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Noncoherent uniform algebras in $\mathbb{C}^n$

by

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Abstract. Let $\mathbf{D} = \overline{\mathbb{D}}$ be the closed unit disk in $\mathbb{C}$ and $\mathbf{B}_n = \overline{\mathbb{B}}_n$ the closed unit ball in $\mathbb{C}^n$. For a compact subset K in $\mathbb{C}^n$ with nonempty interior, let $A(K)$ be the uniform algebra of all complex-valued continuous functions on K that are holomorphic in the interior of K. We give short and non-technical proofs of the known facts that $A(\overline{\mathbb{D}}^n)$ and $A(\mathbf{B}_n)$ are noncoherent rings. Using, additionally, Earl's interpolation theorem in the unit disk and the existence of peak functions, we also establish with the same method the new result that $A(K)$ is not coherent. As special cases we obtain Hickel's theorems on the noncoherence of $A(\overline{\Omega})$, where Ω runs through a certain class of pseudoconvex domains in $\mathbb{C}^n$, results that were obtained with deep and complicated methods. Finally, using a refinement of the interpolation theorem we show that no uniformly closed subalgebra A of $C(K)$ with $P(K) \subseteq A \subseteq C(K)$ is coherent provided the polynomial convex hull of K has no isolated points.

1. Introduction. In this paper we are interested in a certain algebraic property of some standard Banach algebras of holomorphic functions of several complex variables. By introducing new methods we are able to solve a fourty-year old problem first considered by McVoy and Rubel in the realm of uniform algebras appearing in approximation theory and complex analysis of several variables.

Let us start by recalling the notion of a coherent ring.

Definition 1.1. A commutative unital ring $\mathcal{A}$ is said to be *coherent* if the intersection of any two finitely generated ideals in $\mathcal{A}$ is finitely generated.

We refer the reader to the article [6] for the relevance of the property of coherence in commutative algebra.

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Definition 1.2. Let $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ be the open unit disk in $\mathbb{C}$ and $\mathbb{B}_n = \{\boldsymbol{z} = (z_1, \ldots, z_n) \in \mathbb{C}^n : \sum_{j=1}^n |z_j|^2 < 1\}$ the open unit ball in $\mathbb{C}^n$. Their Euclidean closures are denoted by $\mathbf{D}$ and $\mathbf{B}_n$, respectively.

For a bounded open set Ω in $\mathbb{C}^n$, let $H^\infty(\Omega)$ be the Banach algebra of all bounded and holomorphic functions $f : \Omega \to \mathbb{C}$, with pointwise addition and multiplication, and the supremum norm:

$$\|f\|_\infty := \sup_{z \in \Omega} |f(z)|, \qquad f \in H^\infty(\Omega).$$

For a compact set $K \subset \mathbb{C}^n$, let $A(K)$ be the uniform algebra of all complex-valued continuous functions on K that are holomorphic in the interior K° of K. If $K = \overline{\Omega}$, then we view $A(K)$ as a subalgebra of $H^\infty(\Omega)$.

If $K = \mathbf{D}^n$, then $A(K)$ is called the *polydisk algebra*; if $K = \mathbf{B}_n$, then $A(K)$ is the *ball algebra*.

In the context of function algebras of holomorphic functions in the unit disk $\mathbb{D}$ in $\mathbb{C}$, we mention [11], where it was shown that the Hardy algebra $H^\infty(\mathbb{D})$ is coherent, while the disk algebra $A(\overline{\mathbb{D}})$ is not. For $n \geq 3$, Amar [1] showed that the Hardy algebras $H^\infty(\mathbb{D}^n)$, $H^\infty(\mathbb{B}_n)$, the polydisk algebra $A(\mathbf{D}^n)$ and the ball algebra $A(\mathbf{B}_n)$ are not coherent.

The missing $n = 2$ case for the bidisk algebra $A(\mathbf{D}^2)$ (respectively the ball algebra $A(\mathbf{B}_2)$) follows as a special case of a general result due to Hickel [8] on the noncoherence of the algebra $A(\overline{\Omega})$ of continuous functions on $\overline{\Omega}$ that are holomorphic in Ω, where $\Omega \subset \mathbb{C}^n$ ($n \geq 2$) is a bounded strictly pseudoconvex domain with a C^∞ boundary. But the proof in [8] is technical. To illustrate our subsequent methods, we first give a short, elegant proof of the noncoherence of $A(\mathbf{D}^n)$ and $A(\mathbf{B}_n)$. Let me mention that an entirely elementary proof, developed **after** this manuscript had been written in 2013, has been published in [15].

