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Hereditarily infinite-dimensional property for asymptotic dimension and graphs with large girth

by

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Abstract. It is shown that every subspace of a coarse disjoint union of a sequence of graphs with large girth has infinite asymptotic dimension or asymptotic dimension at most one.

1. Introduction. The notion of asymptotic dimension $\operatorname{asdim} X$ for a metric space X was introduced by Gromov [10] as a large scale analogue of covering dimension. Among the unsolved problems in asymptotic dimension theory is the following one, posed by Dranishnikov [4, Problem 1].

Problem 1.1 ([4]). *For a proper metric space X, is it true that* $\operatorname{asdim} X = \dim \nu X$*?*

Here, a metric space is said to be *proper* if every closed bounded subspace is compact. By νX we denote the *Higson corona* of a proper metric space X, that is, $\nu X = hX \setminus X$, where hX is the Higson compactification of X (see [16, Section 2.3] or [5]), and $\dim \nu X$ stands for the covering dimension of νX (see [7] or [12]).

Dranishnikov, Keesling and Uspenskij [5, Theorem 1.1] proved that $\dim \nu X \leq \operatorname{asdim} X$ for every proper metric space X, and Dranishnikov [4, Theorem 6.2] proved that $\dim \nu X = \operatorname{asdim} X$ for every proper metric space X with $\operatorname{asdim} X < \infty$. Thus, there exists a counterexample to Problem 1.1 if and only if there exists a proper metric space X such that $\operatorname{asdim} X = \infty$ and $\dim \nu X < \infty$.

Assume there exists a proper metric space X satisfying $\operatorname{asdim} X = \infty$ and $\dim \nu X < \infty$, and let Y be a closed subspace of X with $\operatorname{asdim} Y < \infty$.

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According to [5, proof of Theorem 1.4], the closure $\mathrm{Cl}_{hX} Y$ of Y in hX is the Higson compactification hY of Y. From this and Dranishnikov's theorem [4, Theorem 6.2] stated above, we have

$$\operatorname{asdim} Y = \dim \nu Y = \dim(\mathrm{Cl}_{hX} Y \setminus Y) \leq \dim \nu X.$$

This shows the following.

FACT 1.2. *If X is a proper metric space satisfying* $\operatorname{asdim} X = \infty$ *and* $\dim \nu X = k < \infty$, *then every closed subspace Y of X satisfies* $\operatorname{asdim} Y = \infty$ *or* $\operatorname{asdim} Y \leq k$.

In order to find candidates for a counterexample to Problem 1.1, we may ask whether there exists a proper metric space X with $\operatorname{asdim} X = \infty$ and $k \in \mathbb{N}$ such that every (closed) subspace Y of X satisfies $\operatorname{asdim} Y = \infty$ or $\operatorname{asdim} Y \leq k$. The purpose of this paper is to show the following.

THEOREM 1.3. *Let $\bigsqcup G_i$ be a coarse disjoint union of a sequence $\{G_i\}$ of graphs with large girth. Then every $Y \subset \bigsqcup G_i$ satisfies* $\operatorname{asdim} Y = \infty$ *or* $\operatorname{asdim} Y \leq 1$.

All graphs in this paper are assumed to be finite and connected. For the relevant definitions, see Section 2. Let $\{G_i\}$ be a sequence of graphs with large girth such that for some $D \geq 3$ every vertex in G_i, $i \in \mathbb{N}$, has degree between 3 and D. Note that such a sequence $\{G_i\}$ exists (see [15, Chapter 5]), and a coarse disjoint union $\bigsqcup G_i$ of $\{G_i\}$ is a proper metric space. Willett [19, Theorem 1.2] proved that $\bigsqcup G_i$ does not have Yu's property A [20], and hence $\operatorname{asdim}(\bigsqcup G_i) = \infty$ due to [11, Lemma 4,3]. Thus, $\bigsqcup G_i$ can be a candidate for a counterexample to Problem 1.1.

