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WEIGHTED INEQUALITIES FOR THE DYADIC MAXIMAL OPERATOR INVOLVING AN INFINITE PRODUCT

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Abstract. We define a generalized dyadic maximal operator involving an infinite product. We get adapted A_p and S_p weighted inequalities for this operator. A version of the Carleson embedding theorem is also proved. Our results heavily depend on a generalized Hölder inequality.

1. Introduction. The *Hardy–Littlewood maximal operator* M is an operator acting on real valued Lebesgue measurable functions f by the formula

$$Mf(x) = \sup_{x\in Q} \frac{1}{|Q|} \int_Q |f(y)|\,dy,$$

where Q is a nondegenerate cube in the n-dimensional real Euclidean space $\mathbb{R}^n$ with sides parallel to the coordinate axes, and $|Q|$ is the Lebesgue measure of Q.

Let u, v be two *weights*, i.e., positive locally integrable functions. As is well known, for $p \geq 1$, Muckenhoupt [M] showed that the inequality

$$\lambda^p \int_{\{Mf>\lambda\}} u(x)\,dx \leq C \int_{\mathbb{R}^n} |f(x)|^p v(x)\,dx, \qquad \forall\lambda > 0, \forall f \in L^p(v),$$

holds if and only if $(u, v) \in A_p$, i.e., for any cube Q in $\mathbb{R}^n$ with sides parallel to the axes,

$$\left(\frac{1}{|Q|}\int_Q u(x)\,dx\right)\left(\frac{1}{|Q|}\int_Q v(x)^{-1/(p-1)}\,dx\right)^{p-1} < C, \qquad p > 1;$$

$$\frac{1}{|Q|}\int_Q u(x)\,dx \leq C \operatorname*{ess\,inf}_Q v(x), \qquad p = 1.$$

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Suppose that $u = v$ and $p > 1$. Muckenhoupt [M] also proved that

$$\int_{\mathbb{R}^n} (Mf(x))^p v(x)\,dx \le C \int_{\mathbb{R}^n} |f(x)|^p v(x)\,dx, \qquad \forall f \in L^p(v),$$

holds if and only if v satisfies

$$\left(\frac{1}{|Q|}\int_Q v(x)\,dx\right)\left(\frac{1}{|Q|}\int_Q v(x)^{-1/(p-1)}\,dx\right)^{p-1} < C, \qquad \forall Q.$$

But the problem of finding all u and v such that

$$\int_{\mathbb{R}^n} (Mf(x))^p u(x)\,dx \le C \int_{\mathbb{R}^n} |f(x)|^p v(x)\,dx, \qquad \forall f \in L^p(v),$$

is much harder and complicated. In order to solve the problem, Sawyer [S] established the testing *condition* $S_{p,q}$: for any cube Q in $\mathbb{R}^n$ with sides parallel to the axes,

$$\left(\int_Q \big(M(\chi_Q v^{1-p'})(x)\big)^q u(x)\,dx\right)^{1/q} \le C\left(\int_Q v(x)^{1-p'}\,dx\right)^{1/p},$$

where $1 < p \le q < \infty$. The condition $S_{p,q}$ is a sufficient and necessary condition for the weighted inequality

$$\left(\int_{\mathbb{R}^n} (Mf(x))^q u(x)\,dx\right)^{1/q} \le C\left(\int_{\mathbb{R}^n} |f(x)|^p v(x)\,dx\right)^{1/p}, \qquad \forall f \in L^p(v),$$

to hold. Motivated by these results, the theory of weighted inequalities developed rapidly in the last years, not only for the Hardy–Littlewood maximal operator but also for other operators in harmonic analysis like Calderón-Zygmund operators (see [CMP] and [GR] for more information).

Recently, the multisublinear maximal function

$$\mathcal{M}(f_1,\dots,f_m)(x) = \sup_{x\in Q}\prod_{i=1}^m \frac{1}{|Q|}\int_Q |f_i(y_i)|\,dy_i,$$

associated with cubes with sides parallel to the coordinate axes, was studied in [LOPTT]. The importance of this operator is that it generalizes the Hardy–Littlewood maximal function (case $m = 1$) and it controls the class of multilinear Calderón–Zygmund operators in several ways, as shown in [LOPTT]. The relevant class of multiple weights for $\mathcal{M}$ is given by the *condition* $A_{\vec{p}}$: for $\vec{p} = (p_1,\dots,p_m)$, $\vec{\omega} = (\omega_1,\dots,\omega_m)$ and a weight v, the weight vector $(v,\vec{\omega})$ is in $A_{\vec{p}}$ if

$$\sup_Q \frac{v(Q)}{|Q|}\prod_{i=1}^m\left(\frac{1}{|Q|}\int_Q \omega_i(y_i)^{-1/(p_i-1)}\,dy_i\right)^{p/p_i'} < \infty,$$

where $1/p = \sum_{i=1}^m 1/p_i$ and $1 \le p_1,\dots,p_m < \infty$.

It is easy to see that in the linear case (that is, if $m = 1$), condition $A_{\vec{p}}$ is the usual A_p. In [LOPTT] the following multilinear extension of the Muckenhoupt A_p theorem for the maximal function was obtained: the inequality

$$\|\mathcal{M}(\vec{f})\|_{L^{p,\infty}(v)} \le C \prod_{i=1}^{m} \|f_i\|_{L^{p_i}(\omega_i)}, \quad \forall f_i \in L^{p_i}(\omega_i),$$

holds if and only if $(v, \vec{\omega}) \in A_{\vec{p}}$. Moreover, if $1 < p_1, \dots, p_m < \infty$ and $v = \prod_{i=1}^{m} w_i^{p/p_i}$, then the inequality

$$\|\mathcal{M}(\vec{f})\|_{L^{p}(v)} \le C \prod_{i=1}^{m} \|f_i\|_{L^{p_i}(\omega_i)}, \quad \forall f_i \in L^{p_i}(\omega_i), \tag{1.1}$$

holds if and only if $(v, \vec{\omega}) \in A_{\vec{p}}$. The more general case was extensively discussed in [GLY, GLPT].