Using techniques from the theory of Banach algebras which are based on peak functions, bounded approximate identities and Cohen's factorization theorem (cf. [13]), and, additionally, function-theoretic tools, like Earl's interpolation theorem for $H^\infty(\mathbb{D})$ in the unit disk (a refinement of Carleson's interpolation theorem) [5, p. 309], we succeed in showing the noncoherence of $A(K)$ for every compact set K in $\mathbb{C}^n$.

Finally, by replacing Earl's theorem with a result on asymptotic interpolation, we can handle for compact sets $K \subseteq \mathbb{C}^n$ without isolated points the case of any uniformly closed algebra A with $P(K) \subseteq A \subseteq C(K)$, where $P(K)$ is the smallest closed subalgebra of $C(K)$ containing the polynomials.

To conclude, let me point out that the coherence of rings of stable transfer functions of multidimensional systems, such as $A(\mathbf{B}_n)$ or $A(\mathbf{D}^n)$, plays a role in the stabilization problem in control theory via the factorization approach (see [17]).

2. Preliminaries. In this section we collect some technical results which we will use in our proofs.

LEMMA 2.1. *Let $\mathcal{A}$ be a commutative unital ring and M an ideal in $\mathcal{A}$ such that $M \neq \mathcal{A}$. Suppose that I is a finitely generated ideal of $\mathcal{A}$ which satisfies $I = IM$. Then there exists $m \in M$ such that $(1+m)I = 0$. If $\mathcal{A}$ has no zero divisors, then $I = 0$.*

Proof. This follows from Nakayama's lemma [10, Theorem 76]. ∎

LEMMA 2.2. *Let I be a non-finitely generated ideal in a commutative unital ring $\mathcal{A}$. Suppose that $a \in \mathcal{A}$ is not a zero-divisor. Then aI is not finitely generated either.*

Proof. Suppose, on the contrary, that $aI = (G_1, \dots, G_m)$ for some $G_1, \dots, G_m$ in $\mathcal{A}$. Then there exist $F_1, \dots, F_m \in I$ such that $G_j = aF_j$, $j = 1, \dots, m$. We claim that $I = (F_1, \dots, F_m)$. Indeed, trivially $(F_1, \dots, F_m) \subseteq I$. Also, for any $f \in I$, $af \in aI = (G_1, \dots, G_m)$ gives the existence of $\alpha_1, \dots, \alpha_m$ in $\mathcal{A}$ such that

$$af = \alpha_1 G_1 + \dots + \alpha_m G_m = \alpha_1 a F_1 + \dots + \alpha_m a F_m.$$

Since a is not a zero-divisor, it follows that

$$f = \alpha_1 F_1 + \dots + \alpha_m F_m \in (F_1, \dots, F_m).$$

This shows that the reverse inclusion $I \subseteq (F_1, \dots, F_m)$ is true, too. But this means that I, which coincides with $(F_1, \dots, F_m)$, is finitely generated, a contradiction. ∎

Here is an example that shows that the condition of a being a non-zero-divisor is necessary:

EXAMPLE 2.3. Let D_1 and D_2 be two disjoint copies of the unit disk, say $D_1 = \{|z-0.5| < 0.5\}$ and $D_2 = \{|z+0.5| < 0.5\}$, and let $\mathcal{A}$ be the algebra of bounded analytic functions on $D_1 \cup D_2$. Let $S(z) = \exp(-(1+z)/(1-z))$ be the atomic inner function. Consider the associated elements f_n of $\mathcal{A}$ given by

$$f_n(z) = \begin{cases} S^{1/n}(z) & \text{if } z \in D_1, \\ S(z) & \text{if } z \in D_2. \end{cases}$$

and let $a \in \mathcal{A}$ be defined as

$$a(z) = \begin{cases} 0 & \text{if } z \in D_1, \\ 1 & \text{if } z \in D_2. \end{cases}$$

Then the ideal $I = (f_1, f_2, \dots,)$ generated by the functions f_n in $\mathcal{A}$ is not finitely generated, although the ideal aI is finitely generated.

DEFINITION 2.4. Let X be a metrizable space.

(1) $C_b(X, \mathbb{C})$ denotes the space of bounded, complex-valued continuous functions on X.