QUESTION 1.4. *Does there exist a sequence $\{G_i\}$ of graphs with large girth such that for some $D \geq 3$ every vertex in G_i, $i \in \mathbb{N}$, has degree between 3 and D, and* $\dim \nu(\bigsqcup G_i) < \infty$ *for a coarse disjoint union $\bigsqcup G_i$?*

In (topological) dimension theory, a metric space X is said to be *hereditarily infinite-dimensional* if $\dim X = \infty$ and every $Y \subset X$ satisfies $\dim Y = \infty$ or $\dim Y \leq 0$ (see [12, Section 4.8]). Such a compact metric space was first constructed by Walsh [18]. In this context, we may ask the following (see also Remark 3.3).

QUESTION 1.5. *Does there exist a metric space X with* $\operatorname{asdim} X = \infty$ *such that every $Y \subset X$ satisfies* $\operatorname{asdim} Y = \infty$ *or* $\operatorname{asdim} Y \leq 0$*?*

2. Notation and terminology. Let $\mathbb{N}$ denote the set of positive integers. By a (finite connected) *graph* $G = (V(G), E(G))$ we mean a finite set $V(G)$ and $E(G) \subset \{\{x, y\} : x, y \in V(G),\ x \neq y\}$ such that for every $x, y \in V(G)$ with $x \neq y$ there exits a finite sequence $\{x_i\}_{i=0}^{n}$ in $V(G)$ such that $x_0 = x$, $x_n = y$ and $\{x_{i-1}, x_i\} \in E(G)$ for every $i \in \{1, \ldots, n\}$. Such a sequence $\{x_i\}_{i=0}^n$ is called a *path* between x and y. Below, $V(G)$ is also denoted by G for simplicity, and every graph G is assumed to be a metric space with the *edge metric d_G* defined by

$$d_G(x,y) = \min\{n \in \mathbb{N} : \text{there exists a path } \{x_i\}_{i=0}^n \text{ between } x \text{ and } y\}$$

if $x \neq y$ and $d_G(x,x) = 0$ for $x, y \in G$. For a graph G with cycles, the *girth* of G, denoted by $g(G)$, is the length of a shortest cycle in G. A sequence $\{G_i\}$ of graphs is said to have *large girth* if $g(G_i) \to \infty$ as $i \to \infty$. For a metric space (X,d) and $U, U' \subset X$, set

$$\begin{aligned}\operatorname{diam} U &= \sup\{d(x,y) : x, y \in U\},\\ d(U,U') &= \inf\{d(x,x') : x \in U,\ x' \in U'\}.\end{aligned}$$

A metric space $(\bigsqcup X_i, d)$ is said to be a *coarse disjoint union* of a sequence $\{(X_i, d_i)\}$ of metric spaces if $\bigsqcup X_i = \bigcup_{i \in \mathbb{N}} X_i$, $d|_{X_i \times X_i} = d_i$ for every $i \in \mathbb{N}$, and for every $R > 0$ there exists $k \in \mathbb{N}$ such that $d(X_i, X_j) > R$ if $i > j \geq k$. For $R > 0$, a family $\mathcal{U}$ of subsets of a metric space (X,d) is said to be *R-disjoint* if $d(U,U') \geq R$ for any distinct $U, U' \in \mathcal{U}$.

Definition 2.1. For $R, S > 0$, the *dimension of a metric space X on a scale R with control S* (or (R,S)-$\dim X$ for short) [6] (see also [1]) is the minimum integer n such that there exist families $\mathcal{U}_0, \mathcal{U}_1, \dots, \mathcal{U}_n$ of subsets of X with the following properties:

1. $\bigcup_{j=0}^n \mathcal{U}_j$ covers X,
2. $\mathcal{U}_j$ is R-disjoint for every $j \in \{0, 1, \dots, n\}$, and
3. $\operatorname{diam} U \leq S$ for every $U \in \mathcal{U}_j$ and $j \in \{0, 1, \dots, n\}$.

