STUDIA MATHEMATICA 234 (1) (2016)

A Katznelson–Tzafriri type theorem for Cesàro bounded operators

by

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Abstract. We extend the well-known Katznelson–Tzafriri theorem, originally stated for power-bounded operators, to the case of Cesàro bounded operators of any order $\alpha > 0$. For this purpose, we use a functional calculus between a new class of fractional Wiener algebras and the algebra of bounded linear operators, defined for operators with the corresponding Cesàro boundedness. Finally, we apply the main theorem to get ergodicity results for the Cesàro means of bounded operators.

1. Introduction. Let $A(\mathbb{T})$ be the convolution Wiener algebra formed by all continuous periodic functions $\mathfrak{f}(t) = \sum_{n=-\infty}^{\infty} a(n)e^{int}$, for $t \in [0, 2\pi)$, with the norm $\|\mathfrak{f}\|_{A(\mathbb{T})} := \sum_{n=-\infty}^{\infty} |a(n)|$. This algebra is regular. We denote by $A_+(\mathbb{T})$ the convolution closed subalgebra of $A(\mathbb{T})$ where the functions satisfy $a(n) = 0$ for $n < 0$. Note that $A(\mathbb{T})$ and $\ell^1(\mathbb{Z})$ are isometrically isomorphic. The same holds for $A_+(\mathbb{T})$ and $\ell^1(\mathbb{N}_0)$, where $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. In the above, the sequence $(a(n))_{n\in\mathbb{Z}}$ corresponds to the Fourier coefficients of $\mathfrak{f}$, that is

$$a(n) := \widehat{\mathfrak{f}}(n) = \frac{1}{2\pi}\int_0^{2\pi} \mathfrak{f}(t)e^{-int}\,dt.$$

Let E be a closed subset of $\mathbb{T}$ and $\mathfrak{f} \in A(\mathbb{T})$. We recall that $\mathfrak{f}$ is *of spectral synthesis with respect to* E if for every $\varepsilon > 0$ there exists $\mathfrak{f}_\varepsilon \in A(\mathbb{T})$ such that $\|\mathfrak{f} - \mathfrak{f}_\varepsilon\|_{A(\mathbb{T})} < \varepsilon$ with $\mathfrak{f}_\varepsilon = 0$ in a neighborhood of E. The above definition is valid in any regular Banach algebra. For more details see [K, Chapter VIII, Section 7]. Since $\sup_{t\in[0,2\pi)} |\mathfrak{f}(t)| \le \|\mathfrak{f}\|_{A(\mathbb{T})}$, if $\mathfrak{f}$ is of spectral synthesis with respect to E, then $\mathfrak{f}$ vanishes on E.

2010 *Mathematics Subject Classification*: 47A35, 47A10, 43A45, 26A33.

Key words and phrases: Cesàro bounded, functional calculus, convolution algebras, spectral synthesis, Weyl differences.
Received 30 October 2015; revised 23 May 2016.
Published online 4 July 2016.

DOI: 10.4064/sm8436-5-2016

Let X be a complex Banach space and $\mathcal{B}(X)$ the Banach algebra of all bounded linear operators on X. An operator $T \in \mathcal{B}(X)$ is *power-bounded* if $\sup_{n\geq 0}\|T^n\| < \infty$. In 1986, Y. Katznelson and L. Tzafriri proved that if T is a power-bounded operator on X and $\mathfrak{f} \in A_+(\mathbb{T})$ is of spectral synthesis in $A(\mathbb{T})$ with respect to $\sigma(T)\cap\mathbb{T}$, then

$$\lim_{n\to\infty}\|T^n\theta(\widehat{\mathfrak{f}})\| = 0,$$

where $\sigma(T)$ denotes the spectrum of T and $\theta : \ell^1(\mathbb{N}_0) \to \mathcal{B}(X)$ is the functional calculus given by

$$\theta(f) := \sum_{j=0}^{\infty} f(j)T^j, \qquad x \in X,\ f \in \ell^1(\mathbb{N}_0)$$

(see [KT, Theorem 5]). Moreover, for $T \in \mathcal{B}(X)$ power-bounded, we have $\lim_{n\to\infty}\|T^n - T^{n+1}\| = 0$ if and only if $\sigma(T)\cap\mathbb{T} \subseteq \{1\}$ (see [KT, Theorem 1]).

Léka [L1] proved that for T power-bounded in a Hilbert space H, the result of [KT, Theorem 5] still holds if $\mathfrak{f} \in A_+(\mathbb{T})$ just vanishes on the peripheral spectrum. For contractions on H this had been proved in [ESZ2]. When the Fourier coefficients of $\mathfrak{f}$ satisfy $\sum_{j\geq 0} j|\widehat{\mathfrak{f}}(j)| < \infty$, the same holds in any Banach space [AOR]. If T is $(C,1)$-bounded and $\sigma(T)\cap\mathbb{T} = \{1\}$, but T is not power-bounded, then $\|T^n - T^{n+1}\|$ need not converge to zero. The first counter-examples were given in [TZ]. There is a counter-example in a Hilbert space with $\sigma(T) = \{1\}$ in [L2].

A similar result for C_0-semigroups was proved simultaneously in [ESZ1] and [V2]. The result states that if $(T(t))_{t\geq 0} \subset \mathcal{B}(X)$ is a bounded C_0-semigroup generated by A, and $\mathfrak{f} \in L^1(\mathbb{R}_+)$ is of spectral synthesis in $L^1(\mathbb{R})$ with respect to $i\sigma(A)\cap\mathbb{R}$, then

$$\lim_{t\to\infty}\|T(t)\Theta(\mathfrak{f})\| = 0,$$

where $\Theta : L^1(\mathbb{R}_+) \to \mathcal{B}(X)$ is the Hille functional calculus given by

$$\Theta(\mathfrak{f})x := \int_0^{\infty} \mathfrak{f}(t)T(t)x, \qquad x \in X,\ \mathfrak{f} \in L^1(\mathbb{R}_+).$$

In [CT, Theorem 5.5], there is a nice proof of this result, which has inspired the proof of the main theorem of this paper (Theorem 3.4).

In [GMM], the authors give a similar theorem for α-times integrated semigroups: Let $\alpha > 0$, let $(T_\alpha(t))_{t\geq 0} \subset \mathcal{B}(X)$ be an α-times integrated semigroup generated by A such that $\sup_{t>0} t^{-\alpha}\|T_\alpha(t)\| < \infty$, and let $\mathfrak{f} \in \mathcal{T}_+^{(\alpha)}(t^\alpha)$ be of spectral synthesis in $\mathcal{T}^{(\alpha)}(|t|^\alpha)$ (both are Sobolev subalgebras of $L^1(\mathbb{R}_+)$ and $L^1(\mathbb{R})$ respectively which have been studied in detail in [GM]) with respect to $i\sigma(A)\cap\mathbb{R}$. Then

$$\lim_{t\to\infty} t^{-\alpha}\|T_\alpha(t)\Theta_\alpha(\mathfrak{f})\| = 0,$$

where $\Theta_\alpha : \mathcal{T}_+^{(\alpha)}(t^\alpha) \to \mathcal{B}(X)$ is the bounded algebra homomorphism defined by

$$\Theta_\alpha(\mathfrak{f})x := \int_0^\infty \mathcal{W}_+^\alpha \mathfrak{f}(t) T_\alpha(t)x, \qquad x \in X,\ \mathfrak{f} \in \mathcal{T}_+^{(\alpha)}(t^\alpha),$$

and $\mathcal{W}_+^\alpha \mathfrak{f}$ is the Weyl fractional derivative of order α of $\mathfrak{f}$.