(2) A *function algebra* A on X is a uniformly closed, point separating subalgebra of $C_b(X,\mathbb{C})$, containing the constants.
(3) A point $x_0 \in X$ is called a *peak point* for A if there is a function $p \in A$ (called a *peak function*) with $p(x_0) = 1$ and

$$\sup_{x\in X\setminus U} |p(x)| < 1 \tag{2.1}$$

for every open neighborhood U of x.

Note that in case X is compact, condition (2.1) is equivalent to

$$|p(x)| < 1 \quad \text{for all } x \in X, x \neq x_0.$$

DEFINITION 2.5. Let $\mathcal{A}$ be a commutative Banach algebra (without an identity element), and M a closed ideal of $\mathcal{A}$. Then a bounded sequence $(e_n)_{n\in\mathbb{N}}$ in M is called a (strong) *approximate identity for M* if

$$\lim_{n\to\infty} \|e_n f - f\| = 0 \quad \text{for all } f \in M.$$

For compact spaces, the following is [2, p. 74, Corollary 1.6.4].

PROPOSITION 2.6. *Let X be a metric space and $x_0 \in X$ a peak point for the function algebra A on X. If p is an associated peak function, then the sequence (e_n) defined by*

$$e_n = 1 - p^n$$

is a bounded approximate identity for the maximal ideal

$$M(x_0) = \{f \in A : f(x_0) = 0\}.$$

Proof. For the reader's convenience here is the outline. For $f \in A$,

$$|e_k f - f| = |p|^k |f|.$$

Let $\epsilon > 0$. As $f(x_0) = 0$, there is an open neighbourhood U of x_0 such that $|f| < \epsilon$ on U. By assumption,

$$m := \sup_{X\setminus U} |p| < 1.$$

Now choose $k_0 \in \mathbb{N}$ large enough so that for $k > k_0$, $m^k\|f\|_\infty < \epsilon$. Thus for $k > k_0$,

$$|e_k f - f| = |p^k f| \leq \begin{Bmatrix} m^k\|f\|_\infty & \text{on } X \setminus U \\ 1^k \cdot \epsilon & \text{on } U \end{Bmatrix} < \epsilon.$$

Hence $\|e_k f - f\|_\infty \leq \varepsilon$ for $k > k_0$. ∎

Our central Banach-algebraic tool will be Cohen's Factorization Theorem (see [2, p. 74, Theorem 1.6.5]).

PROPOSITION 2.7. *Let $\mathcal{A}$ be a commutative unital real or complex Banach algebra, I a closed ideal of $\mathcal{A}$, and suppose that I has an approximate identity. Then every $f \in I$ can be decomposed as a product $f = gh$ of two functions $g, h \in I$.*

The main function-theoretic tool for the construction of our ideals in general uniform algebras in $\mathbb{C}^n$ will be the following result on asymptotic interpolation given in [12, p. 515], with predecessors in [7] and [3]. Recall that $\rho(z,w) = |(z-w)/(1-\overline{z}w)|$ is the pseudohyperbolic distance between z and w in $\mathbb{D}$.

THEOREM 2.8. *Let (a_n) be a thin sequence in $\mathbb{D}$, that is, a sequence such that the associated Blaschke product b satisfies*

$$\lim_n (1-|a_n|^2)|b'(a_n)| = 1.$$

Then for any sequence $(w_n) \in \ell^\infty$ with $\sup_n |w_n| \le 1$ there exists a Blaschke product B and a sequence of positive numbers $\tau_n \to 1$ such that for any $0 \le \tau_n' \le \tau_n$ with $\tau_n' \to 1$, the zeros of B can be chosen to be contained in the union of the pseudohyperbolic disks $\{z \in \mathbb{D} : \rho(z,a_n) \le \tau_n'\}$ and to satisfy

$$|B(a_n) - w_n| \to 0.$$

If the interpolating nodes (a_n) cluster only at the point 1, then the zeros of B can be chosen so that they cluster also only at 1.

Proof. It remains to verify the assertion on the zeros of B whenever (a_n) clusters only at 1. Since the pseudohyperbolic disk $D_\rho(a,r)$ coincides with the Euclidean disk $D(C,R)$ where

$$C = \frac{1-r^2}{1-r^2|a|^2}a, \qquad R = \frac{1-|a|^2}{1-r^2|a|^2}r$$

(see [5]), it suffices to choose $\tau_n' := \min\{\tau_n, r_n\}$ where

$$r_n = \sqrt{\frac{1-\sqrt{1-|a_n|^2}}{|a_n|^2}}$$

and to verify that in that case $R_n \to 0$ and $C_n \to 1$. ∎

3. A sufficient criterion for noncoherence. The following concept of multipliers is new and is the key for our short proofs of the noncoherence results.

