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RIEMANNIAN MANIFOLDS WITH HARMONIC CURVATURE

BY

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Abstract. We prove an integral inequality for compact n-dimensional manifolds with harmonic curvature tensor and positive scalar curvature, generalizing a recent result of Catino that deals with the conformally flat case, and classify those manifolds for which our inequality is an equality: they are either Einstein, $\mathbb{S}^1 \times \mathbb{S}^{n-1}$ with the product metric, or $\mathbb{S}^1 \times \mathbb{S}^{n-1}$ with a rotationally symmetric Derdziński metric.

1. Introduction. In the last years, much attention has been paid to the classification of conformally flat Riemannian manifolds under topological and/or geometrical assumptions. Tani [T] proved that any compact conformally flat n-dimensional manifold with positive Ricci curvature and constant scalar curvature is covered isometrically by $\mathbb{S}^n$ with the round metric. Carron and Herzlich [CH] classified complete conformally flat manifolds with dimension $n \geq 3$ and non-negative Ricci curvature. Gursky [G1] and Hebey and Vaugon [HV1, HV2] classified compact conformally flat manifolds satisfying an integral pinching condition. For related research and some improvements in this direction, see for instance [C, G2, HL, PRS, XZ, DG, D, Z] and references therein.

Let (M^n, g) be an n-dimensional Riemannian manifold. Denote by R and R_{ij} the scalar curvature and the Ricci curvature respectively. We let $E_{ij} = R_{ij} - \frac{R}{n} g_{ij}$ be the traceless Ricci tensor. Recently, Catino [C1] studied compact conformally flat n-dimensional manifolds with constant positive scalar curvature and satisfying an optimal integral pinching condition. He proved the following

THEOREM A. *Let (M^n, g) be a compact conformally flat n-dimensional Riemannian manifold with constant positive scalar curvature. Then*

$$\int_M (R - \sqrt{n(n-1)}\,|E|)|E|^{(n-2)/n} \leq 0,$$

and equality occurs if and only if (M^n, g) is covered isometrically by either

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$\mathbb{S}^n$ with the round metric, $\mathbb{S}^1 \times \mathbb{S}^{n-1}$ with the product metric, or $\mathbb{S}^1 \times \mathbb{S}^{n-1}$ with a rotationally symmetric Derdziński metric.

We say that (M^n, g) has *harmonic curvature* if the divergence of the Riemannian curvature vanishes (that is, $R_{ijkl,}{}^{l} = 0$). Obviously, every manifold with a parallel Ricci curvature tensor has harmonic curvature (see (2.1) below). However, there are examples of compact and noncompact Riemannian manifolds with harmonic curvature and the Ricci curvature tensor not parallel (see [D] and [G, Theorem 5.2]). That is, the conditions of harmonic curvature are weaker than those of parallel Ricci curvature tensor or conformal flatness. Therefore, it is natural to ask what is the classification for manifolds with harmonic curvature. In the following we show a new classification result for compact Riemannian manifolds with harmonic curvature.

THEOREM 1.1. *Let (M^n, g) be a compact n-dimensional Riemannian manifold, $n \geq 3$, with harmonic curvature tensor. If the scalar curvature is positive, then*

$$\int_M (R - \sqrt{n(n-1)}\,|E|)|E|^{(n-2)/n} \leq \sqrt{\frac{(n-1)(n-2)}{2}} \int_M |W|\,|E|^{(n-2)/n}, \tag{1.1}$$

and equality occurs if and only if (M^n, g) is either Einstein or isometrically covered by one of:

(1) *$\mathbb{S}^1 \times \mathbb{S}^{n-1}$ with the product metric;*
(2) *$\mathbb{S}^1 \times \mathbb{S}^{n-1}$ with a rotationally symmetric Derdziński metric.*

REMARK 1.2. The *Cotton tensor* is defined by

$$C_{ijk} = A_{kj,i} - A_{ki,j}, \tag{1.2}$$

where A_{ij} is called the *Schouten tensor* given by

$$A_{ij} = R_{ij} - \frac{R}{2(n-1)} g_{ij}.$$

The divergence of the Weyl curvature tensor is related to the Cotton tensor by (cf. [HW])

$$-\frac{n-3}{n-2} C_{ijk} = W_{ijkl,}{}^{l}. \tag{1.3}$$

By virtue of (2.1), we have $C_{ijk} = 0$ for Riemannian manifolds with harmonic curvature tensor. Hence, we conclude from (1.3) that the Weyl curvature tensor is harmonic.