Let $\alpha > 0$ and $T \in \mathcal{B}(X)$. We denote $\mathcal{T}(n) := T^n$ for $n \in \mathbb{N}_0$. The *Cesàro sum* of order $\alpha > 0$ of T is the family of operators $(\Delta^{-\alpha}\mathcal{T}(n))_{n\in\mathbb{N}_0} \subset \mathcal{B}(X)$ defined by

$$\Delta^{-\alpha}\mathcal{T}(n)x := \sum_{j=0}^{n} k^\alpha(n-j)T^j x, \qquad x \in X,\ n \in \mathbb{N}_0,$$

and the *Cesàro mean* of order $\alpha > 0$ of T is the family of operators $(M_T^\alpha(n))_{n\in\mathbb{N}_0}$ given by

$$M_T^\alpha(n)x := \frac{1}{k^{\alpha+1}(n)}\Delta^{-\alpha}\mathcal{T}(n)x, \qquad x \in X,\ n \in \mathbb{N}_0,$$

where

$$k^\alpha(n) := \frac{\Gamma(\alpha+n)}{\Gamma(\alpha)\Gamma(n+1)} = \binom{n+\alpha-1}{\alpha-1}, \qquad n \in \mathbb{N}_0,$$

is the *Cesàro kernel* of order α. When the Cesàro mean of order α of T is uniformly bounded, that is,

$$\sup_n \|M_T^\alpha(n)\| < \infty,$$

the operator T is said to be *Cesàro bounded* of order α or simply (C,α)-*bounded*. We extend the Cesàro kernel for $\alpha = 0$ using that $k^0(n) := \lim_{\alpha\to 0^+} k^\alpha(n) = \delta_{n,0}$ for $n \in \mathbb{N}_0$, where $\delta_{n,j}$ is the Kronecker delta. Then $(C,0)$-boundedness is equivalent to power-boundedness, and for $\alpha = 1$ the operator T is said to be *Cesàro mean bounded* (or simply Cesàro bounded). From [Z, formulas (1.10) and (1.17), p. 77] it can be proved that for $\beta > \alpha \geq 0$ we have $\sup_n\|M_T^\beta(n)\| \leq \sup_n\|M_T^\alpha(n)\| \leq \sup_n\|T^n\|$; in particular if T is a power-bounded operator then T is (C,α)-bounded for any $\alpha > 0$. The converse is not true in general (see [LSS, Propositions 4.3 and 4.4]). The Assani matrix

$$T = \begin{pmatrix} -1 & 2 \\ 0 & -1 \end{pmatrix}$$

is $(C,1)$-bounded but it is not power-bounded since

$$T^n = \begin{pmatrix} (-1)^n & (-1)^{n+1}2n \\ 0 & (-1)^n \end{pmatrix}, \qquad n \in \mathbb{N}_0$$

(see [Em, Section 4.7] and [SZ, Remark 2.3]).

LEMMA 1.1. *Let $\alpha > 0$. If T is (C,α)-bounded, then it has spectral radius $r(T) \leq 1$.*

Proof. This is a straightforward consequence of [SZ, Lemma 2.1] since T is $(C,[\alpha]+1)$-bounded. ∎

The study of mean ergodic theorems for operators which are not power-bounded started with [H]. There are many results concerning ergodicity [D, ED, Em, SZ, TZ, Y] and growth [LSS, S] of Cesàro sums and of Cesàro means of order α.

In a recent paper [AL$^+$1], it is proved that the algebraic structure of the Cesàro sum of order α of a bounded operator is similar to the algebraic structure of α-times integrated semigroups [AL$^+$1, Theorem 3.3]. In [AL$^+$1, Section 2], certain weighted convolution algebras, denoted by $\tau^\alpha(k^{\alpha+1})$ and contained in $\ell^1(\mathbb{N}_0)$, are constructed for any $\alpha > 0$. Then (C,α)-boundedness is characterized by the existence of an algebra homomorphism from $\tau^\alpha(k^{\alpha+1})$ into $\mathcal{B}(X)$ [AL$^+$1, Corollary 3.7].

The outline of this paper is as follows: In Section 2 we use Weyl fractional differences to construct Banach algebras $\tau^\alpha(|n|^\alpha)$ contained in $\ell^1(\mathbb{Z})$ (Theorem 2.11). The techniques used are similar to those in [AL$^+$1, Section 2], and we follow the same steps as in the continuous case [GM], adapting the proofs. In Section 3 we define fractional Wiener algebras of periodic continuous functions, $A^\alpha_+(\mathbb{T})$ and $A^\alpha(\mathbb{T})$, which are isometrically isomorphic via the Fourier transform to $\tau^\alpha(k^{\alpha+1})$ and $\tau^\alpha(|n|^\alpha)$, respectively. These algebras allow us to state the main theorem of this paper (Theorem 3.4): Let $\alpha > 0$, let $T \in \mathcal{B}(X)$ be a (C,α)-bounded operator and let $\mathfrak{f} \in A^\alpha_+(\mathbb{T})$ be of spectral synthesis in $A^\alpha(\mathbb{T})$ with respect to $\sigma(T)\cap\mathbb{T}$. Then

$$\lim_{n\to\infty}\|M^\alpha_T(n)\theta_\alpha(\widehat{\mathfrak{f}})\| = 0,$$

where $\theta_\alpha : \tau^\alpha(k^{\alpha+1}) \to \mathcal{B}(X)$ is the bounded algebra homomorphism defined by

$$\theta_\alpha(f)x := \sum_{n=0}^{\infty} W^\alpha_+ f(n)\Delta^{-\alpha}\mathcal{T}(n)x, \qquad x \in X,\ f \in \tau^\alpha(k^{\alpha+1}),$$

and $W^\alpha_+ f$ is the Weyl fractional difference of order α of f (see [AL$^+$1, Theorem 3.5]). Finally, in Section 4 we give two applications of ergodicity for (C,α)-bounded operators (Theorems 4.1 and 4.3).

Notation. We denote by $\ell^1(\mathbb{Z})$ the set of complex sequences $f : \mathbb{Z} \to \mathbb{C}$ such that $\sum_{n=-\infty}^{\infty}|f(n)| < \infty$, and by $c_{0,0}(\mathbb{Z})$ the set of complex sequences with finite support. It is well known that $\ell^1(\mathbb{Z})$ is a Banach algebra with the usual (commutative and associative) convolution product

$$(f * g)(n) = \sum_{j=-\infty}^{\infty} f(n-j)g(j), \qquad n \in \mathbb{Z}.$$

The above is valid for sequences defined in $\mathbb{N}_0$ instead $\mathbb{Z}$, and the corresponding convolution product is

$$(f * g)(n) = \sum_{j=0}^{n} f(n-j)g(j), \qquad n \in \mathbb{N}_0.$$

Moreover, if f is a sequence defined in $\mathbb{N}_0$, we can see it as a sequence defined in $\mathbb{Z}$ with $f(n) = 0$ for all $n < 0$.

Throughout the paper, we use the variable constant convention, in which C denotes a constant which may not be the same from line to line. The constant is frequently written with subscripts to emphasize that it depends on some parameters.