DEFINITION 3.1. Let A be a function algebra on a metrizable space X and $x_0 \in X$ a nonisolated point ([1]). A function $S \in C_b(X \setminus \{x_0\}, \mathbb{C})$ is called

([1]) This means that there is a sequence of distinct points in X converging to x_0.

a *multiplier* for the maximal ideal

$$M(x_0) = \{f \in A : f(x_0) = 0\},$$

if the ideal

$$L := L_S := \{f \in A : Sf \in A\}$$

coincides with $M(x_0)$ and if there exists $p \in M(x_0)$ such that pS is not a zero-divisor (2).

As a canonical example we mention the atomic inner function

$$S(z) = \exp\left(-\frac{1+z}{1-z}\right),$$

which is a multiplier for the maximal ideal $M(1)$ of the disk algebra $A(\mathbf{D})$.

THEOREM 3.2. *Let A be a function algebra on a metrizable space X. Suppose that $x_0 \in X$ is a nonisolated peak point for A and that the function $S \in C_b(X \setminus \{x_0\}, \mathbb{C})$ is a multiplier for the maximal ideal $M(x_0)$. Then A is not coherent.*

Proof. We shall exhibit two principal ideals whose intersection is not finitely generated. By assumption, S is a multiplier for $M(x_0)$. In particular, there is a function $p \in M(x_0)$ such that $pS \in A$ is not a zero-divisor. This implies that p is not a zero-divisor either. Let

$$I := (p), \qquad K := \{pSf : f \in A \text{ and } Sf \in A\},$$
$$J := (pS), \qquad L := \{f \in A : Sf \in A\}.$$

We claim that $K = I \cap J$. Trivially $K \subseteq I \cap J$. On the other hand, if $g \in I \cap J$, then there exist $f, h \in A$ such that $g = ph = pSf$, and so $Sf = h \in A$. In other words, $g \in K$. Thus also $I \cap J \subseteq K$.

It remains to show that K is not finitely generated. Note that by definition, $K = pSL$. Moreover, since S is a multiplier for M, we have $M = L$.

Let $f \in L$. Since M has an approximate identity (by Proposition 2.6), we may apply Cohen's Factorization Theorem (Proposition 2.7) to conclude that there exist $g, h \in M$ such that

$$f = hg.$$

Consequently, $L = LM$. Assuming that L is finitely generated, by Nakayama's Lemma 2.1 there exists $m \in M$ such that $(1 + m)L = 0$. Note that $L = M$. Since A is point separating, there exists for every $x_1 \in X \setminus \{x_0\}$ a function $f \in M = L$ such that $f(x_1) \neq 0$. Hence $(1 + m(x_1))f(x_1) = 0$ implies that $m(x_1) = -1$. Since, by assumption, x_0 is not an isolated point in X, the

(2) The notation $Sf \in A$ is to be interpreted in the usual way that $Sf : X \setminus \{x_0\} \to \mathbb{C}$ has a continuous extension F to X with $F \in A$.

continuity of m on X implies that $m(x_0) = -1$, contradicting the fact that $m \in M(x_0)$. Thus we conclude that L cannot be finitely generated.

Because S is a multiplier, $pS \in M(x_0)$. Moreover, pS is not a zero-divisor. Hence, by Lemma 2.2, $K = pSL$ is not finitely generated either. ∎

In the next sections we apply Theorem 3.2 to concrete function algebras of several complex variables.

4. The noncoherence of the ball and polydisk algebra. In view of Theorem 3.2, to prove the noncoherence, it suffices to exhibit a peak function and a multiplier for some distinguished maximal ideal.

