COLLOQUIUM MATHEMATICUM

VOL. 145 2016 NO. 2

ON AN INTRINSIC CHARACTERIZATION OF SELF-ADJOINT C^*-SEGAL ALGEBRAS

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Abstract. An intrinsic characterization of self-adjoint C^*-Segal algebras among Banach $*$-algebras is obtained in terms of Pták's spectral function. A connection between order structure in C^*-Segal algebras and Grothendieck's well known dual characterization of C^*-algebras is revealed.

Recently there has been considerable interest in Banach algebras that are dense ideals in Banach algebras (respectively C^*-algebras), called Segal algebras (respectively C^*-Segal algebras). A commutative C^*-Segal algebra incorporates Nachbin's weighted function algebras in its Gelfand–Naimark framework [1]; and at the non-commutative level, this leads to weighted C^*-algebras (a non-commutative analogue of Nachbin algebras) [7]. The present note is aimed at discussing an intrinsic characterization of self-adjoint C^*-Segal algebras. A relation to Grothendieck's dual characterization of a C^*-algebra [6] is also discussed.

Following [7], a faithful Banach algebra $\mathcal{A}$ is a *C^*-Segal algebra* if $\mathcal{A}$ continuously sits in a C^*-algebra $\mathcal{B}$ as a dense ideal under an injective homomorphism i. Also, $\mathcal{A}$ is *self-adjoint* if $i(\mathcal{A})$ is closed under the involution of $\mathcal{B}$. In this case, $\mathcal{A}$ becomes a Banach $*$-algebra. We address the issue of intrinsically characterizing self-adjoint C^*-Segal algebras. Let $s_{\mathcal{A}}(x) = r(x^*x)^{1/2}$ be the celebrated *Pták function* of Banach $*$-algebra theory [4, p. 224], where $r(\cdot)$ denotes the spectral radius. We show that $(\mathcal{A}, \|\cdot\|_{\mathcal{A}})$ is a self-adjoint C^*-Segal algebra if and only if $\mathcal{A}$ is hermitian, $*$- semisimple, and for some $l > 0$, the inequalities $\|ab\|_{\mathcal{A}} \le l s_{\mathcal{A}}(a)\|b\|_{\mathcal{A}}$, $\|ab\|_{\mathcal{A}} \le l\|a\|_{\mathcal{A}} s_{\mathcal{A}}(b)$ hold for all $a, b \in \mathcal{A}$. This is used to show that if $\mathcal{A}$ is a self-adjoint C^*-Segal algebra in a C^*-algebra $\mathcal{B}$, then $\mathcal{B}$ is necessarily unique and is the enveloping C^*-algebra $C^*(\mathcal{A})$ of $\mathcal{A}$. This leaves open the question of describing intrinsically faithful Banach algebras that are not necessarily self-adjoint C^*-Segal algebras.

2010 *Mathematics Subject Classification*: Primary 46H05; Secondary 46K05.
Key words and phrases: Banach algebra, Banach $*$-algebra, C^*-algebra, Segal algebra, C^*-Segal algebra.
Received 24 September 2015; revised 21 March 2016.
Published online 4 July 2016.

DOI: 10.4064/cm6774-4-2016

The *multiplier algebra* $M(\mathcal{A})$ of a Banach algebra $(\mathcal{A}, \|\cdot\|_{\mathcal{A}})$ consists of pairs $m = (m_l, m_r)$ of bounded linear maps from $\mathcal{A}$ to $\mathcal{A}$ such that $m_l(ab) = m_l(a)b$, $m_r(ab) = am_r(b)$, $am_l(b) = m_r(a)b$ for all $a, b \in \mathcal{A}$. The Banach algebra norm on $M(\mathcal{A})$ is $\|m\|_M = \max\{\|m_r\|_{\text{op}}, \|m_l\|_{\text{op}}\}$, where $\|\cdot\|_{\text{op}}$ denotes the operator norm $\|T\|_{\text{op}} = \sup\{\|Ta\|_{\mathcal{A}} : \|a\|_{\mathcal{A}} \leq 1\}$. If $\mathcal{A}$ is a Banach $*$-algebra, then $M(\mathcal{A})$ is a Banach $*$-algebra with involution $m \to m^* = (m_r^*, m_l^*)$, $m_l^*(x) = m_l(x^*)^*$, $m_r^*(x) = m_r(x^*)^*$ for all $x \in \mathcal{A}$.