The *asymptotic dimension* of X ($\operatorname{asdim} X$ for short) [10] is the minimum integer n such that for every $R > 0$ there exists $S > 0$ with (R,S)-$\dim X \leq n$. A family $\{X_\alpha : \alpha \in A\}$ of metric spaces is said to satisfy $\operatorname{asdim}\{X_\alpha\} \leq n$ *uniformly* [2] if for every $R > 0$ there exists $S > 0$ such that (R,S)-$\dim X_\alpha \leq n$ for every $\alpha \in A$.

For undefined terminology we refer to [13].

3. Proof of Theorem 1.3.

As a corollary of [2, Theorem 1], we have the following.

Lemma 3.1. *If $\bigsqcup X_i$ is a coarse disjoint union of a sequence $\{X_i\}$ of metric spaces such that* $\operatorname{asdim}\{X_i\} \leq n$ *uniformly, then* $\operatorname{asdim}(\bigsqcup X_i) \leq n$.

Lemma 3.2. *Let G be a graph and let $X \subset G$ with* $\operatorname{diam} X < g(G)/4$. *Then* $(R, 3R)$-$\dim X \leq 1$ *for every $R > 0$.*

Proof. We may assume X is nonempty. Fix $x_0 \in X$. For every $x \in X$, take a shortest path $[x_0, x]$ in G between x_0 and x. Set $T = \bigcup_{x \in X} [x_0, x]$ and let d be the edge metric on G. Since $\operatorname{diam} X < g(G)/4$, T is a tree and

$d|_{T\times T}$ is the edge metric on T. Thus, by [16, proof of Proposition 9.8], we have $(R,3R)$-$\dim T \le 1$ for every $R > 0$, which implies the conclusion since $X \subset T$. ∎

Proof of Theorem 1.3. We may assume $g(G_i) \le g(G_{i+1})$ for every $i \in \mathbb{N}$. Let $Y \subset \bigsqcup G_i$ with $\operatorname{asdim} Y = n < \infty$. Take an increasing sequence $0 < S_1 < S_2 < \cdots$ of real numbers such that (k, S_k)-$\dim Y \le n$ for every $k \in \mathbb{N}$, and take families $\mathcal{U}_0^k, \mathcal{U}_1^k, \ldots, \mathcal{U}_n^k$ of subsets of Y satisfying properties (1)–(3) in Definition 2.1 with $R = k$ and $S = S_k$. Choose an increasing sequence $i_1 < i_2 < \cdots$ in $\mathbb{N}$ such that $S_k < g(G_{i_k})/4$ and $d(G_i, G_{i'}) > k$ if $i > i' \ge i_k$ for every $k \in \mathbb{N}$.

Fix $j \in \{0, 1, \ldots, n\}$. For every $k \in \mathbb{N}$ set

$$Y_k^j = \bigcup \mathcal{U}_j^k \cap \bigcup_{i=i_k}^{i_{k+1}-1} G_i, \qquad Y^j = \bigcup_{k\in\mathbb{N}} Y_k^j,$$
$$m_k = |\{U \cap G_i : U \in \mathcal{U}_j^k,\ i_k \le i < i_{k+1}\}|,$$

where $|A|$ denotes the cardinality of the set A. Then $m_k \in \mathbb{N}$ since $\bigcup_{i=i_k}^{i_{k+1}-1} G_i$ is finite. Let $N_k = \{m \in \mathbb{N} : \sum_{i=1}^{k-1} m_i < m \le \sum_{i=1}^{k} m_i\}$ and let $\{V_m : m \in N_k\} = \{U \cap G_i : U \in \mathcal{U}_j^k,\ i_k \le i < i_{k+1}\}$ be such that $V_m \ne V_{m'}$ if $m \ne m'$. Since $\{V_m : m \in N_k\}$ is a k-disjoint finite family of subsets of Y_k^j for every $k \in \mathbb{N}$, Y^j is a coarse disjoint union of the family $\{V_m : m \in \mathbb{N}\}$. On the other hand, for every $k \in \mathbb{N}$ and $m \in N_k$, we have $V_m \subset G_i$ for some $i \ge i_k$, and hence $\operatorname{diam} V_m \le S_k < g(G_{i_k})/4 \le g(G_i)/4$. Thus, by Lemma 3.2, $(R,3R)$-$\dim V_m \le 1$ for every $R > 0$. This shows that $\operatorname{asdim}\{V_m\} \le 1$ uniformly. By Lemma 3.1, we have $\operatorname{asdim} Y^j \le 1$.