In order to establish a generalization of Sawyer's theorem to the multilinear setting, Chen and Damián [CD] introduced a reverse Hölder inequality $RH_{\vec{p}}$ on weights and established a multilinear version of Sawyer's result; however, the method does not work without $RH_{\vec{p}}$. In our opinion, it is difficult to establish a multilinear version of Sawyer's result without any assumptions. In fact, we also found that Li, Xue and Yan [LXY] introduced a kind of monotonicity property and established a multilinear version of Sawyer's result.

In this paper, for suitable $\vec{f} = (f_1, f_2, \dots)$ (see Remarks 2.4 and 3.1 for two kinds of suitable conditions), we define a new generalized dyadic maximal function

$$\mathfrak{M}_d(\vec{f})(x) \triangleq \sup_{x \in B \in \mathcal{D}} \prod_{i=1}^{\infty} \frac{1}{|B|} \int_B |f_i(y_i)|\, dy_i,$$

where $\mathcal{D}$ is the family of dyadic cubes in $\mathbb{R}^n$. This operator involves an infinite product. With some assumptions and notation described in Section 3, our main results are weighted inequalities for the operator $\mathfrak{M}_d$.

Firstly, we have weighted weak type inequalities.

THEOREM 1.1. *Let $1 < p_i < \infty$ for $i \in \mathbb{N}$, and $1/p = \sum_{i=1}^{\infty} 1/p_i$. Let v and ω_i be weights. Then the following statements are equivalent:*

(1) *There exists a positive constant C such that*

$$\left(\frac{1}{|B|} \int_B v(x)\, dx\right)^{1/p} \prod_{i=1}^{\infty} \left(\frac{1}{|B|} \int_B \omega_i(y_i)^{-1/(p_i-1)}\, dy_i\right)^{1/p_i'} \le C, \quad \forall B \in \mathcal{D}.$$

(2) *There exists a positive constant C such that*

$$v(B)^{1/p} \prod_{i=1}^{\infty} \left(\frac{1}{|B|} \int_B f_i(y_i)\, dy_i\right) \le C \prod_{i=1}^{\infty} \|f_i \chi_B\|_{L^{p_i}(\omega_i)}$$

for any $\vec{f} \in \prod_{i=1}^{\infty} L^{p_i}(\omega_i)$ and $B \in \mathcal{D}$.

(3) *There exists a positive constant C such that*

$$\lambda v(\{\mathfrak{M}_d(\vec{f}) \geq \lambda\})^{1/p} \leq C \prod_{i=1}^{\infty} \|f_i\|_{L^{p_i}(\omega_i)}$$

for any $\vec{f} \in \prod_{i=1}^{\infty} L^{p_i}(\omega_i)$ *and* $\lambda > 0$.

(4) *There exists a positive constant C such that*

$$\lambda v(\{\mathfrak{M}_d(\vec{f}) > \lambda\})^{1/p} \leq C \prod_{i=1}^{\infty} \|f_i\|_{L^{p_i}(\omega_i)}, \quad \forall \vec{f} \in \prod_{i=1}^{\infty} L^{p_i}(\omega_i), \forall \lambda > 0.$$

Moreover, if we denote the smallest constants C in (1)–(4) *by* $[v, \vec{\omega}]_{A_{\vec{p}}}$, $[v, \vec{\omega}]'_{A_{\vec{p}}}$, $\|\mathfrak{M}_d\|'$ *and* $\|\mathfrak{M}_d\|$, *respectively, then*

$$[v, \vec{\omega}]_{A_{\vec{p}}} = [v, \vec{\omega}]'_{A_{\vec{p}}} = \|\mathfrak{M}_d\|' = \|\mathfrak{M}_d\|.$$

Secondly, we give a strong type inequality which partially generalizes (1.1). Let $1 < p_i < \infty$ for $i \in \mathbb{N}$, and $1/p = \sum_{i=1}^{\infty} 1/p_i$. Let $\omega_i \in A_{p_i}$ for $i \in \mathbb{N}$. We say that the weight vector $\vec{\omega}$ satisfies the *condition* $A^*_{\vec{p}}$ if

$$\prod_{i=1}^{\infty} [\omega_i]_{A_{p_i}}^{1/p_i} < \infty.$$

Proposition 1.2. *Let $1 < p_i < \infty$ for $i \in \mathbb{N}$, and $1/p = \sum_{i=1}^{\infty} 1/p_i$. If $\sum_{i=1}^{\infty} (\ln p_i)/p_i < \infty$ and the weight vector $\vec{\omega}$ satisfies the condition $A^*_{\vec{p}}$, then*

$$\|\mathfrak{M}_d(\vec{f})\|_{L^p(v)} \leq C \prod_{i=1}^{\infty} \|f_i\|_{L^{p_i}(\omega_i)}, \quad \forall \vec{f} \in \prod_{i=1}^{\infty} L^{p_i}(\omega_i),$$

where

$$v = \prod_{i=1}^{\infty} \omega_i^{1/p_i} \quad \text{and} \quad C = \prod_{i=1}^{\infty} [\omega_i]_{A_{p_i}}^{p'_i/p_i} \prod_{i=1}^{\infty} p_i^{p'_i/p_i} \prod_{i=1}^{\infty} p'_i.$$