THEOREM 4.1. *The ball algebra $A(\mathbf{B}_n)$ is not coherent for any $n = 1, 2, \ldots$.*

Proof. Indeed,

$$P(z_1, \ldots, z_n) = \frac{1 + z_1}{2}$$

is a peak function at $(1, 0, \ldots, 0)$ for $A(\mathbf{B}_n)$ (note that if $|1 + z_1| = 2$, then $z_1 = 1$ and the remaining coordinates $z_2, \ldots, z_n$ are automatically zero because $(z_1, \ldots, z_n) \in \mathbf{B}_n$), and

$$S(z_1, \ldots, z_n) = \exp\left(-\frac{1 + z_1}{1 - z_1}\right)$$

is a multiplier for $M(1, 0, \ldots, 0)$. ∎

THEOREM 4.2. *The polydisk algebra $A(\mathbf{D}^n)$ is not coherent for any $n = 1, 2, \ldots$.*

Proof. Indeed,

$$P_1(z_1, \ldots, z_n) = \left(\frac{1 + z_1}{2}\right) \cdots \left(\frac{1 + z_n}{2}\right)$$

is a peak function at $a = (1, \ldots, 1)$ for $A(\mathbf{D}^n)$ and

$$S(z_1, \ldots, z_n) = \exp\left(-\frac{1 + z_1}{1 - z_1}\right) \cdots \exp\left(-\frac{1 + z_n}{1 - z_n}\right)$$

is a multiplier for $M(1, \ldots, 1)$. ∎

Thus we have obtained a short proof of the result by Amar and Hickel [1, 8].

5. The noncoherence of $P(K) \subseteq A \subseteq C(K)$. For a compact set $K \subset \mathbb{C}^n$, let $C(K)$ denote the uniform algebra of complex-valued continuous functions on K, let $A(K)$ be the uniform algebra of all functions continuous on K and holomorphic in K°, and let $P(K)$ be the subalgebra of those functions in $A(K)$ that can be uniformly approximated on K by holomorphic polynomials.

Let us recall the following well-known result:

THEOREM 5.1. *Let $K \subseteq \mathbb{C}^n$ be a compact set. Then*

(1) *Endowed with the usual pointwise operations* ([3]) *and the supremum norm*

$$\|f\|_\infty = \sup\{|f(z)| : z \in K\}$$

$A(K) = A(+, \cdot, \bullet, \|\cdot\|_\infty)$ and $P(K) = A(+, \cdot, \bullet, \|\cdot\|_\infty)$ are uniformly closed point separating subalgebras of $C(K)$.

(2) *Let A be $A(K)$ or $P(K)$. Standard maximal ideals in A are given by*

$$M(z_0) := \{f \in A : f(z_0) = 0\}$$

for a unique $z_0 \in K$ ([4]).

(3) *The spectrum* (*or maximal ideal space*) *of $P(K)$ coincides with the polynomial convex hull $\widehat{K}$ of K.*

(4) *The Shilov boundary, ∂A, of A is a nonvoid closed subset of ∂K.*

(5) *The set $\Pi(A)$ of peak points for A is a nonvoid dense subset of ∂A.*

(6) *For each $z_0 \in \Pi(A)$, the associated maximal ideal $M(z_0)$ has a bounded approximate identity.*

Proof. (1) is elementary; (2)–(5) are standard facts in the theory of uniform algebras (see for instance [2] and [4]); note that the Shilov boundary is the closure of the set of weak peak points and that for function algebras on metrizable spaces every weak peak point is actually a peak point [2, p. 96]. Finally, (3) is in [4, p. 67], and (6) follows from Proposition 2.6. ∎

We note that if $x_0 \in \partial K$ is a peak point for $P(K)$, then it is a peak point for any uniformly closed algebra A with $P(K) \subseteq A \subseteq C(K)$.

LEMMA 5.2. *Let $K \subseteq \mathbb{C}^n$ be compact with $K^\circ \neq \emptyset$. If $z_0 \in \partial(K^\circ)$ is a peak point for $P(\overline{K^\circ})$, then it is a peak point for $A(K)$.*

Proof. Let $f \in P(\overline{K^\circ})$ peak at z_0. Then $f(\overline{K^\circ}) \subseteq \mathbb{D} \cup \{1\} \subseteq \overline{\mathbb{D}}$. Let $F : \mathbb{C}^n \to \mathbb{C}$ be a continuous extension of f to $\mathbb{C}^n$. Since $\overline{\mathbb{D}}$ is a retract for $\mathbb{C}$, there is a retraction r of $\mathbb{C}$ onto $\overline{\mathbb{D}}$ with $r(z) = z$ for $z \in \overline{\mathbb{D}}$. Hence $r \circ F$ is an extension of f with target space $\overline{\mathbb{D}}$.

([3]) Addition $+$, multiplication $\cdot$ and multiplication $\bullet$ by complex scalars.