2. Fractional differences and convolution Banach algebras. For $\alpha > 0$, the Cesàro kernel of order α, $(k^\alpha(n))_{n\in\mathbb{N}_0}$, plays a key role in the main results of this paper. Many of its properties can be found in [Z, Vol. I, p. 77]. We quote some of them: the semigroup property, $k^\alpha * k^\beta = k^{\alpha+\beta}$ for $\alpha, \beta > 0$; for $\alpha > 0$,

$$k^\alpha(n) = \frac{n^{\alpha-1}}{\Gamma(\alpha)}(1 + O(1/n)), \qquad n \in \mathbb{N} \tag{2.1}$$

(see [Z, Vol. I, (1.18)]); k^α is increasing (as a function of n) for $\alpha > 1$, decreasing for $0 < \alpha < 1$ and $k^1(n) = 1$ for $n \in \mathbb{N}$ [Z, Chapter III, Theorem 1.17]; $k^\alpha(n) \le k^\beta(n)$ for $\beta \ge \alpha > 0$ and $n \in \mathbb{N}_0$; finally, for $\alpha > 0$, there exists $C_\alpha > 0$ such that

$$k^\alpha(2n) \le C_\alpha k^\alpha(n), \qquad n \in \mathbb{N}_0 \tag{2.2}$$

(see [AL$^+$1, Lemma 2.1]). The Cesàro kernel k^α has been introduced in order to study fractional difference problems, see [AL], [AL$^+$2] and [Li].

As mentioned in the introduction, for each $\alpha > 0$ there exists a convolution Banach algebra $\tau^\alpha(k^{\alpha+1})$ contained in $\ell^1(\mathbb{N}_0)$ with continuous inclusions

$$\tau^\beta(k^{\beta+1}) \hookrightarrow \tau^\alpha(k^{\alpha+1}) \hookrightarrow \ell^1(\mathbb{N}_0), \qquad \beta > \alpha > 0,$$

and $\tau^0(k^1) \equiv \ell^1(\mathbb{N}_0)$ (see [AL$^+$1]). Now we are interested in obtaining some similar spaces contained in $\ell^1(\mathbb{Z})$. For convenience, we denote $\tau^\alpha(n^\alpha) := \tau^\alpha(k^{\alpha+1})$ for $\alpha > 0$.

In the following, let $(f(n))_{n\in\mathbb{Z}}$ be a sequence of complex numbers. Some results in this section can be extended immediately to vector-valued sequences, that is, when f takes values in a complex Banach space X. For $n \in \mathbb{Z}$ we consider the usual forward and backward difference operators,

$\Delta f(n) = f(n+1) - f(n)$ and $\nabla f(n) = f(n) - f(n-1)$, and the natural powers

$$\Delta^m f(n) = \sum_{j=0}^{m} (-1)^{m-j} \binom{m}{j} f(n+j), \quad n \in \mathbb{Z},$$
$$\nabla^m f(n) = \sum_{j=0}^{m} (-1)^{j} \binom{m}{j} f(n-j), \quad n \in \mathbb{Z},$$

for $m \in \mathbb{N}_0$ (see for example [E, (2.1.1)] for Δ^m; for ∇^m it is a simple check using Δ^m). Observe that $\Delta^m, \nabla^m : c_{0,0}(\mathbb{Z}) \to c_{0,0}(\mathbb{Z})$ for $m \in \mathbb{N}_0$.

For convenience, following the notation of [AL$^+$1], we write $W_+ = -\Delta$ and $W_- = \nabla$, $W_+^m = (-1)^m \Delta^m$ and $W_-^m = \nabla^m$ for $m \in \mathbb{N}$. The inverse operators of W_+ and W_-, and their powers in $c_{0,0}(\mathbb{Z})$, are given by

$$W_+^{-m} f(n) = \sum_{j=n}^{\infty} k^m(j-n) f(j), \quad n \in \mathbb{Z},$$
$$W_-^{-m} f(n) = \sum_{j=-\infty}^{n} k^m(n-j) f(j), \quad n \in \mathbb{Z},$$

for $m \in \mathbb{N}$ (see for example [GW, p. 307] in the case of W_+ for sequences defined in $\mathbb{N}_0$).

Definition 2.1. Let $(f(n))_{n\in\mathbb{Z}}$ be a complex sequence and $\alpha > 0$. The *Weyl sums* of order α of f are given by

$$W_+^{-\alpha} f(n) := \sum_{j=n}^{\infty} k^\alpha(j-n) f(j), \quad n \in \mathbb{Z},$$
$$W_-^{-\alpha} f(n) := \sum_{j=-\infty}^{n} k^\alpha(n-j) f(j), \quad n \in \mathbb{Z},$$

whenever the sums make sense, and the *Weyl differences* are

$$W_+^{\alpha} f(n) := W_+^m W_+^{-(m-\alpha)} f(n) = (-1)^m \Delta^m W_+^{-(m-\alpha)} f(n), \quad n \in \mathbb{Z},$$
$$W_-^{\alpha} f(n) := W_-^m W_-^{-(m-\alpha)} f(n) = \nabla^m W_-^{-(m-\alpha)} f(n), \quad n \in \mathbb{Z},$$

for $m = [\alpha] + 1$, whenever the right hand sides converge. In particular $W_+^\alpha, W_-^\alpha : c_{0,0}(\mathbb{Z}) \to c_{0,0}(\mathbb{Z})$ for $\alpha \in \mathbb{R}$.

The above definitions have been considered in more restrictive contexts in some papers [AL$^+$1, GW]. The natural properties that are satisfied in those contexts are generalized below, and the proof is similar to the proof of [AL$^+$1, Proposition 2.4].

Proposition 2.2. *Let $f \in c_{0,0}(\mathbb{Z})$ and $\alpha, \beta \in \mathbb{R}$. Then:*

(i) $W_+^{\alpha+\beta} f = W_+^\alpha W_+^\beta f$.

(ii) $W_-^{\alpha+\beta} f = W_-^\alpha W_-^\beta f$.
(iii) $\lim_{\alpha\to 0} W_+^\alpha f = \lim_{\alpha\to 0} W_-^\alpha f = f$.

Note that the Cesàro kernel can be considered in a more general setting. For $\alpha \in \mathbb{R}$,

$$k^\alpha(n) = \frac{\alpha(\alpha+1)\cdots(\alpha+n-1)}{n!} \quad \text{for } n \in \mathbb{N}, \qquad k^\alpha(0) = 1.$$

Also if $\alpha < 0$ with $\alpha \neq \{0, -1, -2, \ldots\}$ we can write $k^\alpha(n) = (-1)^n \binom{-\alpha}{n}$ and (2.1) is valid too. It is known that

$$\sum_{n=0}^{\infty} k^\alpha(n) z^n = (1-z)^{-\alpha}, \qquad |z| < 1.$$

Then we can deduce that $k^\alpha * k^\beta = k^{\beta+\alpha}$ for $\alpha, \beta \in \mathbb{R}$. This allows us to represent the Weyl differences in the following way.

Proposition 2.3. *Let $(f(n))_{n\in\mathbb{Z}}$ be a complex sequence and $\alpha \in \mathbb{R}$. Then*

$$W_+^\alpha f(n) = \sum_{j=n}^{\infty} k^{-\alpha}(j-n) f(j), \quad W_-^\alpha f(n) = \sum_{j=-\infty}^{n} k^{-\alpha}(n-j) f(j), \qquad n \in \mathbb{Z},$$

whenever the Weyl differences of order α of f make sense.