If the Weyl curvature tensor is zero and the scalar curvature is constant, we easily see from (1.2) and (1.3) that the Ricci curvature is a Codazzi tensor for $n \geq 4$ and hence the curvature tensor is harmonic. Therefore, our Theorem 1.1 obviously generalizes Theorem A of Catino [C1].

2. Proof of Theorem 1.1. Throughout this paper, we use moving frames in all calculations and the Einstein convention of summing over the repeated indices. First we have the following identities, the validity of which is well-known (see for instance [MMP]):

$$R_{ij,k} - R_{ik,j} = R_{jkil,l}, \tag{2.1}$$

$$R_{ij,kl} - R_{ij,lk} = R_{lj}R_{likl} + R_{il}R_{ljkl},$$

$$R_{ij,j} = \tfrac{1}{2}R_{,i}, \tag{2.2}$$

where R_{ijkl} denotes the components of the Riemannian curvature. Since the curvature tensor is harmonic, we deduce from (2.1) that the Ricci tensor is a Codazzi tensor. Thus, from (2.2) we derive

$$R_{,i} = R_{jj,i} = R_{ij,j} = \tfrac{1}{2}R_{,i},$$

which shows that $R_{,i} = 0$ and the scalar curvature R is constant. We let $E_{ij} = R_{ij} - \frac{R}{n}g_{ij}$ be the traceless Ricci tensor. Then E_{ij} is also a Codazzi tensor, that is,

$$E_{ij,k} = E_{ik,j}. \tag{2.3}$$

The Laplacian of E_{ij} is

$$\Delta E_{ij} = E_{ij,kk} = E_{ik,jk} = -R_{ikjl}E_{kl} + R_{jk}E_{ik},$$

where we have used (2.3) and $E_{ik,k} = E_{kk,i} = 0$. In particular, the following Weitzenböck formula holds:

$$\tfrac{1}{2}\Delta|E|^2 = |\nabla E|^2 - R_{ikjl}E_{ij}E_{kl} + R_{jk}E_{ij}E_{ik}. \tag{2.4}$$

For $n \geq 3$, the Weyl curvature tensor is defined by

$$\begin{aligned} W_{ijkl} &= R_{ijkl} - \frac{1}{n-2}(R_{ik}g_{jl} - R_{il}g_{jk} + R_{jl}g_{ik} - R_{jk}g_{il}) \\ &\quad + \frac{R}{(n-1)(n-2)}(g_{ik}g_{jl} - g_{il}g_{jk}) \\ &= R_{ijkl} - \frac{1}{n-2}(E_{ik}g_{jl} - E_{il}g_{jk} + E_{jl}g_{ik} - E_{jk}g_{il}) \\ &\quad - \frac{R}{n(n-1)}(g_{ik}g_{jl} - g_{il}g_{jk}) \end{aligned} \tag{2.5}$$

and a simple computation shows

$$\begin{aligned} &- R_{ikjl}E_{ij}E_{kl} + R_{jk}E_{ij}E_{ik} \\ &\qquad = -W_{ikjl}E_{ij}E_{kl} + \frac{R}{n-1}|E|^2 + \frac{n}{n-2}E_{ij}E_{jk}E_{ki}. \end{aligned} \tag{2.6}$$

Inserting (2.6) into (2.4) gives

$$\tag{2.7} \tfrac{1}{2}\Delta|E|^2 = |\nabla E|^2 - W_{ikjl}E_{ij}E_{kl} + \frac{R}{n-1}|E|^2 + \frac{n}{n-2}E_{ij}E_{jk}E_{ki}.$$

We recall the following inequality which was first proved by Huisken (cf. [H, Lemma 3.4]):

$$\tag{2.8} |W_{ikjl}E_{ij}E_{kl}| \le \sqrt{\frac{n-2}{2(n-1)}}\,|W|\,|E|^2,$$

and

$$\tag{2.9} E_{ij}E_{jk}E_{ki} \ge -\frac{n-2}{\sqrt{n(n-1)}}\,|E|^3,$$

with equality in (2.9) at some point $p \in M$ if and only if E can be diagonalized at p and the eigenvalue multiplicity of E is at least $n-1$ (see also [C2] or [O]). As first observed by Bourguignon [B], any traceless Codazzi tensor satisfies the following sharp inequality (for a proof, see e.g. [HV2]):

$$\tag{2.10} |\nabla E|^2 \ge \frac{n+2}{n}\big|\nabla|E|\big|^2.$$

Applying (2.8)–(2.10) to (2.7) yields

$$\tag{2.11} \begin{aligned} \frac{1}{2}\Delta|E|^2 \ge{}& \frac{n+2}{n}\big|\nabla|E|\big|^2 - \sqrt{\frac{n-2}{2(n-1)}}\,|W|\,|E|^2 \\ &+ \frac{R}{n-1}|E|^2 - \sqrt{\frac{n}{n-1}}\,|E|^3. \end{aligned}$$