Finally, we can get S_p weighted inequalities involving infinite products. Let $1 < p_i < \infty$ for $i \in \mathbb{N}$, and $1/p = \sum_{i=1}^{\infty} 1/p_i$. Let ω_i be weights and let $\sigma_i = \omega_i^{-1/(p_i-1)}$ for $i \in \mathbb{N}$. We say that the weight vector $\vec{\omega}$ satisfies the *reverse Hölder condition* $RH_{\vec{p}}$ if there exists a positive constant C such that

$$\prod_{i=1}^{\infty} \left(\frac{1}{|B|} \int_B \sigma_i \, dx \right)^{p/p_i} \leq C \frac{1}{|B|} \int_B \prod_{i=1}^{\infty} \sigma_i^{p/p_i} \, dx, \quad \forall B \in \mathcal{D}. \tag{1.2}$$

Moreover, we denote the smallest constant C in (1.2) by $[\vec{\omega}]_{RH_{\vec{p}}}$.

Theorem 1.3. *Let $1 < p_i < \infty$ for $i \in \mathbb{N}$, and $1/p = \sum_{i=1}^{\infty} 1/p_i$. If $(\omega_1, \omega_2, \dots) \in RH_{\vec{p}}$ then the following statements are equivalent:*

(1) *There exists a positive constant C such that*

$$\|\mathfrak{M}_d(\vec{f})\|_{L^p(v)} \le C \prod_{i=1}^{\infty} \|f_i\|_{L^{p_i}(\omega_i)}, \quad \forall \vec{f} \in \prod_{i=1}^{\infty} L^{p_i}(\omega_i).$$

(2) *There exists a positive constant C such that*

$$\Big(\int_B (\mathfrak{M}_d(\overrightarrow{\sigma\chi_B})(x))^p v(x)\,dx\Big)^{1/p} \le C \prod_{i=1}^{\infty} \Big(\int_B \sigma_i(x)\,dx\Big)^{1/p_i}, \quad \forall B \in \mathcal{D}.$$

Moreover, if we denote the smallest constants C in (1) *and* (2) *by $\|\mathfrak{M}_d\|$ and $[v,\vec{\omega}]_{S_{\vec{p}}}$, respectively, then*

$$[v,\vec{\omega}]_{S_{\vec{p}}} \le \|\mathfrak{M}_d\| \le [v,\vec{\omega}]_{S_{\vec{p}}} [\vec{\omega}]_{RH_{\vec{p}}}^{1/p}.$$

The paper is organized as follows. Section 2 contains a generalized Hölder inequality proved in [CJJ]. Theorems 1.1 and 1.3 are proved in Section 3, and Proposition 1.2 is deduced from Lemma 3.3.

Throughout this paper, the letter C will denote a positive constant which may change from one instance to another.

2. Preliminaries

2.1. Generalized Hölder inequality for integrals. In this subsection, we suppose that $(\Omega, \mathcal{F}, \mu)$ is a measure space and $\{f_i\}$ is a sequence of nonnegative measurable functions on $(\Omega, \mathcal{F}, \mu)$. We recall the following lemma which is a generalized Hölder inequality (see, e.g., [CJJ, Theorem 2.11]). This kind of inequality is also discussed in the context of σ-finite measure spaces in [K].

LEMMA 2.1. *Let $0 < p_i < \infty$ for $i \in \mathbb{N}$, and $\sum_{i=1}^{\infty} 1/p_i = 1/p$. If $\prod_{i=1}^{\infty} \|f_i\|_{L^{p_i}} < \infty$, then the function $\prod_{i=1}^{\infty} f_i$ is well defined and $\|\prod_{i=1}^{\infty} f_i\|_{L^p} \le \prod_{i=1}^{\infty} \|f_i\|_{L^{p_i}}$.*

2.2. Dyadic cubes and the dyadic maximal function in $\mathbb{R}^n$. For a given $k \in \mathbb{Z}$, let $\mathcal{D}_k$ be the collection of all cubes of the form

$$[m_1 2^{-k}, (m_1+1)2^{-k}) \times \cdots \times [m_n 2^{-k}, (m_n+1)2^{-k}),$$

where $m_1, \dots, m_n$ run over the set of integers. The elements of $\mathcal{D} = \bigcup_{k\in\mathbb{Z}} \mathcal{D}_k$ are called *dyadic cubes*. Given a cube $B \in \mathcal{D}$, we denote its Lebesgue measure by $|B|$. Observe that two dyadic cubes are either disjoint, or one is contained in the other. For each $x \in \mathbb{R}^n$ and $k \in \mathbb{Z}$, there is a unique element of $\mathcal{D}_k$ containing x. Moreover, the σ-algebra $\sigma(\mathcal{D}_k)$ of measurable subsets of $\mathbb{R}^n$ formed by countable unions and complements of elements of $\mathcal{D}_k$ increases as k increases.

Given a locally integrable function on $\mathbb{R}^n$, we define its *dyadic maximal function* $M_d(f)$ by

$$M_d(f)(x) = \sup_{x\in B\in\mathcal{D}} \frac{1}{|B|}\int_B |f(y)|\,dy.$$

Recall that the conditional expectation of a locally integrable function f on $\mathbb{R}^n$ with respect to the family of the σ-algebras $\sigma(\mathcal{D}_k)$ is defined as (see, e.g., [G, p. 384])

$$E_k(f)(x) = \sum_{B\in\mathcal{D}_k}\left(\frac{1}{|B|}\int_B f(y)\,dy\right)\chi_B(x).$$

Then we have $M_d(f)(x) = \sup_{k\in\mathbb{Z}} E_k(|f|)(x)$. Moreover, for a suitable $\vec{f} = (f_1, f_2, \dots)$, we also have $\mathfrak{M}_d(\vec{f})(x) = \sup_{k\in\mathbb{Z}}\prod_{i=1}^{\infty} E_k(|f_i|)(x)$.