Proof. We only prove the result for W_+; the proof for W_- is analogous. If $\alpha \in \mathbb{N}$, then

$$W_+^\alpha f(n) = \sum_{j=0}^{\alpha} (-1)^j \binom{\alpha}{j} f(n+j) = \sum_{j=0}^{\infty} k^{-\alpha}(j) f(n+j) = \sum_{j=n}^{\infty} k^{-\alpha}(n-j) f(j).$$

Now let $m-1 < \alpha < m$ with $m \in \mathbb{N}$. Then

$$\begin{aligned}
W_+^\alpha f(n) &= W_+^m W_+^{-(m-\alpha)} f(n) = \sum_{j=0}^{m} (-1)^j \binom{m}{j} \sum_{l=n+j}^{\infty} k^{m-\alpha}(l-n-j) f(l) \\
&= \sum_{l=n}^{n+m} f(l) \sum_{j=0}^{l-n} (-1)^j \binom{m}{j} k^{m-\alpha}(l-n-j) \\
&\quad + \sum_{l=n+m+1}^{\infty} f(l) \sum_{j=0}^{m} (-1)^j \binom{m}{j} k^{m-\alpha}(l-n-j) \\
&= \sum_{l=n}^{n+m} f(l)(k^{-m} * k^{m-\alpha})(l-n) + \sum_{l=n+m+1}^{\infty} f(l)(k^{-m} * k^{m-\alpha})(l-n) \\
&= \sum_{l=n}^{\infty} k^{-\alpha}(l-n) f(l). \ \blacksquare
\end{aligned}$$

Remark 2.4. The operators W_+^α and $W_+^{-\alpha}$ for $\alpha \in (0,1)$ are tightly connected to the definition of $(I-T)^\alpha$ for any contraction T in a Banach space, given in [DL]. If we denote by S the shift operator on $\ell^1(\mathbb{Z})$, that is, $Sf(n) = f(n+1)$ for $n \in \mathbb{Z}$, then $W_+^\alpha = (I-S)^\alpha$ (compatible with $I-S = -\Delta$), well-defined on the whole space $\ell^1(\mathbb{Z})$, and $W_+^{-\alpha} = [(I-S)^\alpha]^{-1}$, defined on the range of $(I-S)^\alpha$. Also these identities are valid for $\alpha > 0$. The author is currently studying these fractional powers of the operator $I-S$ as fractional powers of the generator of a uniformly bounded C_0-semigroup on a Banach space; the results will appear in a forthcoming paper.

Remark 2.5. Note that

$$W_+^m f(n) = \sum_{j=0}^m (-1)^j \binom{m}{j} f(n+j),$$
$$W_-^m f(n) = \sum_{j=0}^m (-1)^j \binom{m}{j} f(n-j)$$

for $m \in \mathbb{N}$ and $n \in \mathbb{Z}$, therefore in general $W_+^\alpha f(n) \neq W_-^\alpha f(n)$ for $\alpha > 0$ and $n \in \mathbb{Z}$ (it suffices take $0 < \alpha < 1$ and the sequence given by $f(n) = 1$ for $n = 0, 1$, and $f(n) = 0$ otherwise). However, we have the following link between W_+^α and W_-^α; its proof is left to the reader.

Proposition 2.6. *Let α be a positive real number and let $f \in c_{0,0}(\mathbb{Z})$ be such that $f(n) = f(-n)$ for all $n \in \mathbb{Z}$. Then*

$$W_+^\alpha f(n) = W_-^\alpha f(-n), \qquad n \in \mathbb{Z}.$$

In particular $W_+^\alpha f(0) = W_-^\alpha f(0)$.

Let $(f(n))_{n\in\mathbb{Z}}$ be a complex sequence. We denote by $(f_+(n))_{n\in\mathbb{Z}}$, $(f_-(n))_{n\in\mathbb{Z}}$ and $(\tilde{f}(n))_{n\in\mathbb{Z}}$ the sequences given by

$$f_+(n) := \begin{cases} f(n), & n \geq 0, \\ 0, & n < 0, \end{cases} \qquad f_-(n) := \begin{cases} 0, & n \geq 0, \\ f(n), & n < 0, \end{cases}$$

and $\tilde{f}(n) = f(-n)$ for $n \in \mathbb{Z}$. It is a simple check that $(W_+^{-\alpha} f)\tilde{}(n) = W_-^{-\alpha} \tilde{f}(n)$, $n \in \mathbb{Z}$, for $\alpha > 0$ and $f \in c_{00}(\mathbb{Z})$. Then the following result is a straight consequence.

Proposition 2.7. *Let $f \in c_{0,0}(\mathbb{Z})$ and $\alpha > 0$. Then:*

(i) $W_+^\alpha f_+(n) = W_+^\alpha f(n)$, $n \geq 0$.
(ii) $W_-^\alpha f_-(n) = W_-^\alpha f(n)$, $n < 0$.
(iii) $(W_+^\alpha f)\tilde{}(n) = W_-^\alpha \tilde{f}(n)$, $n \in \mathbb{Z}$.

DEFINITION 2.8. Let $\alpha > 0$. We denote by $W^\alpha : c_{0,0}(\mathbb{Z}) \to c_{0,0}(\mathbb{Z})$ the operator given by

$$W^\alpha f(n) := \begin{cases} W_+^\alpha f(n), & n \geq 0, \\ W_-^\alpha f(n), & n < 0, \end{cases}$$

for $f \in c_{0,0}(\mathbb{Z})$.

We are interested in the relation between the convolution product and the fractional Weyl differences. If $f, g \in c_{0,0}(\mathbb{Z})$ then it is known that $f * g \in c_{0,0}(\mathbb{Z})$. In [AL+1, Lemma 2.7], the following equality is proved:

$$\begin{aligned} W_+^\alpha(f_+ * g_+)(n) &= \sum_{j=0}^{n} W_+^\alpha g(j) \sum_{p=n-j}^{n} k^\alpha(p-n+j) W_+^\alpha f(p) \\ &\quad - \sum_{j=n+1}^{\infty} W_+^\alpha g(j) \sum_{p=n+1}^{\infty} k^\alpha(p-n+j) W_+^\alpha f(p), \qquad n \geq 0, \end{aligned} \tag{2.3}$$

for $f, g \in c_{0,0}(\mathbb{Z})$ and $\alpha \geq 0$. The rest of this section is inspired by the continuous case (see [GM]).

LEMMA 2.9. *Let $f, g \in c_{0,0}(\mathbb{Z})$ and $\alpha > 0$. Then:*

(i) $W_+^\alpha(f_+ * g_-)(n) = (W_+^\alpha f_+ * g_-)(n)$, $n \geq 0$.
(ii) $W_-^\alpha(f_- * g_+)(n) = (W_-^\alpha f_- * g_+)(n)$, $n < 0$.

Proof. (i) Let $n \geq 0$. Then

$$\begin{aligned} (f_+ * g_-)(n) &= \sum_{j=n+1}^{\infty} W_+^{-\alpha} W_+^\alpha f_+(j) g_-(n-j) \\ &= \sum_{j=n+1}^{\infty} W_+^\alpha f_+(j) \sum_{i=n+1}^{j} k^\alpha(j-i) g_-(n-i) \\ &= \sum_{j=n+1}^{\infty} W_+^\alpha f_+(j) \sum_{u=n}^{j-1} k^\alpha(u-n) g_-(u-j) \\ &= \sum_{u=n}^{\infty} k^\alpha(u-n) \sum_{j=u+1}^{\infty} W_+^\alpha f_+(j) g_-(u-j) = W_+^{-\alpha}(W_+^\alpha f_+ * g_-)(n), \end{aligned}$$

where we have used Fubini's Theorem and a change of variables, and then $W_+^\alpha(f_+ * g_-)(n) = W_+^\alpha f_+ * g_-(n)$.