THEOREM 1. *Let $(\mathcal{A}, \|\cdot\|_{\mathcal{A}})$ be a faithful Banach $*$-algebra. Then the following are equivalent:*

(1) *$\mathcal{A}$ is a self-adjoint C^*-Segal algebra.*
(2) *The Banach $*$-algebra $M(\mathcal{A})$ contains a $*$-subalgebra which is a C^*-algebra containing $\mathcal{A}$, and there is $l > 0$ such that $\|ab\|_{\mathcal{A}} \leq ls_{\mathcal{A}}(a)\|b\|_{\mathcal{A}}$ and $\|ab\|_{\mathcal{A}} \leq l\|a\|_{\mathcal{A}}s_{\mathcal{A}}(b)$ for all $a, b \in \mathcal{A}$.*
(3) *$\mathcal{A}$ is hermitian, $*$-semisimple and there is $l > 0$ such that $\|ab\|_{\mathcal{A}} \leq l\|a\|_{\mathcal{A}}s_{\mathcal{A}}(b)$ and $\|ab\|_{\mathcal{A}} \leq ls_{\mathcal{A}}(a)\|b\|_{\mathcal{A}}$ for all $a, b \in \mathcal{A}$.*

Proof. (1)⇒(2), (1)⇒(3). Let $\mathcal{A}$ be a self-adjoint C^*-Segal algebra. Let $(\mathcal{C}, \|\cdot\|_0)$ be a C^*-algebra such that $\mathcal{A}$ is a Segal $*$-algebra in $\mathcal{C}$. Therefore $\mathcal{A}$ is a dense $*$-ideal in $\mathcal{C}$, and there is $l > 0$ such that $\|a\|_0 \leq l\|a\|_{\mathcal{A}}$ $(a \in \mathcal{A})$. Since $\mathcal{A}$ is an ideal in $\mathcal{C}$, it is spectrally invariant in $\mathcal{C}$. Hence $s_{\mathcal{A}}(a) = s_{\mathcal{C}}(a) = \|a\|_0$ $(a \in \mathcal{A})$. It follows from [2, Theorem 2.3] that $\|ab\|_{\mathcal{A}} \leq l\|a\|_{\mathcal{A}}s_{\mathcal{A}}(b)$ and $\|ab\|_{\mathcal{A}} \leq ls_{\mathcal{A}}(a)\|b\|_{\mathcal{A}}$ for every $a, b \in \mathcal{A}$. It follows that (1)⇒(3). Further, for $c \in \mathcal{C}$, consider $T_c = (l_c, r_c) : \mathcal{A} \to \mathcal{A}$. Then T_c is a continuous multiplier on $\mathcal{A}$. The map $c \mapsto (l_c, r_c)$ is an embedding of $\mathcal{C}$ into $M(\mathcal{A})$. Thus (1)⇒(2).

(2)⇒(1). Let $(\mathcal{C}, \|\cdot\|_0)$ be a C^*-algebra such that $\mathcal{A} \subset \mathcal{C} \subset M(\mathcal{A})$ as $*$-subalgebras, and let $l > 0$ be such that $\|ab\|_{\mathcal{A}} \leq l\|a\|_{\mathcal{A}}s_{\mathcal{A}}(b)$ and $\|ab\|_{\mathcal{A}} \leq ls_{\mathcal{A}}(a)\|b\|_{\mathcal{A}}$ for all $a, b \in \mathcal{A}$. Since $\mathcal{A}$ is a $*$-ideal in $M(\mathcal{A})$, $\mathcal{A}$ is a $*$-ideal in $\mathcal{C}$. This implies that $\mathcal{A}$ is spectrally invariant in $\mathcal{C}$. Therefore $s_{\mathcal{A}}(a) = s_{\mathcal{C}}(a) = \|a\|_0$ $(a \in \mathcal{A})$. Thus the closure of $\mathcal{A}$ in $\mathcal{C}$ is a C^*-algebra in which $\mathcal{A}$ is a dense $*$-ideal.