REMARK 2.2. Let $q > 1$. We have (see, e.g., [G, p. 94])

$$\|M_d f\|_{L^p} \le \frac{q}{q-1}\|f\|_{L^p}.$$

REMARK 2.3. Let $1 < p_i < \infty$ for $i \in \mathbb{N}$, and $\sum_{i=1}^{\infty} 1/p_i = 1/p$. Then

$$\prod_{i=1}^{\infty} p_i' < \infty,$$

where $1/p_i + 1/p_i' = 1$, $i \in \mathbb{N}$. This can be checked easily (see, e.g., [CJJ, Theorem 2.12]).

REMARK 2.4. Let $p_i > 1$ for $i \in \mathbb{N}$. If $\prod_{i=1}^{\infty}\|f_i\|_{L^{p_i}} < \infty$ and $\sum_{i=1}^{\infty} 1/p_i = 1/p$, then

$$\|\mathfrak{M}_d(\vec{f})\|_{L^p} \le \Big\|\prod_{i=1}^{\infty} M_d f_i\Big\|_{L^p} \le \prod_{i=1}^{\infty}\|M_d f_i\|_{L^{p_i}} \le \Big(\prod_{i=1}^{\infty} p_i'\Big)\prod_{i=1}^{\infty}\|f_i\|_{L^{p_i}} < \infty,$$

where we have used Lemma 2.1 and Remarks 2.2 and 2.3.

3. Main results and proofs. There are a lot of assumptions and notation which will be used in this section. For convenience, we state them at the beginning of this part.

Assumptions and notation. Let $\omega_i \in L^1_{\mathrm{loc}}$ and $1 < p_i < \infty$ for $i \in \mathbb{N}$, and let $\{f_i\}$ be a sequence of nonnegative measurable functions on $\mathbb{R}^n$. Write $\vec{p} = (p_1, p_2, \dots)$, $\vec{\omega} = (\omega_1, \omega_2, \dots)$, $\vec{f} = (f_1, f_2, \dots)$ and $\sigma_i = \omega_i^{-1/(p_i-1)}$ for $i \in \mathbb{N}$. In addition, we also write $\overrightarrow{f\chi_G} = (f_1\chi_G, f_2\chi_G, \dots)$ and $\overrightarrow{\sigma\chi_G} = (\sigma_1\chi_G, \sigma_2\chi_G, \dots)$, where G is a measurable set.

Assume that $\prod_{i=1}^{\infty} E_k(\omega_i^{1-p_i'})^{1/p_i'} < \infty$. We suppose that $1/p = \sum_{i=1}^{\infty} 1/p_i$ and $\prod_{i=1}^{\infty}\sigma_i^{1/p_i} > 0$. We also suppose that $\prod_{i=1}^{\infty}\|f_i\|_{L^{p_i}(\omega_i)} < \infty$, written

$\vec{f} \in \prod_{i=1}^{\infty} L^{p_i}(\omega_i)$. Moreover, we assume that $\overrightarrow{\sigma\chi_B} \in \prod_{i=1}^{\infty} L^{p_i}(\omega_i)$ for all $B \in \mathcal{D}$, and denote this by $\vec{\sigma} \in \prod_{i=1}^{\infty} L^{p_i}_{\mathrm{loc}}(\omega_i)$.

Remark 3.1. It follows from the generalized Hölder inequality for integrals that

$$\int_{\mathbb{R}^n} \prod_{i=1}^{\infty} E_k(f_i^{p_i}\omega_i)^{p/p_i}\,dx \le \prod_{i=1}^{\infty}\Big(\int_{\mathbb{R}^n} E_k(f_i^{p_i}\omega_i)\,dx\Big)^{p/p_i} = \prod_{i=1}^{\infty}\Big(\int_{\mathbb{R}^n} f_i^{p_i}\omega_i\,dx\Big)^{p/p_i} < \infty.$$

Hence, $\prod_{i=1}^{\infty} E_k(f_i^{p_i}\omega_i)^{1/p_i} < \infty$. By Hölder's inequality, we have

$$\prod_{i=1}^{\infty} E_k(f_i) \le \prod_{i=1}^{\infty} E_k(f_i^{p_i}\omega_i)^{1/p_i} E_k(\omega_i^{-1/(p_i-1)})^{1/p_i'} = \prod_{i=1}^{\infty} E_k(f_i^{p_i}\omega_i)^{1/p_i} \prod_{i=1}^{\infty} E_k(\omega_i^{-1/(p_i-1)})^{1/p_i'} < \infty.$$

Thus $\mathfrak{M}_d(\vec{f})$ is well defined. Moreover, for $B \in \mathcal{D}$, we have $\prod_{i=1}^{\infty} E_k(\sigma_i\chi_B) < \infty$ and $\mathfrak{M}_d(\overrightarrow{\sigma\chi_B})$ is well defined.

3.1. Generalized A_p weights involving infinite products. Now, we can give the proof of our main results relating to A_p weights involving infinite products.

Proof of Theorem 1.1. We shall follow the scheme: (1)⇔(2), (3)⇔(4) and (3)⇒(2)⇒(4). Obviously, the equivalence (3)⇔(4) is trivial.