(ii) Using Proposition 2.7 and part (i) we see for $n < 0$ that

$$\begin{aligned} W_-^\alpha(f_- * g_+)(n) &= W_+^\alpha(f_- * g_+)\tilde{\ }(-n) = W_+^\alpha((f_-)\tilde{\ } * (g_+)\tilde{\ })(-n) \\ &= W_+^\alpha(\tilde{f}_+ * \tilde{g}_-)(-n) = (W_+^\alpha \tilde{f}_+ * \tilde{g}_-)(-n) \\ &= ((W_+^\alpha \tilde{f}_+)\tilde{\ } * (\tilde{g}_-)\tilde{\ })(n) = (W_-^\alpha f_- * g_+)(n). \quad \blacksquare \end{aligned}$$

LEMMA 2.10. *Let* $f, g \in c_{0,0}(\mathbb{Z})$ *and* $\alpha > 0$. *Then*

$$W^\alpha(f * g)(n) = (W_+^\alpha f_+ * g_-)(n) + W_+^\alpha(f_+ * g_+)(n) + (f_- * W_+^\alpha g_+)(n),$$

for $n \geq 0$, *and*

$$W^\alpha(f * g)(n) = (W_-^\alpha f_- * g_+)(n) + W_-^\alpha(f_- * g_-)(n) + (f_+ * W_-^\alpha g_-)(n),$$

for $n < 0$.

Proof. It is a simple check that

$$\begin{aligned}(f * g)(n) &= (f_+ * g_-)(n) + (f_+ * g_+)(n) + (f_- * g_+)(n), \quad && n \geq 0,\\ (f * g)(n) &= (f_- * g_+)(n) + (f_- * g_-)(n) + (f_+ * g_-)(n), \quad && n < 0.\end{aligned}$$

Then by Lemma 2.9 we get the result. ∎

For $\alpha \geq 0$ we define $q_\alpha : c_{0,0}(\mathbb{Z}) \to [0, \infty)$ by

$$q_\alpha(f) := \sum_{n=-\infty}^{\infty} k^{\alpha+1}(|n|)|W^\alpha f(n)|, \qquad f \in c_{0,0}(\mathbb{Z}).$$

Observe that for $\alpha = 0$ this is the usual norm in $\ell^1(\mathbb{Z})$.

The following theorem is the main result of this section; it extends [AL$^+$1, Theorem 2.11] and [GW, Theorem 4.5].

THEOREM 2.11. *Let* $\alpha > 0$. *The map* q_α *defines a norm in* $c_{0,0}(\mathbb{Z})$ *and*

$$q_\alpha(f * g) \leq C_\alpha q_\alpha(f) q_\alpha(g), \qquad f, g \in c_{0,0}(\mathbb{Z}),$$

with $C_\alpha > 0$ *independent of* f *and* g. *Denote by* $\tau^\alpha(|n|^\alpha)$ *the Banach algebra obtained as the space of complex sequences* f *such that* $\lim_{n\to\infty} f(n) = 0$ *and the norm* $q_\alpha(f)$ *converges. Furthermore, these spaces are continuously embedded in each other in the following way:*

$$\tau^\beta(|n|^\beta) \hookrightarrow \tau^\alpha(|n|^\alpha) \hookrightarrow \ell^1(\mathbb{Z})$$

for $\beta > \alpha > 0$, *and* $\lim_{\alpha\to 0^+} q_\alpha(f) = \|f\|_1$ *for* $f \in c_{0,0}(\mathbb{Z})$.

Proof. It is clear that q_α is a norm in $c_{0,0}(\mathbb{Z})$. We write

$$\begin{aligned} q_\alpha(f) &= \sum_{n=-\infty}^{-1} k^{\alpha+1}(-n)|W_-^\alpha f_-(n)| + \sum_{n=0}^{\infty} k^{\alpha+1}(n)|W_+^\alpha f_+(n)| \\ &:= q_\alpha^-(f_-) + q_\alpha^+(f_+). \end{aligned}$$

We have to show that q_α defines a Banach algebra. First we prove that

$$q_\alpha^+((f * g)_+) \leq C_\alpha q_\alpha(f) q_\alpha(g).$$

By Lemma 2.10,

$$W^\alpha(f * g)(n) = (W_+^\alpha f_+ * g_-)(n) + W_+^\alpha(f_+ * g_+)(n) + (f_- * W_+^\alpha g_+)(n)$$

for $n \geq 0$. Now we work with each summand separately. For the first,

$$\begin{aligned}\sum_{n=0}^{\infty} k^{\alpha+1}(n)|(W_+^\alpha f_+ * g_-)(n)| &\leq \sum_{n=0}^{\infty} k^{\alpha+1}(n) \sum_{j=n+1}^{\infty} |W_+^\alpha f_+(j)|\,|g_-(n-j)| \\ &= \sum_{j=1}^{\infty} |W_+^\alpha f_+(j)| \sum_{n=0}^{j-1} k^{\alpha+1}(n)|g_-(n-j)| \\ &\leq \sum_{j=1}^{\infty} |W_+^\alpha f_+(j)| k^{\alpha+1}(j) \sum_{u=-j}^{-1} |g_-(u)| \\ &\leq q_\alpha^+(f_+) q_\alpha^-(g_-) \leq q_\alpha(f) q_\alpha(g),\end{aligned}$$

where we have used Fubini's Theorem, a change of variables and the fact that $k^{\alpha+1}$ is increasing (as a function of n) for $\alpha > 0$. The bound on the third summand is clear using the commutativity of the convolution and the bound on the first summand. The bound on the second summand is a consequence of Proposition 2.7(i) and [AL$^+$1, Theorem 2.11].

To finish we have to estimate $q_\alpha^-((f*g)_-)$. By Proposition 2.7(ii) we have, for $n < 0$,

$$\begin{aligned}W_-^\alpha(f*g)(n) &= W_+^\alpha (f*g)\tilde{\ }(-n) = W_+^\alpha(\tilde f * \tilde g)(-n) \\ &= W_+^\alpha((\tilde f * \tilde g)_+)(-n),\end{aligned}$$

so

$$\begin{aligned}q_\alpha^-((f*g)_-) &\leq \sum_{n=0}^{\infty} k^{\alpha+1}(n)|W_+^\alpha(\tilde f * \tilde g)_+(n)| \leq C_\alpha q_\alpha(\tilde f) q_\alpha(\tilde g) \\ &= C_\alpha q_\alpha(f) q_\alpha(g).\end{aligned}$$

Finally note that if $f \in \tau^\beta(|n|^\beta)$, then

$$\begin{aligned}q_\alpha(f) &= \sum_{n=-\infty}^{\infty} |W^\alpha f(n)| k^{\alpha+1}(n) = \sum_{n=0}^{\infty} k^{\alpha+1}(n) \Big|\sum_{j=n}^{\infty} k^{\beta-\alpha}(j-n) W_+^\beta f(j)\Big| \\ &\quad + \sum_{n=-\infty}^{-1} k^{\alpha+1}(-n) \Big|\sum_{j=-\infty}^{n} k^{\beta-\alpha}(n-j) W_-^\beta f(j)\Big| \\ &\leq \sum_{j=0}^{\infty} |W_+^\beta f(j)| k^{\beta+1}(j) + \sum_{j=-\infty}^{-1} |W_-^\beta f(j)| k^{\beta+1}(-j) \\ &= \sum_{j=-\infty}^{\infty} k^{\beta+1}(|j|)|W^\beta f(j)| = q_\beta(f),\end{aligned}$$

where we have applied Proposition 2.2 and the semigroup property of k^α. ∎

REMARK 2.12. Note that by (2.1) the norm q_α is equivalent to the norm $\overline{q_\alpha}$ where

$$\overline{q_\alpha}(f) := \sum_{n=1}^{\infty} n^\alpha |W_-^\alpha f(-n)| + |f(0)| + \sum_{n=1}^{\infty} n^\alpha |W_+^\alpha f(n)|$$
$$= |f(0)| + \sum_{n=1}^{\infty} n^\alpha \big(|W_+^\alpha f(n)| + |W_+^\alpha \tilde{f}(n)|\big).$$