Since $Y = \bigcup_{i=1}^{i_1-1} G_i \cup \bigcup_{j=0}^{n} Y^j$ and $\bigcup_{i=1}^{i_1-1} G_i$ is finite, the conclusion that $\operatorname{asdim} Y \le 1$ follows from [2, Finite Union Theorem]. ∎

REMARK 3.3. Let $(\bigsqcup G_i, d)$ be a coarse disjoint union of a sequence $\{G_i\}$ of graphs with large girth. Then there is $Y \subset \bigsqcup G_i$ such that $\operatorname{asdim} Y = 1$. Indeed, for every $i \in \mathbb{N}$, take $x_i, y_i \in G_i$ such that $\lfloor g(G_i)/8 \rfloor \le d(x_i, y_i) \le g(G_i)/4$, where $\lfloor x \rfloor$ denotes the largest integer at most x. Let I_i be the set of points of the shortest path between x_i and y_i. Then their union $Y = \bigcup_{i\in\mathbb{N}} I_i$ is a coarse disjoint union of $\{I_i\}$, and hence $\operatorname{asdim} Y \le 1$ by Lemmas 3.2 and 3.1. Since $\operatorname{diam} I_i \to \infty$ as $i \to \infty$, we have $\operatorname{asdim} Y \not\le 0$. Thus $\operatorname{asdim} Y = 1$.

REMARK 3.4. Let Γ be a countable group. Note that every countable group has a left-invariant uniformly discrete proper metric which is unique up to coarse equivalence (see [13, Proposition 1.2.2 and Example 1.4.7]). If the nth product $\mathbb{Z}^n$ of the group of integers can be coarsely embedded into Γ for every $n \in \mathbb{N}$, then we have $\dim \nu\Gamma = \infty$ by Fact 1.2 and the fact that $\operatorname{asdim} \mathbb{Z}^n = n$ (see [13, Examples 2.2.3 and 2.2.6]). In particular, if Γ is the countable direct sum $\bigoplus_{i=1}^{\infty} \mathbb{Z}$ (see [13, Example 2.6.1]), the wreath product $\mathbb{Z} \wr \mathbb{Z}$ (see [13, Proposition 2.6.3]), the first Grigorchuk group [8], [9] (see [17, Theorem 3 and Corollary 4]), or the Thompson group F (see [3, Theorem 4.8]), then $\dim \nu\Gamma = \infty$, and hence Γ cannot be a candidate for a counterexample to Problem 1.1.

REMARK 3.5. Let $\{G_i\}$ be a sequence of graphs with large girth satisfying the following property: There exist $D > 0$, $A > 0$ and $\lambda \in (0, 1/24]$ such that, for every $i \in \mathbb{N}$, every vertex in G_i has degree between 3 and D, $\operatorname{diam} G_i \leq Ag(G_i)$, and $1 < \lfloor \lambda g(G_i) \rfloor < \lfloor \lambda g(G_{i+1}) \rfloor$. Osajda [14, Theorems 2.7 and 3.2] proved that there exists a $C'(\lambda)$-small cancelation labeling of $\{G_i\}$ over finite sets of labels, and for the finitely generated group Γ defined by the graphical presentation with respect to the labeling, each G_i can be embedded into Γ with respect to the word metric given by the presentation isometrically, and hence Γ contains a coarse disjoint union of $\{G_i\}$. The author does not know whether Γ has metric subspaces with arbitrarily large finite asymptotic dimension.

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