(1)⇒(2). For $B \in \mathcal{D}$, it follows from Hölder's inequality and (1) that

$$\begin{aligned}
&v(B)^{1/p}\prod_{i=1}^{\infty}\Big(\frac{1}{|B|}\int_B f_i(y_i)\,dy_i\Big)\\
&\le v(B)^{1/p}\prod_{i=1}^{\infty}\Big(\frac{1}{|B|}\int_B f_i(y_i)^{p_i}\omega_i(y_i)\,dy_i\Big)^{1/p_i}\Big(\frac{1}{|B|}\int_B \omega_i(y_i)^{-p_i'/p_i}\,dy_i\Big)^{1/p_i'}\\
&= \prod_{i=1}^{\infty}\Big(\int_B f_i(y_i)^{p_i}\omega_i(y_i)\,dy_i\Big)^{1/p_i}\\
&\quad\times\Big(\Big(\frac{1}{|B|}\int_B v(x)\,dx\Big)^{1/p}\prod_{i=1}^{\infty}\Big(\frac{1}{|B|}\int_B \omega_i(y_i)^{-1/p_i-1}\,dy_i\Big)^{1/p_i'}\Big)\\
&\le [v,\vec{\omega}]_{A_{\vec{p}}}\prod_{i=1}^{\infty}\|f_i\chi_B\|_{L^{p_i}(\omega_i)}.
\end{aligned}$$

(2)⇒(1). Let $f_i = \omega_i^{-1/(p_i-1)}$. For $B \in \mathcal{D}$, we have

$$\begin{aligned}\left(\frac{1}{|B|}\int_B v(x)\,dx\right)^{1/p} &\prod_{i=1}^{\infty}\left(\frac{1}{|B|}\int_B \omega_i(y_i)^{-1/(p_i-1)}\,dy_i\right)\\ &= \left(\frac{1}{|B|}\right)^{1/p} v(B)^{1/p}\prod_{i=1}^{\infty}\left(\frac{1}{|B|}\int_B f_i(y_i)\,dy_i\right)\\ &\le [v,\vec{\omega}]'_{A_{\vec{p}}}\left(\frac{1}{|B|}\right)^{1/p}\prod_{i=1}^{\infty}\left(\int_B \omega_i(y_i)^{-1/(p_i-1)}\,dy_i\right)^{1/p_i}\\ &= [v,\vec{\omega}]'_{A_{\vec{p}}}\prod_{i=1}^{\infty}\left(\frac{1}{|B|}\int_B \omega_i(y_i)^{-1/(p_i-1)}\,dy_i\right)^{1/p_i}.\end{aligned}$$

It follows that

$$\left(\frac{1}{|B|}\int_B v(x)\,dx\right)^{1/p}\prod_{i=1}^{\infty}\left(\frac{1}{|B|}\int_B \omega_i(y_i)^{-1/(p_i-1)}\,dy_i\right)^{1/p_i'} \le [v,\vec{\omega}]'_{A_{\vec{p}}}.$$

(3)⇒(2). Let $B \in \mathcal{D}$. For $x \in B$, we have

$$\prod_{i=1}^{\infty}\left(\frac{1}{|B|}\int_B f_i(y_i)\,dy_i\right) \le \mathfrak{M}_d(\overrightarrow{f\chi_B})(x).$$

It follows from (3) that

$$\begin{aligned}\prod_{i=1}^{\infty}\left(\frac{1}{|B|}\int_B f_i(y_i)\,dy_i\right)v(B)^{1/p} &\le \lambda v(\{\mathfrak{M}_d(\overrightarrow{f\chi_B}) \ge \lambda\})^{1/p}\\ &\le \|\mathfrak{M}_d\|'\prod_{i=1}^{\infty}\|f_i\chi_B\|_{L^{p_i}(\omega_i)},\end{aligned}$$

where $\lambda = \prod_{i=1}^{\infty}(|B|^{-1}\int_B f_i(y_i)\,dy_i)$.

(2)⇒(4). For $R > 0$, we shall denote by $\mathfrak{M}_d^{(R)}(\vec{f})$ the maximal operator obtained by taking in the corresponding definition just those cubes whose side length is less than or equal to R. We have $\mathfrak{M}_d(\vec{f})(x) = \lim_{R\to\infty}\mathfrak{M}_d^{(R)}(\vec{f})(x)$ and $\mathfrak{M}_d^{(R)}(\vec{f})(x)$ increases with R, so it will be enough to prove the inequality for $\mathfrak{M}_d^{(R)}$ with constant independent of R. But, after fixing $R > 0$ and $\lambda > 0$, observe that $\{x \in \mathbb{R}^n : \mathfrak{M}_d^{(R)}(\vec{f})(x) > \lambda\} = \bigcup_j Q_j$, where Q_j are the maximal dyadic cubes of side length $\le R$ for which

$$\prod_{i=1}^{\infty}\left(\frac{1}{|Q_j|}\int_{Q_j} f_i(y_i)\,dy_i\right) > \lambda.$$