Let $\Omega_0 = \{p \in M : |E| \ne 0\}$. Then $\mathrm{Vol}(M \setminus \Omega_0) = 0$ from [K, Theorem 1.8] (for details see [C1, Lemma 2.2]). For any $\varepsilon > 0$, we define $\Omega_\varepsilon = \{p \in M : |E| \ge \varepsilon\}$ and

$$f_\varepsilon(p) = \begin{cases} |E|(p) & \text{if } p \in \Omega_\varepsilon, \\ \varepsilon & \text{if } p \in M \setminus \Omega_\varepsilon. \end{cases}$$

We assume that ε is a regular value of $|E|$. Integration by parts gives

$$\tag{2.12} \begin{aligned} &\int_M \Big(-\frac{1}{2}\Delta|E|^2 + \frac{n+2}{n}\big|\nabla|E|\big|^2\Big) f_\varepsilon^{-(n+2)/n} \\ &= -\frac{n+2}{n}\int_M \langle \nabla|E|, \nabla f_\varepsilon\rangle |E| f_\varepsilon^{-(n+2)/n-1} + \frac{n+2}{n}\int_M \big|\nabla|E|\big|^2 f_\varepsilon^{-(n+2)/n} = 0, \end{aligned}$$

where in the last equality we have used $f_\varepsilon = |E|$ on Ω_ε and $\nabla f_\varepsilon = 0$ on $M \setminus \Omega_\varepsilon$. Multiplying both sides of (2.11) by $f_\varepsilon^{-(n+2)/n}$ and applying (2.12), we get

$$
\begin{aligned}
(2.13)\qquad 0 \geq & \int_M \left(-\tfrac{1}{2}\Delta|E|^2 + \frac{n+2}{n}\big|\nabla|E|\big|^2\right) f_\varepsilon^{-(n+2)/n} \\
& + \int_M \left(-\sqrt{\frac{n-2}{2(n-1)}}\,|W|\,|E|^2 + \frac{R}{n-1}|E|^2 - \sqrt{\frac{n}{n-1}}\,|E|^3\right) f_\varepsilon^{-(n+2)/n} \\
= & \int_M \left(-\sqrt{\frac{n-2}{2(n-1)}}\,|W|\,|E|^2 + \frac{R}{n-1}|E|^2 - \sqrt{\frac{n}{n-1}}\,|E|^3\right) f_\varepsilon^{-(n+2)/n} \\
= & \int_M |E|^{(n-2)/n}\left(-\sqrt{\frac{n-2}{2(n-1)}}\,|W| + \frac{R}{n-1} - \sqrt{\frac{n}{n-1}}\,|E|\right)|E|^{(n+2)/n} f_\varepsilon^{-(n+2)/n}.
\end{aligned}
$$

Letting $\varepsilon \to 0$ in (2.13), we see that $|E|^{(n+2)/n} f_\varepsilon^{-(n+2)/n} \to 1$ a.e. on M. We have thus proved (1.1).

Let us now assume the equality case in (1.1), that is,

$$\int_M \left(-\sqrt{\frac{n-2}{2(n-1)}}\,|W| + \frac{R}{n-1} - \sqrt{\frac{n}{n-1}}\,|E|\right)|E|^{(n-2)/n} = 0.$$

The three inequalities (2.8)–(2.10) that led to (2.11) and (2.13) must now all be equalities. Hence, as stated in the lines following (2.9), E has, at each point p, an eigenvalue of multiplicity $n-1$ or n. Writing $E_{ij} = ag_{ij} + bv_iv_j$ at p, with some scalars a, b and a vector v, we see that the left-hand side of (2.8) is zero at every point p. As (2.8) holds with equality and, according to [DG], g is real-analytic, (M, g) must be conformally flat or Einstein. Our claim about the equality case now follows from Catino's Theorem A. This completes the proof of Theorem 1.1.

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