([4]) Note that, in general, there are many more maximal ideals than those given by point evaluation at points in K; even in the case where $K = \overline{\Omega}$, Ω a bounded pseudoconvex domain in $\mathbb{C}^n$, $n \geq 2$, every function $f \in H^\infty(\Omega)$ (a fortiori $f \in A(\overline{\Omega})$) may have a bounded holomorphic extension to a strictly larger domain Ω' (see [9]). Hence, in that case, the spectrum of $A(\overline{\Omega})$ is strictly larger than $\overline{\Omega}$ itself.

By Urysohn's Lemma in metric spaces, there is a continuous function $u : \mathbb{C}^n \to [0,1]$ such that

$$\{z \in \mathbb{C}^n : u(z) = 1\} = \overline{K^\circ}.$$

Now consider $\phi(z) = (1+z)/2$ that maps $\overline{\mathbb{D}}$ onto $|z - 1/2| \le 1/2$. We claim that

$$g := \phi \circ (u \cdot (r \circ F)) : K \to \mathbb{D} \cup \{1\}$$

is a peak function at z_0 that belongs to $A(K)$.

To see this, we note that $u(z) \cdot (r(F(z)) \in \overline{\mathbb{D}}$ for every $z \in K$. Moreover, for $z \in K^\circ$, $F(z) = f(z) \in \overline{\mathbb{D}}$; hence $r(F(z)) = f(z)$ and so $u(z)r(F(z)) = f(z)$. Since ϕ and f are holomorphic, we deduce that g is holomorphic in K°. Thus $g \in A(K)$. Now if for some $z_1 \in K$, $g(z_1) = 1$, then necessarily $u(z_1)\, r(F(z_1)) = 1$. Now $|r(F(z_1))| \le 1$; hence $|u(z_1)| = u(z_1) = 1$. We conclude that $z_1 \in \overline{K^\circ}$. Therefore, as shown previously, we have $u(z_1)r(F(z_1)) = f(z_1) = 1$. Since $f \in P(\overline{K^\circ})$ peaks at z_0, we finally obtain $z_1 = z_0$. ∎

Definition 5.3. Let $\Omega \subset \mathbb{C}^n$ be a bounded open set. For $a \in \partial\Omega$ and $f \in H^\infty(\Omega)$, let $\mathrm{Cl}(f,a)$ denote the *cluster set* of f at a, that is, the set of all points $w \in \mathbb{C}$ such that there exists a sequence (z_n) in Ω such that $(f(z_n))$ converges to w.

It is obvious that $\mathrm{Cl}(f,a)$ is a compact, nonvoid subset of $\mathbb{C}$. In fact

$$\mathrm{Cl}(f,a) = \bigcap_{0<r\le 1} \overline{f(\Omega \cap B(a,r))}.$$

In the case of the polydisk or the unit ball, $\mathrm{Cl}(f,a)$ is connected.

The proof of the following fundamental lemma was motivated by parts of the proof of [14, Theorem 3.1] concerning the pseudo-Bézout property for $P(K)$ and its siblings, where $K \subset \mathbb{C}$ is compact. It gives us the possibility to construct multipliers for maximal ideals.

Lemma 5.4. *For a compact set $K \subseteq \mathbb{C}^n$, let A be a uniformly closed algebra with $P(K) \subseteq A \subseteq C(K)$. Let $x_0 \in \partial K$ be a nonisolated peak point for $P(K)$ and $p \in P(K)$ an associated peak function. Then there exists a function $S \in C_b(K \setminus \{x_0\})$ such that*

$$0 \in \mathrm{Cl}(S,x_0) \quad \textit{but} \quad \mathrm{Cl}(S,x_0) \ne \{0\},$$

and

$$(1-p)S \in A.$$

Moreover, S is a multiplier for the maximal ideal $M(x_0) = \{f \in A : f(x_0) = 0\}$.

Proof. We consider two cases:

CASE 1: *K is the closure of a domain D in $\mathbb{C}^n$* ([5]). Let $(z_n) \in K$ be a sequence of distinct points in K converging to x_0. Then $p(z_n) \to 1$ and $p(z_n) \in \mathbb{D}$. By passing to a subsequence if necessary, we may assume that $(p(z_n))$ is a (thin) interpolating sequence for $H^\infty(\mathbb{D})$. By Earl's interpolation theorem [5, p. 309], there is an interpolating Blaschke product B satisfying

$$B(p(z_{2n})) = 0 \quad \text{and} \quad B(p(z_{2n+1})) = \delta \tag{5.1}$$

for all n and some constant $\delta > 0$ and such that the zeros of B cluster only at 1. Hence $B \circ p$ is discontinuous at x_0.