These maximal dyadic cubes do exist because of the restriction on their size. Moreover, they are disjoint. It follows from (2) and the generalized Hölder inequality that

$$\lambda^p v(\{x\in\mathbb{R}^n : \mathfrak{M}_d^{(R)}(\vec f)>\lambda\}) = \lambda^p \sum_j v(Q_j)$$
$$\le \sum_j v(Q_j)\Big(\prod_{i=1}^{\infty}\Big(\frac{1}{|Q_j|}\int_{Q_j} f_i(y_i)\,dy_i\Big)\Big)^p \le ([v,\vec\omega]'_{A_{\vec p}})^p \sum_j \prod_{i=1}^{\infty} \|f_i\chi_{Q_j}\|^p_{L^{p_i}(\omega_i)}$$
$$\le ([v,\vec\omega]'_{A_{\vec p}})^p \prod_{i=1}^{\infty}\Big(\sum_j \int_{Q_j} f_i(y_i)^{p_i}\omega_i(y_i)\,dy_i\Big)^{p/p_i} \le \Big([v,\vec\omega]'_{A_{\vec p}}\prod_{i=1}^{\infty}\|f_i\|_{L^{p_i}(\omega_i)}\Big)^p.$$

Thus

$$\lambda v(\{\mathfrak{M}_d(\vec f)>\lambda\})^{1/p} \le [v,\vec\omega]'_{A_{\vec p}}\prod_{i=1}^{\infty}\|f_i\|_{L^{p_i}(\omega_i)}. \ \blacksquare$$

Lemma 3.2. *Let $1<p_i<\infty$ for $i\in\mathbb{N}$, and $1/p=\sum_{i=1}^{\infty}1/(p_i)$. If $\sum_{i=1}^{\infty}(\ln p_i)/p_i<\infty$, then $\prod_{i=1}^{\infty}p_i^{p_i'/p_i}p_i'<\infty$.*

Proof. (1) We first prove $\prod_{i=1}^{\infty}p_i'<\infty$. It suffices to prove $\sum_{i=1}^{\infty}\ln p_i' <\infty$. As $p_i'=(1-1/p_i)^{-1}$, we should prove $\sum_{i=1}^{\infty}\ln(1-1/p_i)^{-1}<\infty$. Since

$$\lim_{i\to\infty}\frac{\ln(1-1/p_i)^{-1}}{1/p_i}=1$$

and $\sum_{i=1}^{\infty}1/p_i=1/p$, we have $\sum_{i=1}^{\infty}\ln(1-1/p_i)^{-1}<\infty$ by the Limit Comparison Test.

(2) We now prove $\prod_{i=1}^{\infty}p_i^{p_i'/p_i}<\infty$. It suffices to prove $\sum_{i=1}^{\infty}(p_i'/p_i)\ln p_i <\infty$. As $1/p=\sum_{i=1}^{\infty}1/p_i$, we have $\lim_{i\to\infty}p_i=\infty$. Hence $\lim_{i\to\infty}p_i'=1$. It follows that $M\triangleq\sup_{i\ge1}p_i'<\infty$. Thus,

$$\sum_{i=1}^{\infty}\frac{p_i'}{p_i}\ln p_i \le M\sum_{i=1}^{\infty}\frac{\ln p_i}{p_i}<\infty.$$

It follows from (1) and (2) that $\prod_{i=1}^{\infty}p_i^{p_i'/p_i}p_i'<\infty$. $\blacksquare$

In order to prove Proposition 1.2, we give the following lemma which is essentially taken from A. K. Lerner [L]. Combining it with Remark 3.2, we get Proposition 1.2.

Lemma 3.3. *Let ω be a weight and $1<p<\infty$. Suppose that $\sigma=\omega^{-1/(p-1)}\in L^1_{\mathrm{loc}}$. Then the following statements are equivalent:*

(1) *There exists a positive constant C such that*

$$\|M_d(f)\|_{L^p(\omega)}\le C\|f\|_{L^p(\omega)},\qquad \forall f\in L^p(\omega).$$

(2) *There exists a positive constant* C *such that*

$$\frac{\omega(B)}{|B|}\left(\frac{\sigma(B)}{|B|}\right)^{p-1} \le C, \qquad \forall B \in \mathcal{D}.$$

Moreover, if we denote the smallest constants C *in* (1) *and* (2) *by* $\|M_d\|$ *and* $[\omega]_{A_p}$*, respectively, then*

$$[\omega]_{A_p}^{1/p} \le \|M_d\| \le [\omega]_{A_p}^{p'/p} p^{p'/p} p'.$$

3.2. Generalized S_p weights involving infinite products. In this final part of the paper, we will establish generalized S_p weighted inequalities.

In order to prove Theorem 1.3, we need a lemma related to the Carleson embedding theorem. Its linear version can be found in [HP].

LEMMA 3.4. *Let* $1 < p_i < \infty$ *for* $i \in \mathbb{N}$*, and* $1/p = \sum_{i=1}^{\infty} 1/p_i$*. Let* ω_i *be weights and let* $\sigma_i = \omega_i^{-1/(p_i-1)}$ *for* $i \in \mathbb{N}$*. Suppose* $\{a_B\}_{B\in\mathcal{D}}$ *are nonnegative numbers that satisfy*

$$\sum_{B\subset G} a_B \le A \int_G \prod_{i=1}^{\infty} \sigma_i^{p/p_i}\, dx, \qquad \forall G \in \mathcal{D}.$$

Then for all $\vec{f} \in \prod_{i=1}^{\infty} L^{p_i}(\sigma_i)$*, we have*

$$\left(\sum_{B\in\mathcal{D}} a_B \left(\prod_{i=1}^{\infty} \frac{1}{\sigma_i(B)} \int_B f_i(y_i)\sigma_i(y_i)dy_i\right)^p\right)^{1/p}$$
$$\le A^{1/p} \|\mathfrak{M}_d^{\vec{\sigma}}(\vec{f})\|_{L^p(\nu_{\vec{\sigma}})} \le A^{1/p} \prod_{i=1}^{\infty} p_i' \|f_i\|_{L^{p_i}(\sigma_i)},$$

where

$$\nu_{\vec{\sigma}} = \prod_{i=1}^{\infty} \sigma_i^{p/p_i} \quad \textit{and} \quad \mathfrak{M}_d^{\vec{\sigma}}(\vec{f})(x) = \sup_{x\in B\in\mathcal{D}} \prod_{i=1}^{\infty} \frac{1}{\sigma_i(B)} \int_B f_i(y_i)\sigma_i(y_i)\, dy_i.$$