3. A Katznelson–Tzafriri type theorem for (C,α)-bounded operators. For $\alpha > 0$, we denote by $A^\alpha(\mathbb{T})$ a new Wiener algebra formed by all continuous periodic functions $\mathfrak{f}(t) = \sum_{n=-\infty}^{\infty} \widehat{\mathfrak{f}}(n) e^{int}$, for $t \in [0, 2\pi]$, with the norm

$$\|\mathfrak{f}\|_{A^\alpha(\mathbb{T})} := \sum_{n=-\infty}^{\infty} |W^\alpha \widehat{\mathfrak{f}}(n)| k^{\alpha+1}(|n|) < \infty.$$

This algebra is regular since its characters are all defined on $\mathbb{T}$, just as for $\ell^1(\mathbb{Z})$, which is $\mathbb{T}$. Similarly to the case $\alpha = 0$, we denote by $A_+^\alpha(\mathbb{T})$ the convolution closed subalgebra of $A^\alpha(\mathbb{T})$ where the coefficients $\widehat{\mathfrak{f}}(n) = 0$ for $n < 0$. Note that $A^\alpha(\mathbb{T})$ and $\tau^\alpha(|n|^\alpha)$ are isometrically isomorphic via Fourier coefficients. The same holds for $A_+^\alpha(\mathbb{T})$ and $\tau^\alpha(n^\alpha)$.

REMARK 3.1. Recall that in [AOR] a version of the original Katznelson–Tzafriri Theorem is proved for periodic functions $\mathfrak{f}$ whose Fourier coefficients satisfy $\sum_{j\geq 0} j|\widehat{\mathfrak{f}}(j)| < \infty$. Note that $A_+^1(\mathbb{T})$ includes these functions. Also, if $\mathfrak{f} \in A_+(\mathbb{T})$ has decreasing Fourier coefficients then $\mathfrak{f} \in A_+^1(\mathbb{T})$, by the remark at the beginning of the proof of [K, Theorem 4.1, Chapter 1] for the sequence $a_n = \sum_{j\geq n} \widehat{\mathfrak{f}}(j)$. This class of functions is studied in [Z].

More generally, the subalgebras $A_+^m(\mathbb{T})$ for $m \in \mathbb{N}$ are larger than the Korenblyum subalgebras defined in [GW]. In fact, $\mathfrak{f}(t) = \sum_{n\geq 1} \frac{1}{n^{m+1}} e^{int} \in A_+^m(\mathbb{T})$ and $\mathfrak{f}$ does not belong to the corresponding Korenblyum subalgebra.

REMARK 3.2. The spaces $A_+^\alpha(\mathbb{T})$ decrease as α increases, and they are dense in $A_+(\mathbb{T})$, since by [GW] those with integer α are. Furthermore we have the following:

(i) The proof of [AL$^+$1, Theorem 2.10(iii)] shows that $\|\mathfrak{f}\|_{A_+^\alpha(\mathbb{T})} \leq \|\mathfrak{f}\|_{A_+^\beta(\mathbb{T})}$ for $0 \leq \alpha < \beta$, and so these spaces are continuously embedded in each other with norm 1.

(ii) For $0 \leq \alpha < \beta$ we have $A_+^\beta(\mathbb{T}) \subsetneq A_+^\alpha(\mathbb{T})$. This is a consequence of the characterization of (C,α)-boundedness by means of homomorphisms defined on these spaces [AL$^+$1, Corollary 3.7] and the existence of operators which are (C,β)-bounded but not (C,α)-bounded [LSS, Propositions 4.3 and 4.4].

(iii) For $\alpha > 0$, the functions $\mathfrak{f}$ such that $\sum_{j\ge 0} j^\alpha |\widehat{\mathfrak{f}}(j)| < \infty$ are in $A^\alpha_+(\mathbb{T})$. In fact, there exists a sequence $c \in \ell^\infty(\mathbb{N}_0)$ such that $|\widehat{\mathfrak{f}}(n+j)| < c(n)|\widehat{\mathfrak{f}}(n)|$ for all $j \ge 0$. By Proposition 2.3 and (2.1) for $k^{-\alpha}$ we have

$$\overline{q_\alpha}(\widehat{\mathfrak{f}}) \le |\widehat{\mathfrak{f}}(0)| + \|c\|_\infty \sum_{j=0}^{\infty} |k^{-\alpha}(j)| \sum_{n=0}^{\infty} n^\alpha |\widehat{\mathfrak{f}}(n)| < \infty.$$

Let E be a closed subset of $\mathbb{T}$ and $\mathfrak{f} \in A^\alpha(\mathbb{T})$. We recall that $\mathfrak{f}$ is of spectral synthesis with respect to E if for every $\varepsilon > 0$ there exists $\mathfrak{f}_\varepsilon \in A^\alpha(\mathbb{T})$ such that $\|\mathfrak{f} - \mathfrak{f}_\varepsilon\|_{A^\alpha(\mathbb{T})} < \varepsilon$ with $\mathfrak{f}_\varepsilon = 0$ in a neighborhood of E.

Let $T \in \mathcal{B}(X)$ and $\alpha > 0$. We can express the (C,α)-boundedness of T in the following way: there exists a constant $C > 0$ such that

$$\|\Delta^{-\alpha}\mathcal{T}(n)\| \le C k^{\alpha+1}(n), \qquad n \in \mathbb{N}_0.$$

Furthermore, we have mentioned in the introduction that for $\alpha > 0$ and $T \in \mathcal{B}(X)$ a (C,α)-bounded operator, there exists a bounded algebra homomorphism $\theta_\alpha : \tau^\alpha(n^\alpha) \to \mathcal{B}(X)$ given by

$$\theta_\alpha(f)x = \sum_{n=0}^{\infty} W^\alpha_+ f(n) \Delta^{-\alpha}\mathcal{T}(n)x, \qquad x \in X,\ f \in \tau^\alpha(n^\alpha),$$

(see [AL$^+$1, Theorem 3.5]).

Remark 3.3. Let $T \in \mathcal{B}(X)$ and $\alpha > 0$. We can write $T^j = (k^{-\alpha} * \Delta^{-\alpha}\mathcal{T})(j)$.

(i) Let m be a positive integer and T a (C,m)-bounded operator. Since $k^{-m} \in c_{0,0}(\mathbb{N}_0)$, we have $\|T^j\| = O(j^m)$. So, if $\mathfrak{f}$ belongs to the Korenblyum subalgebra $(\sum_{j\ge 0} j^m |\widehat{\mathfrak{f}}(j)| < \infty)$ then $\sum_{j\ge 0} \widehat{\mathfrak{f}}(j)T^j$ converges in operator norm. Moreover, by induction,

$$\begin{aligned}
\theta_m(\widehat{\mathfrak{f}}) &= \lim_{N\to\infty} \sum_{n=0}^{N} W^m_+ \widehat{\mathfrak{f}}(n) \Delta^{-m}\mathcal{T}(n) \\
&= \lim_{N\to\infty} \Big(\sum_{j=0}^{N} \widehat{\mathfrak{f}}(j)T^j + (-1)^{m+1} \sum_{j=0}^{m-1} W^j_+ \widehat{\mathfrak{f}}(N+1) \Delta^{-(j+1)}\mathcal{T}(N) \Big) \\
&= \lim_{N\to\infty} \Big(\sum_{j=0}^{N} \widehat{\mathfrak{f}}(j)T^j \\
&\qquad + (-1)^{m+1} \sum_{j=0}^{m-1} \Big(\sum_{l=0}^{j} (-1)^l \binom{j}{l} \widehat{\mathfrak{f}}(N+1+l) \Big) \Delta^{-(j+1)}\mathcal{T}(N) \Big) \\
&= \sum_{j=0}^{\infty} \widehat{\mathfrak{f}}(j)T^j.
\end{aligned}$$