(3)⇒(1). Let $\mathcal{A}$ be hermitian and $*$-semisimple, and let $l > 0$ be such that $\|ab\|_{\mathcal{A}} \leq l\|a\|_{\mathcal{A}}s_{\mathcal{A}}(b)$ and $\|ab\|_{\mathcal{A}} \leq ls_{\mathcal{A}}(a)\|b\|_{\mathcal{A}}$. Since $\mathcal{A}$ is hermitian and $*$-semisimple, $s_{\mathcal{A}}(\cdot)$ is a C^*-norm on $\mathcal{A}$. Let $\mathcal{C} = (\mathcal{A}, s_{\mathcal{A}}(\cdot))^\sim$. Then $\mathcal{C}$ is a C^*-algebra. Since $\mathcal{A}$ is a Banach $*$-algebra and $\mathcal{C}$ is a C^*-algebra, the inclusion map from $\mathcal{A}$ to $\mathcal{C}$ is continuous. Therefore there is $k > 0$ such that $s_{\mathcal{A}}(a) \leq k\|a\|_{\mathcal{A}}$ $(a \in \mathcal{A})$. Now it follows that $\mathcal{A}$ is ideal in $\mathcal{C}$. Indeed, for $a, b_n \in \mathcal{A}$ with $b_n \to b \in \mathcal{C}$ in $s_{\mathcal{A}}(\cdot)$, we have $ab_n \to ab$ in $s_{\mathcal{A}}(\cdot)$; and since (ab_n) is Cauchy in $\|\cdot\|_{\mathcal{A}}$, it converges to d in $\mathcal{A}$, and hence also in $s_{\mathcal{A}}(\cdot)$. Thus $ab = d \in \mathcal{A}$. Similarly $ba \in \mathcal{A}$, and $\mathcal{A}$ is an ideal in $\mathcal{C}$. ∎

COROLLARY 2. *Let $\mathcal{A}$ be a self-adjoint C^*-Segal algebra in a C^*-algebra $\mathcal{B}$. Then $\mathcal{B}$ is unique; $\mathcal{B}$ is the enveloping C^*-algebra $C^*(\mathcal{A})$ of $\mathcal{A}$; and $\mathcal{A}$ is a dense ideal in $C^*(\mathcal{A})$.*

Proof. Let $\mathcal{A}$ be a self-adjoint C^*-Segal algebra in a C^*-algebra $\mathcal{B}$. Then $\mathcal{A}$ is spectrally invariant in $\mathcal{B}$; and for all $a \in \mathcal{A}$, we have $\|a\|_{\mathcal{B}}^2 = \|a^*a\|_{\mathcal{B}} = r_{\mathcal{B}}(a^*a) = r_{\mathcal{A}}(a^*a) = s_{\mathcal{A}}(a)^2$. Since $\mathcal{A}$ is hermitian, $s_{\mathcal{A}}(\cdot)$ is the greatest C^*-seminorm [4, Cor. 8, p. 227]. Thus $\mathcal{B} = (\mathcal{A}, s_{\mathcal{A}}(\cdot))^{\sim} = C^*(\mathcal{A})$. ∎

We end the present note with a couple of remarks on regularity properties and order structure of a C^*-Segal algebra.

(1) A *weighted C^*-algebra* [1, 7] is a pair $(\mathcal{A}, \pi)$ where $\mathcal{A}$ is a self-adjoint C^*-Segal algebra in a C^*-algebra $\mathcal{B}$ and $\pi : \mathcal{A} \to M(\mathcal{B})$ is a positive isometric $\mathcal{B}$-module homomorphism. Notice that π is never a $*$-homomorphism. For otherwise, $\mathcal{A}$ becomes a C^*-algebra, hence norm regular, and so never a Segal algebra by [7, Cor. 2.7], since a faithful Banach algebra is a Segal algebra if and only if it is norm irregular. Thus (ironically) a weighted C^*-algebra is never a C^*-algebra. It also follows that $\mathcal{A}$ is a weighted C^*-algebra if and only if $\mathcal{A}$ is a self-adjoint C^*-Segal algebra that admits a positive isometric $C^*(\mathcal{A})$-module homomorphism $\pi : \mathcal{A} \to M(C^*(\mathcal{A}))$. A typical example is Nachbin's weighted function algebra [7]. Let $\nu : X \to \mathbb{R}$, $\nu(t) \geq 1$, be a continuous function on a locally compact space X. Let $\mathcal{A} = C_0^{\nu}(X) = \{f \in C(X) : \nu f \in C_0(X)\}$ with the norm $\|f\|_{\mathcal{A}} = \sup\{|f(t)\nu(t)| : t \in X\}$. Then $C^*(\mathcal{A}) = C_0(X)$, $M(C^*(\mathcal{A})) = C_b(X)$, the collection of bounded continuous functions on X, and $\pi(f) = \nu f$. On the other hand, the Segal algebra $\mathcal{A} = C^p(H)$, $1 \leq p < \infty$, the Schatten–von Neumann class of operators on a Hilbert space H, with $C^*(\mathcal{A}) = K(H)$, the C^*-algebra of compact operators, fails to be a weighted C^*-algebra.