Now let $S := B \circ p$. Since $|p| < 1$ everywhere on $K \setminus \{x_0\}$ and B is continuous on $\overline{\mathbb{D}} \setminus \{1\}$, it follows that $S = B \circ p$ is continuous on $K \setminus \{x_0\}$. Moreover, since x_0 is not an isolated point, we have $0 \in \mathrm{Cl}(S, x_0)$ and $\delta \in \mathrm{Cl}(S, x_0)$.

It remains to show that $(1 - p)S \in A$ and that S is the multiplier we are looking for. Let us point out that for any $q \in C(\overline{\Omega})$ with $q(x_0) = 0$, the function $qS = q \cdot (B \circ p)$ is continuous at x_0. We claim that if $q \in A$ and $q(x_0) = 0$, then $q(B \circ p) \in A$.

To see this, consider the partial products $B_n := \prod_{j=1}^n L_j$ of the Blaschke product B. Then B_n converges locally uniformly (in $\mathbb{D}$) to B. Since B_n is analytic in a neighbourhood of the $P(K)$-spectrum $\sigma(p)$ of p, where $\sigma(p) \subseteq \overline{\mathbb{D}}$, we see that $B_n \circ p \in P(K) \subseteq A$. Now $q(B_n \circ p)$ converges uniformly in K to $q(B \circ p)$. Hence $q(B \circ p) \in A$. In particular, $(1 - p)(B \circ p) \in A$.

Thus we have shown that

$$M(x_0) \subseteq I_S := \{f \in A : Sf \in A\}.$$

To show the reverse inclusion, let $f \in I_S$. Then the continuity of f and the discontinuity of S at x_0 imply that $f(x_0) = 0$. Hence $f \in M(x_0)$ and so $I_S \subseteq M(x_0)$. Consequently,

$$I_S = \{f \in A : Sf \in A\} = M(x_0).$$

To show that $(1 - p)S$ is not a zero-divisor in A, we have to use the special structure of K, namely that $K = \overline{D}$ for a domain D in $\mathbb{C}^n$. Note that for general K, $S = B \circ p$ may vanish identically on whole components of K° (for example if B has a zero at $p(a)$ and $p \equiv p(a)$ on such a component). Now $(1 - p)S$ is analytic on D; since its zeros are isolated, we deduce that $(1 - p)Sq \equiv 0$ implies $q \equiv 0$ on D for every $q \in A$.

Putting it all together, we have shown that S is a multiplier for $M(x_0)$.

CASE 2: *$K \subseteq \mathbb{C}^n$ is an arbitrary compact set.* To avoid the phenomenon described in the last paragraph, we have to look for a multiplier S that has

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([5]) This is only for the purpose of simplicity, because the tools applied in this case are more elementary than in the general case.

no zeros on $K \setminus \{x_0\}$. It will have the form

$$S = (1 + B) \circ p = 1 + (B \circ p)$$

for some Blaschke product B whose zeros cluster only at 1. Note that $B \circ p$ never takes the value -1 on $K \setminus \{x_0\}$, since $B(\xi) = -1$ only for $\xi \in \mathbb{T} \setminus \{1\}$, and the only unimodular value p takes is 1.

Here is the construction of B. According to the asymptotic interpolation theorem 2.8, there is a Blaschke product B whose zeros cluster only at 1 such that

$$B(p(z_{2n})) \to -1 \quad \text{and} \quad B(p(z_{2n-1})) \to 1. \tag{5.2}$$

Hence $0 \in \mathrm{Cl}(S, x_0)$ and $2 \in \mathrm{Cl}(S, x_0)$. The rest is now clear in view of the proof of Case 1, always having in mind that x_0 is not an isolated point in K. ∎

Theorem 5.5.

(i) *If Ω is a bounded domain in $\mathbb{C}^n$, then $A(\overline{\Omega})$ is not coherent.*
(ii) *If $K \subset \mathbb{C}^n$ is compact with $K^\circ \neq \emptyset$, then $A(K)$ is not coherent.*

Proof. (i) By Theorem 5.1, there exists a peak point $z_0 \in \partial\overline{\Omega}$ for $A(\overline{\Omega})$ and $M(z_0)$ has an approximate identity. Of course $\overline{\Omega}$ is a compact set without isolated points. Hence, by Lemma 5.4, there is a multiplier S for $M(z_0)$. The noncoherence of $A(\overline{\Omega})$ now follows from Theorem 3.2.