Proof. Let us look at the sum

$$\sum_{B\in\mathcal{D}} a_B \left(\prod_{i=1}^{\infty} \frac{1}{\sigma_i(B)} \int_B f_i(y_i)\sigma_i(y_i)\, dy_i\right)^p$$

as an integral on a measure space $(\mathcal{D}, 2^{\mathcal{D}}, \mu)$ built over the set of dyadic cubes $\mathcal{D}$, assigning to each $B \in \mathcal{D}$ the measure a_B. Thus

$$\sum_{B\in\mathcal{D}} a_B\Big(\prod_{i=1}^{\infty}\frac{1}{\sigma_i(B)}\int_B f_i(y_i)\sigma_i(y_i)\,dy_i\Big)^p$$
$$= \int_0^\infty p\lambda^{p-1}\mu\Big\{B\in\mathcal{D}:\prod_{i=1}^{\infty}\frac{1}{\sigma_i(B)}\int_B f_i(y_i)\sigma_i(y_i)\,dy_i>\lambda\Big\}\,d\lambda$$
$$=: \int_0^\infty p\lambda^{p-1}\mu(\mathcal{D}_\lambda)\,d\lambda.$$

Denote by $\mathcal{D}_\lambda(R)$ the dyadic cubes having side length $\le R$ such that

$$\prod_{i=1}^{\infty}\frac{1}{\sigma_i(B)}\int_B f_i(y_i)\sigma_i(y_i)\,dy_i>\lambda.$$

Since the cubes in $\mathcal{D}_\lambda(R)$ have uniformly bounded side length, every cube is contained in a maximal one. Let $\mathcal{D}^*_\lambda(R)$ denote the subfamily formed by these maximal cubes. Then the cubes $Q\in\mathcal{D}^*_\lambda(R)$ are disjoint and their union is contained in the set $\{\mathfrak{M}^{\vec{\sigma}}_d(\vec{f})>\lambda\}$. Thus

$$\mu(\mathcal{D}_\lambda(R)) = \sum_{B\in\mathcal{D}_\lambda(R)} a_B \le \sum_{Q\in\mathcal{D}^*_\lambda(R)}\sum_{B\subset Q} a_B$$
$$\le A\sum_{Q\in\mathcal{D}^*_\lambda(R)}\int_Q\prod_{i=1}^{\infty}\sigma_i^{p/p_i}\,dx \le A\int_{\{\mathfrak{M}^d_{\vec{\sigma}}(\vec{f})>\lambda\}}\prod_{i=1}^{\infty}\sigma_i^{p/p_i}\,dx.$$

It follows that $\mu(\mathcal{D}_\lambda)\le A\int_{\{\mathfrak{M}^d_{\vec{\sigma}}(\vec{f})>\lambda\}}\prod_{i=1}^{\infty}\sigma_i^{p/p_i}\,dx$. Then we obtain

$$\sum_{B\in\mathcal{D}} a_B\Big(\prod_{i=1}^{\infty}\frac{1}{\sigma_i(B)}\int_B f_i(y_i)\sigma_i(y_i)\,dy_i\Big)^p$$
$$\le A\int_0^\infty p\lambda^{p-1}\int_{\{\mathfrak{M}^{\vec{\sigma}}_d(\vec{f})>\lambda\}}\prod_{i=1}^{\infty}\sigma_i^{p/p_i}\,dx\,d\lambda$$
$$= A\int_{\mathbb{R}^n}\mathfrak{M}^{\vec{\sigma}}_d(\vec{f})^p\prod_{i=1}^{\infty}\sigma_i^{p/p_i}\,dx \le A\int_{\mathbb{R}^n}\prod_{i=1}^{\infty}\Big((M^{\sigma_i}_d(f_i))^{p_i}\sigma_i\Big)^{p/p_i}\,dx$$
$$\le A\prod_{i=1}^{\infty}\Big(\int_{\mathbb{R}^n}(M^{\sigma_i}_d(f_i))^{p_i}\sigma_i\,dx\Big)^{p/p_i} \le A\prod_{i=1}^{\infty}(p_i')^p\Big(\int_{\mathbb{R}^n} f_i^{p_i}\sigma_i\,dx\Big)^{p/p_i},$$

where we have used $\mathfrak{M}^{\vec{\sigma}}_d(\vec{f})\le\prod_{i=1}^{\infty}M^{\sigma_i}_d(f_i)$, the generalized Hölder inequality and the boundedness properties of $M^{\sigma_i}_d$ in $L^{p_i}(\sigma_i)$. ∎

Proof of Theorem 1.3. It is clear that (1)⇒(2) without $(v,\vec{\omega})\in RH_{\vec{p}}$, so we omit it.