(2) A self-adjoint C^*-Segal algebra $\mathcal{A}$ satisfies $\|ab\|_{\mathcal{A}} \leq l(\|a\|_{\mathcal{A}} s_{\mathcal{A}}(b) + \|b\|_{\mathcal{A}} s_{\mathcal{A}}(a))$. Thus by [8, Theorem 14], $\mathcal{A}$ is a Banach D_1^*-subalgebra of a C^*-algebra; in fact, of the C^*-algebra $C^*(\mathcal{A})$ by Corollary 2 above. Since $\mathrm{sp}_{\mathcal{A}}(a) = \mathrm{sp}_{C^*(\mathcal{A})}(a)$, $(\mathcal{A}, \|\cdot\|_{C^*(\mathcal{A})})$ is a *Q-normed algebra* [5, §6], in the sense that the quasi-invertible elements of $\mathcal{A}$ form a $\|\cdot\|_{C^*(\mathcal{A})}$-open set. By [3, 8], $\mathcal{A}$ is a smooth subalgebra of $C^*(\mathcal{A})$ whose norm $\|\cdot\|_{\mathcal{A}}$ is a differential norm of order 1.

(3) Let $\mathcal{A}_h$ be the real vector space of all self-adjoint elements of a Banach $*$-algebra $\mathcal{A}$. Let $\mathcal{K}$ consist of all finite sums of elements of the form a^*a, $a \in \mathcal{A}$. Then the relation $a \leq b$ if $b - a \in \mathcal{K}$ $(a, b \in \mathcal{A}_h)$ makes $\mathcal{A}_h$ a partially ordered vector space. An element $u \in \mathcal{K}$ is an *order unit* [10, p. 205] of $\mathcal{A}_h$ if each $x \in \mathcal{A}_h$ satisfies $x \leq lu$ for some scalar $l > 0$. By [7], an *order unit C^*-Segal algebra* is a pair $(\mathcal{A}, u)$ where $\mathcal{A}$ is a self-adjoint C^*-Segal algebra and u is an order unit of $\mathcal{A}_h$ satisfying $\|a\|_{\mathcal{A}} = \inf\{l > 0 : -lu \leq a \leq lu\}$.

Thus $\|\cdot\|_{\mathcal{A}}$ is the Minkowski functional [10, p. 39] of the order interval $[-u, u]$ in $\mathcal{A}_h$.

Let $\mathcal{A}_+ := \mathcal{A} \cap \mathcal{C}_+$, where $\mathcal{C}_+$ is the positive cone of non-negative elements of the C^*-algebra $C^*(\mathcal{A})$. Let $a, b \in \mathcal{A}_+$. Since $\{l > 0 : -lu \leq a + b \leq lu\} \subset \{l > 0 : -lu \leq a \leq lu\}$, it follows that $\|a + b\|_{\mathcal{A}} \geq \|a\|_{\mathcal{A}}$. Hence by [10, Chapter V, Theorem 3.5, p. 215], the positive cone $\mathcal{A}_+$ in $\mathcal{A}_h$ is normal; hence, the $\|\cdot\|_{\mathcal{A}}$-topology on $\mathcal{A}$ is the order topology [10, p. 232]. Since $\mathcal{A}_h = \mathcal{A}_+ - \mathcal{A}_+$ by [7, Lemma 3.8], we deduce from [10, Theorem 5.5, p. 228] that every positive linear functional on $\mathcal{A}_h$ is continuous.

By [10, Corollary 3, p. 220], the normality of $\mathcal{A}_+$ implies that every continuous linear functional on $\mathcal{A}_h$ is a difference of two positive linear functionals. Notice that not every continuous linear functional on a C^*-Segal algebra is a difference of two representable [4, §37] positive linear functionals: otherwise, $\mathcal{A}$ being hermitian, $\mathcal{A}$ would be equivalent to a C^*-algebra, by a well known result of Grothendieck [6], stating that a hermitian Banach $*$-algebra $\mathcal{B}$ is a C^*-algebra if and only if every continuous linear functional on $\mathcal{B}$ is a difference of two representable positive linear functionals. Thus the representability condition in Grothendieck's result cannot be omitted. This also demonstrates the crucial role of the non-existence of a bounded approximate identity (bai) in a C^*-Segal algebra $\mathcal{A}$ to survive, for the presence of a bai in $\mathcal{A}$ implies that every positive linear functional on $\mathcal{A}$ is representable [9, Theorem 4.5.14, p. 214], [11, Theorem 3.2], forcing $\mathcal{A}$ to be a C^*-algebra.

Acknowledgements. The work has been supported by the UGC-SAP-DRS-III provided to the Department of Mathematics, Sardar Patel University, Vallabh Vidyanagar, 388120, India. S. J. Bhatt is grateful to UGC, India for an award of Emeritus Fellowship. The authors are grateful to the referee for very fruitful suggestions on the mathematical aspects as well as organization of the paper.

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