(ii) Similar to (i); just use Lemma 5.2 to get the nonisolated peak point x_0 for $A(K)$. ∎

If $K^\circ = \emptyset$, then $A(K) = C(K)$. In Section 6 we will give a characterization of those compacta in $\mathbb{C}^n$ for which $C(K)$ is coherent. Let us also note that (i) is not a special case of (ii), because there are algebras of the form $A(\overline{\Omega})$ that do not belong to the class of algebras of type $A(K)$: just take as Ω the unit disk with a Cantor set (= compact and totally disconnected set) of positive planar Lebesgue measure deleted.

Definition 5.6. A compact set $K \subseteq \mathbb{C}^n$ is called *admissible* if its polynomial convex hull $\widehat{K}$ contains no isolated points.

Our final theorem contains (more or less) all the preceding ones as special cases.

Theorem 5.7. *Let $K \subseteq \mathbb{C}^n$ be an admissible compact set and let A be a uniformly closed subalgebra of $C(K)$ with $P(K) \subseteq A \subseteq C(K)$. Then A is not coherent.*

Proof. Similar to the proof above; note that Theorem 5.1(5) yields the desired peak point for $P(K)$, and Lemma 5.4 the associated multiplier. ∎

If we are considering algebras of a single complex variable, then we have the following refinement:

Theorem 5.8. *Let $K \subseteq \mathbb{C}$ be an infinite compact set and let A be a uniformly closed subalgebra of $C(K)$ with $P(K) \subseteq A \subseteq C(K)$. Then A is not coherent.*

Proof. The infinity of K implies that the polynomial convex hull $\widehat{K}$ of K is an infinite compact set, too. This in turn implies that its topological boundary $\partial\widehat{K}$ is an infinite compact set. Hence, there exists a nonisolated point $x_0 \in \partial\widehat{K} \subseteq K$. Now by Mergelyan's Theorem, $P(\widehat{K}) = R(\widehat{K}) = A(\widehat{K})$. Using the fact that $\mathbb{C} \setminus \widehat{K}$ is connected, Gonchar's peak point criterion for $R(K)$ (see [4, Corollary 4.4, p. 205]) shows that x_0 is a peak point for $R(\widehat{K}) = P(\widehat{K})$. The noncoherence now follows as in the preceding theorems. ∎

We guess that this result can be extended to the case of several variables.

6. Noncoherence of $C(K)$. Let K be a compact set in $\mathbb{C}^n$. A general result in [16] tells us that for completely regular spaces X, $C(X,\mathbb{R})$ is coherent if and only if X is basically disconnected ([6]). This result can be used to conclude that $C(K,\mathbb{C})$ is coherent if and only if K is finite. Since our compacta K are metrizable, for the reader's convenience we present the following independent easy proof.

Theorem 6.1. *If $K \subseteq \mathbb{C}^n$ is compact, then $C(K)$ is coherent if and only if K is finite.*

Proof. Suppose that K is not finite. Then there is $x_0 \in K$ such that $\lim x_n = x_0$ for some sequence (x_n) of distinct points in K. Let E be a closed subset of K not containing x_0. Then

$$p(x) = \frac{d(x,E)}{d(x,E) + d(x,x_0)}$$

is a peak function for x_0. By passing to a subsequence if necessary, we may assume that $p(x_n) \neq p(x_m)$ for $n \neq m$. Choose a continuous zero-free function $B : [0,1[\to]0,1]$ such that

$$B(p(x_{2n})) = 1 \quad \text{and} \quad B(p(x_{2n-1})) = 1/n \to 0,$$

and let $S := B \circ p$. Then $S \in C_b(K \setminus \{x_0\})$. It is now straightforward to check that the continuous function $(1-p)S$ is not a zero-divisor and that S is a multiplier for $M(x_0)$ (note that the cluster set of S at x_0 is not a singleton and contains 0). Then we apply Theorem 3.2.

If, on the other hand, X is finite, then $C(X)$ is a principal ideal ring. In fact, if $I \subseteq C(X)$ is an ideal, then we define a generator g of I by $g(x) = 1$ if $x \notin Z(I)$ and $g(x) = 0$ if $x \in Z(I)$. Hence $C(X)$ is trivially coherent. ∎

([6]) Recall that X is said to be *basically disconnected* if the closure of $\{x \in X : f(x) \neq 0\}$ is open for every $f \in C(X,\mathbb{R})$.

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