Next, assuming (2), we shall prove (1). Let $\vec{f} \in \prod_{i=1}^{\infty} L^{p_i}(\omega_i)$ and $\alpha > 1$. For every integer k, we shall consider the set

$$S_k = \{x \in \mathbb{R}^n : \alpha^k < \mathfrak{M}_d(\vec{f})(x) \le \alpha^{k+1}\}.$$

From the definition of $\mathfrak{M}_d$, $S_k \subseteq \bigcup_j B_{k,j}$, where for each k, $\{B_{k,j}\}_{j=1,2,\dots}$ is the collection of all dyadic cubes satisfying

$$\prod_{i=1}^{\infty} \frac{1}{|B_{k,j}|} \int_{B_{k,j}} f_i(y_i)\, dy_i > \alpha^k.$$

Define $E_{k,1} = B_{k,1} \cap S_k$ and for $j > 1$,

$$E_{k,j} = \Big(B_{k,j} \setminus \bigcup_{s<j} B_{k,s}\Big) \cap S_k.$$

The sets S_k form a disjoint collection and each S_k is disjoint union of the sets $E_{k,j}$ for varying j. We have

$$\begin{aligned}
\int_{\mathbb{R}^n} \mathfrak{M}_d(\vec{f})^p v\, dx &\le \alpha^p \sum_{k,j\in\mathbb{Z}} v(E_{k,j}) \bigg(\prod_{i=1}^{\infty} \frac{1}{|B_{k,j}|} \int_{B_{k,j}} f_i(y_i)\, dy_i\bigg)^p \\
&= \alpha^p \sum_{k,j\in\mathbb{Z}} v(E_{k,j}) \bigg(\prod_{i=1}^{\infty} \frac{\sigma_i(B_{k,j})}{|B_{k,j}|}\bigg)^p \\
&\quad \times \bigg(\prod_{i=1}^{\infty} \frac{1}{\sigma_i(B_{k,j})} \int_{B_{k,j}} f_i(y_i)\sigma_i(y_i)^{-1}\sigma_i(y_i)\, dy_i\bigg)^p \\
&= \alpha^p \sum_{k,j\in\mathbb{Z}} a_B \bigg(\prod_{i=1}^{\infty} \frac{1}{\sigma_i(B)} \int_B f_i(y_i)\sigma_i(y_i)^{-1}\sigma_i(y_i)\, dy_i\bigg)^p.
\end{aligned}$$

Here $a_B = v(E(B))(\prod_{i=1}^{\infty} \sigma_i(B)/|B|)^p$ if $B = B_{k,j}$ for some (k,j), where $E(B)$ denotes the corresponding set $E_{k,j}$ associated to $B_{k,j}$, and $a_B = 0$ otherwise. If we apply the Carleson embedding theorem to these a_B, we will find the desired result provided that

$$\sum_{B\subset G} a_B \le A \int_G \prod_{i=1}^{\infty} \sigma_i^{p/p_i}\, dx, \qquad G \in \mathcal{D}.$$

However, for $G \in \mathcal{D}$, we obtain

$$\begin{aligned}
\sum_{B\subset G} a_B &= \sum_{B_{k,j}\subset G} v(E_{k,j}) \bigg(\prod_{i=1}^{\infty} \frac{\sigma_i(B_{k,j})}{|B_{k,j}|}\bigg)^p \\
&= \sum_{B_{k,j}\subset G} \int_{E_{k,j}} \bigg(\prod_{i=1}^{\infty} \frac{\sigma_i(B_{k,j})}{|B_{k,j}|}\bigg)^p v(x)\, dx
\end{aligned}$$

$$\leq \sum_{B_{k,j}\subset G} \int_{E_{k,j}} \mathfrak{M}_d(\overrightarrow{\sigma\chi_G})^p v\,dx \leq [v,\vec{w}]^p_{S_{\vec{p}}} \prod_{i=1}^{\infty} \sigma_i(G)^{p/p_i}$$

$$\leq [v,\vec{w}]^p_{S_{\vec{p}}}[\vec{\omega}]_{RH_{\vec{p}}} \int_G \prod_{i=1}^{\infty} \sigma_i^{p/p_i}\,dx,$$

where in the next to last inequality we have used the $S_{\vec{p}}$ condition and in the last inequality we have used the $RH_{\vec{p}}$ condition. Thus, by Lemma 3.4,

$$\|\mathfrak{M}_d(\vec{f})\|_{L^p(v)} \leq \alpha[v,\vec{w}]_{S_{\vec{p}}}[\vec{\omega}]^{1/p}_{RH_{\vec{p}}}\Big(\prod_{i=1}^{\infty} p_i'\Big)\prod_{i=1}^{\infty}\|f_i\|_{L^{p_i}(\omega_i)}.$$

Then we can take the limit $\alpha \to 1$ to obtain

$$\Big(\int_{\mathbb{R}^n} \mathfrak{M}_d(\vec{f})^p v\,dx\Big)^{1/p} \leq [v,\vec{\omega}]_{S_{\vec{p}}}[\vec{\omega}]^{1/p}_{RH_{\vec{p}}}\Big(\prod_{i=1}^{\infty} p_i'\Big)\prod_{i=1}^{\infty}\|f_i\|_{L^{p_i}(\omega_i)}. \blacksquare$$

Remark 3.5. Let $1 < p_i < \infty$ for $i \in \mathbb{N}$, and $1/p = \sum_{i=1}^{\infty} 1/p_i$. It is trivial that $[v,\vec{\omega}]_{S_{\vec{p}}} < \infty$ implies $[v,\vec{\omega}]_{A_{\vec{p}}} < \infty$. Moreover, suppose $\sum_{i=1}^{\infty}(\ln p_i)/p_i < \infty$ and let $v = \prod_{i=1}^{\infty}\omega_i^{1/p_i}$. It follows from Proposition 1.2 and Theorem 1.3 that $\vec{\omega} \in A^*_{\vec{p}}$ implies $[v,\vec{\omega}]_{S_{\vec{p}}} < \infty$.

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