\cdot q!$$

elements.

Obviously, $Y=\bigcup_{p,r\in\omega}Y_{p,r}$, so it is sufficient to prove that $\mu(G+Y_{p,r})=0$ for any $p,r\in\omega$. Notice that for $p,r\in\omega$,

$$\begin{aligned}G+Y_{p,r}&=\bigcup_{y\in Y_{p,r}}G+y=\bigcup_{y\in Y_{p,r}}\bigcap_{m\in\omega}\bigcup_{n\geq m}[F_n+y{\restriction}_n]\subseteq\bigcap_{m\in\omega}\bigcup_{\substack{y\in Y_{p,r}\\ n\geq m}}[F_n+y{\restriction}_n]\\&=\bigcap_{m\in\omega}\bigcup_{\substack{y\in Y_{p,r}\\ q\geq m}}\bigcup_{\alpha(q)\leq n<\alpha(q+1)}[F_n+y{\restriction}_n]\subseteq\bigcap_{m\geq p}\bigcup_{q\geq m}\bigcup_{\substack{\alpha(q)\leq n<\alpha(q+1)\\ w\in K_{p,q,r}}}[F_n+w{\restriction}_n].\end{aligned}$$

Recall that if $w\in K_{p,q,r}$, then $w=y{\restriction}_{\alpha(q+1)}$ for some $y\in Y_{p,r}\subseteq Y_p$, thus for any $k\geq p$, we have $\alpha(k)\geq\varphi(y)(k)$, so $\mu(\bigcup_{n\geq\alpha(k)}[F_n+y{\restriction}_n])\leq 1/(2^{k+1}\cdot k!)$. In particular, $\mu(\bigcup_{n\geq\alpha(q)}[F_n+y{\restriction}_n])\leq 1/(2^{q+1}\cdot q!)$, so

$$\mu\Big(\bigcup_{q\geq m}\bigcup_{\substack{\alpha(q)\leq n<\alpha(q+1)\\ w\in K_{p,q,r}}}[F_n+w{\restriction}_n]\Big)\leq 2^{\alpha(r)}\cdot\sum_{q\geq m}\frac{q!}{2^{q+1}q!}=\frac{2^{\alpha(r)}}{2^m}.$$

Therefore, $\mu(G+Y_{p,r})\leq 2^{\alpha(r)}/2^m$ for any $m\in\omega$, so $\mu(G+Y_{p,r})=0$, for any $p,r\in\omega$. ∎

We say that a set Y is SR^{N} (see [1]) if for every Borel set $H\subseteq 2^\omega\times 2^\omega$ such that $H_x=\{y\in 2^\omega\colon \langle x,y\rangle\in H\}$ is null for any $x\in 2^\omega$, $\bigcup_{x\in Y}H_x$ is null as well.

THEOREM 3.14. *Let $X \in \mathrm{PN}'$, and let Y be an SR^{N} set. Then $X + Y \in \mathrm{PN}$.*

Proof. This is an easy consequence of Lemma 3.13. Indeed, by [1, Theorem 3.8] if Y is an SR^{N} set, then $Y \in \mathrm{N}^*$ and every Borel image of Y into ω^ω is bounded. Let P be perfect. Apply Lemma 3.13 to the measure μ_P, the set Y and a G_δ set G such that $X \subseteq G$ and $\mu_P(G + t) = 0$ for all $t \in 2^\omega$. ∎

In [15], the authors prove that $\mathrm{SN} + \mathrm{PM}' \subseteq s_0$. The question of whether the measure analogue is true remains open.

PROBLEM 3.15. *Is it true that* $\mathrm{SM} + \mathrm{PN}' \subseteq s_0$*?*

REMARK. Notice that a weaker statement, that the algebraic sum of a Sierpiński set and a PN' set is an s_0-set, holds by Proposition 2.21.

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Michał Korch
Institute of Mathematics
University of Warsaw
02-097 Warszawa, Poland
E-mail: m_korch@mimuw.edu.pl

Tomasz Weiss
Institute of Mathematics
Cardinal Stefan Wyszyński University in Warsaw
01-938 Warszawa, Poland
E-mail: tomaszweiss@o2.pl