(ii) Let $\alpha > 0$ be a positive non-integer and T a (C,α)-bounded operator. First observe that the sign of $k^{-\alpha}(j)$ is $(-1)^{[\alpha]+1}$ for all $j \geq [\alpha]+1$. For $j \geq [\alpha]+1$, note that

$$\begin{aligned}
\|T^j\| &\leq C\sum_{n=0}^{j}|k^{-\alpha}(j-n)|k^{\alpha+1}(n)\\
&= C\Big((-1)^{[\alpha]+1}\sum_{n=0}^{j-[\alpha]-1}k^{-\alpha}(j-n)k^{\alpha+1}(n)+\sum_{n=j-[\alpha]}^{j}|k^{-\alpha}(j-n)|k^{\alpha+1}(n)\Big)\\
&= C\Big((-1)^{[\alpha]+1}\sum_{n=0}^{j}k^{-\alpha}(j-n)k^{\alpha+1}(n)\\
&\quad+\sum_{n=j-[\alpha]}^{j}\big(|k^{-\alpha}(j-n)|-(-1)^{[\alpha]+1}k^{-\alpha}(j-n)\big)k^{\alpha+1}(n)\Big)\\
&\leq C_\alpha\big((-1)^{[\alpha]+1}+k^{\alpha+1}(j)\big),
\end{aligned}$$

where we have used the fact that $k^{\alpha+1}$ is increasing and $k^{-\alpha} * k^{\alpha+1} = k^1$. Then $\|T^j\| = O(j^\alpha)$. So, if $\mathfrak{f}$ belongs to the extended Korenblyum subalgebra ($\sum_{j\geq 0} j^\alpha|\widehat{\mathfrak{f}}(j)| < \infty$) then $\sum_{j\geq 0}\widehat{\mathfrak{f}}(j)T^j$ converges in operator norm. Moreover,

$$\begin{aligned}
\theta_\alpha(\widehat{\mathfrak{f}}) &= \lim_{N\to\infty}\sum_{n=0}^{N}W_+^\alpha\widehat{\mathfrak{f}}(n)\Delta^{-\alpha}\mathcal{T}(n)\\
&= \lim_{N\to\infty}\sum_{j=0}^{N}T^j\sum_{n=j}^{N}k^\alpha(n-j)\sum_{l=n}^{\infty}k^{-\alpha}(l-n)\widehat{\mathfrak{f}}(l)\\
&= \lim_{N\to\infty}\sum_{j=0}^{N}T^j\Big(\widehat{\mathfrak{f}}(j)+\sum_{l=N+1}^{\infty}\widehat{\mathfrak{f}}(l)\sum_{n=j}^{N}k^{-\alpha}(l-n)k^\alpha(n-j)\Big)\\
&= \lim_{N\to\infty}\Big(\sum_{j=0}^{N}T^j\widehat{\mathfrak{f}}(j)+\sum_{l=N+1}^{\infty}\widehat{\mathfrak{f}}(l)\sum_{n=0}^{N}k^{-\alpha}(l-n)\Delta^{-\alpha}\mathcal{T}(n)\Big)\\
&= \sum_{j=0}^{\infty}\widehat{\mathfrak{f}}(j)T^j,
\end{aligned}$$

where we have applied the fact that $k^{\alpha+1}$ is increasing, $\sum_{n=0}^{N}|k^{-\alpha}(l-n)| \leq \sum_{n=0}^{\infty}|k^{-\alpha}(n)| \leq C_\alpha$, and

$$\sum_{l=N+1}^{\infty}|\widehat{\mathfrak{f}}(l)|k^{\alpha+1}(N) \leq \sum_{l=N+1}^{\infty}|\widehat{\mathfrak{f}}(l)|k^{\alpha+1}(l) \to 0$$

as $N\to\infty$.

THEOREM 3.4. *Let $\alpha > 0$, let $T \in \mathcal{B}(X)$ be a (C,α)-bounded operator and let $\mathfrak{f} \in A^{\alpha}_{+}(\mathbb{T})$ be of spectral synthesis in $A^{\alpha}(\mathbb{T})$ with respect to $\sigma(T) \cap \mathbb{T}$. Then*

$$\lim_{n\to\infty} \|M^{\alpha}_{T}(n)\theta_{\alpha}(\widehat{\mathfrak{f}})\| = 0.$$

Proof. By assumption, for every $\varepsilon > 0$ there exists $\mathfrak{f}_{\varepsilon} \in A^{\alpha}(\mathbb{T})$ such that $\|\mathfrak{f} - \mathfrak{f}_{\varepsilon}\|_{A^{\alpha}(\mathbb{T})} < \varepsilon$ with $\mathfrak{f}_{\varepsilon} = 0$ in a neighborhood of $\sigma(T) \cap \mathbb{T}$.

Let $(h^{\alpha}_{n}(j))_{j\in\mathbb{Z}}$ for each $n \in \mathbb{N}_0$ be given by

$$h^{\alpha}_{n}(j) := \begin{cases} k^{\alpha}(n-j), & 0 \le j \le n, \\ 0, & \text{otherwise}, \end{cases}$$

the natural extension to $\mathbb{Z}$ of the sequences in $\mathbb{N}_0$ defined in [AL$^+$1, Example 2.5(ii)]. Then

(3.1)

$$\begin{aligned} \Delta^{-\alpha}\mathcal{T}(n)\theta_{\alpha}(\widehat{\mathfrak{f}}) = \theta_{\alpha}(h^{\alpha}_{n})\theta_{\alpha}(\widehat{\mathfrak{f}}) = \theta_{\alpha}(h^{\alpha}_{n} * \widehat{\mathfrak{f}}) = \sum_{j=0}^{\infty} W^{\alpha}_{+}(h^{\alpha}_{n} * \widehat{\mathfrak{f}})(j)\Delta^{-\alpha}\mathcal{T}(j) \\ = \sum_{j=0}^{\infty} W^{\alpha}_{+}(h^{\alpha}_{n} * \widehat{\mathfrak{g}_{\varepsilon}})(j)\Delta^{-\alpha}\mathcal{T}(j) + \sum_{j=0}^{\infty} W^{\alpha}_{+}(h^{\alpha}_{n} * \widehat{\mathfrak{f}_{\varepsilon}})(j)\Delta^{-\alpha}\mathcal{T}(j), \end{aligned}$$

where we have applied [AL$^+$1, Theorem 3.5] and $\mathfrak{g}_{\varepsilon} := \mathfrak{f} - \mathfrak{f}_{\varepsilon}$. For convenience we write $f(n) = \widehat{\mathfrak{f}}(n)$ for $n \in \mathbb{N}_0$, $f_{\varepsilon}(n) = \widehat{\mathfrak{f}_{\varepsilon}}(n)$ and $g_{\varepsilon}(n) = \widehat{\mathfrak{g}_{\varepsilon}}(n) = f(n) - f_{\varepsilon}(n)$ for $n \in \mathbb{Z}$ (note that we suppose that $f(n) = 0$ for $n < 0$, as mentioned in the introduction).

For the first summand, using Lemma 2.10, $W^{\alpha}_{+}(h^{\alpha}_{n}) = e_n$ (see [AL$^+$1, Example 2.5(ii)]), (2.3) and Fubini's Theorem we get

$$\begin{aligned} \sum_{j=0}^{\infty} W^{\alpha}_{+}(h^{\alpha}_{n} * g_{\varepsilon})(j)\Delta^{-\alpha}\mathcal{T}(j) &= \sum_{j=0}^{n-1} g_{\varepsilon}(j-n)\Delta^{-\alpha}\mathcal{T}(j) \\ &+ \Big(\sum_{j=n}^{\infty}\sum_{p=j-n}^{j} - \sum_{j=0}^{n-1}\sum_{p=j+1}^{\infty}\Big) k^{\alpha}(p-j+n)W^{\alpha}_{+}g_{\varepsilon}(p)\Delta^{-\alpha}\mathcal{T}(j) \\ = \sum_{j=0}^{n-1} g_{\varepsilon}(j-n)\Delta^{-\alpha}\mathcal{T}(j) \\ + \Big(\sum_{p=0}^{n}\sum_{j=n}^{p+n} + \sum_{p=n+1}^{\infty}\sum_{j=p}^{p+n} - \sum_{p=1}^{n}\sum_{j=0}^{p-1} - \sum_{p=n+1}^{\infty}\sum_{j=0}^{n-1}\Big) k^{\alpha}(p-j+n)W^{\alpha}_{+}g_{\varepsilon}(p)\Delta^{-\alpha}\mathcal{T}(j). \end{aligned}$$

We now show that each term above, when divided by $k^{\alpha+1}(n)$, tends to 0 as $n \to \infty$, using $\|\Delta^{-\alpha}\mathcal{T}(j)\| \le Ck^{\alpha+1}(j)$ for $j \in \mathbb{N}_0$, the fact that $k^{\alpha+1}(j)$ is increasing as a function of j for $\alpha > 0$, the semigroup property of the kernel k^α and (2.2). By Theorem 2.11, for the first term we have

$$\frac{1}{k^{\alpha+1}(n)}\sum_{j=0}^{n-1}|g_\varepsilon(j-n)|\,\|\Delta^{-\alpha}\mathcal{T}(j)\| \le C\sum_{j=0}^{n-1}|g_\varepsilon(j-n)| \le C\|\mathfrak{g}_\varepsilon\|_{A(\mathbb{T})} \le C\|\mathfrak{g}_\varepsilon\|_{A^\alpha(\mathbb{T})} < C\varepsilon;$$

for the second,

$$\begin{aligned}
&\frac{1}{k^{\alpha+1}(n)}\sum_{p=0}^{n}|W_+^\alpha g_\varepsilon(p)|\sum_{j=n}^{p+n}k^\alpha(p-j+n)\|\Delta^{-\alpha}\mathcal{T}(j)\| \\
&\le C\sum_{p=0}^{n}|W_+^\alpha g_\varepsilon(p)|\frac{k^{\alpha+1}(p+n)}{k^{\alpha+1}(n)}\sum_{j=n}^{p+n}k^\alpha(p-j+n) \\
&= C\sum_{p=0}^{n}|W_+^\alpha g_\varepsilon(p)|\frac{k^{\alpha+1}(p+n)}{k^{\alpha+1}(n)}k^{\alpha+1}(p) \le C\sum_{p=0}^{n}|W_+^\alpha g_\varepsilon(p)|\frac{k^{\alpha+1}(2n)}{k^{\alpha+1}(n)}k^{\alpha+1}(p) \\
&\le C_\alpha\sum_{p=0}^{n}|W_+^\alpha g_\varepsilon(p)|k^{\alpha+1}(p) \le C_\alpha\|\mathfrak{g}_\varepsilon\|_{A^\alpha(\mathbb{T})} < C_\alpha\varepsilon;
\end{aligned}$$

for the third,

$$\begin{aligned}
&\frac{1}{k^{\alpha+1}(n)}\sum_{p=n+1}^{\infty}|W_+^\alpha g_\varepsilon(p)|\sum_{j=p}^{p+n}k^\alpha(p-j+n)\|\Delta^{-\alpha}\mathcal{T}(j)\| \\
&\quad\le C\sum_{p=n+1}^{\infty}|W_+^\alpha g_\varepsilon(p)|\frac{k^{\alpha+1}(p+n)}{k^{\alpha+1}(n)}\sum_{j=p}^{p+n}k^\alpha(p-j+n) \\
&\quad= C\sum_{p=n+1}^{\infty}|W_+^\alpha g_\varepsilon(p)|k^{\alpha+1}(p+n) \le C_\alpha\sum_{p=n+1}^{\infty}|W_+^\alpha g_\varepsilon(p)|k^{\alpha+1}(p) < C_\alpha\varepsilon;
\end{aligned}$$

for the fourth,

$$\begin{aligned}
&\frac{1}{k^{\alpha+1}(n)}\sum_{p=1}^{n}|W_+^\alpha g_\varepsilon(p)|\sum_{j=0}^{p-1}k^\alpha(p-j+n)\|\Delta^{-\alpha}\mathcal{T}(j)\| \\
&\qquad\le C\sum_{p=1}^{n}|W_+^\alpha g_\varepsilon(p)|\frac{k^{\alpha+1}(p)}{k^{\alpha+1}(n)}\sum_{j=0}^{p-1}k^\alpha(p-j+n) \\
&\qquad\le C\sum_{p=1}^{n}|W_+^\alpha g_\varepsilon(p)|\frac{k^{\alpha+1}(p)}{k^{\alpha+1}(n)}\sum_{j=0}^{p+n}k^\alpha(p-j+n)
\end{aligned}$$

$$= C\sum_{p=1}^{n}|W_+^\alpha g_\varepsilon(p)|\frac{k^{\alpha+1}(p)}{k^{\alpha+1}(n)}k^{\alpha+1}(p+n) \le C_\alpha \sum_{p=1}^{n}|W_+^\alpha g_\varepsilon(p)|k^{\alpha+1}(p) < C_\alpha\varepsilon;$$

and for the fifth,

$$\frac{1}{k^{\alpha+1}(n)}\sum_{p=n+1}^{\infty}|W_+^\alpha g_\varepsilon(p)|\sum_{j=0}^{n-1}k^\alpha(p-j+n)\|\Delta^{-\alpha}\mathcal{T}(j)\|$$
$$\le C\sum_{p=n+1}^{\infty}|W_+^\alpha g_\varepsilon(p)|\sum_{j=0}^{n-1}k^\alpha(p-j+n)$$
$$\le C\sum_{p=n+1}^{\infty}|W_+^\alpha g_\varepsilon(p)|\sum_{j=0}^{p+n}k^\alpha(p-j+n) = C\sum_{p=n+1}^{\infty}|W_+^\alpha g_\varepsilon(p)|k^{\alpha+1}(p+n)$$
$$\le C_\alpha\sum_{p=n+1}^{\infty}|W_+^\alpha g_\varepsilon(p)|k^{\alpha+1}(p) < C_\alpha\varepsilon.$$

On the other hand, for the second term in (3.1), we have to prove that

$$\lim_{n\to\infty}\frac{1}{k^{\alpha+1}(n)}\sum_{j=0}^{\infty}W_+^\alpha(h_n^\alpha * f_\varepsilon)(j)\Delta^{-\alpha}\mathcal{T}(j) = 0.$$

It is known that

$$(\lambda - T)^{-1} = \left(\frac{\lambda-1}{\lambda}\right)^\alpha\sum_{n=0}^{\infty}\lambda^{-n-1}\Delta^{-\alpha}\mathcal{T}(n) \qquad \text{for } |\lambda|>1,$$

(see [AL$^+$1, Theorem 4.11(iii)]). Noting that $h_n^\alpha * f_\varepsilon \in \tau^\alpha(|n|^\alpha)$, if $m = [\alpha]+1$ we get

$$\sum_{j=-\infty}^{\infty}W_+^\alpha(h_n^\alpha * f_\varepsilon)(-j)e^{ijt} = \sum_{j=-\infty}^{\infty}W_+^\alpha(h_n^\alpha * f_\varepsilon)(j)e^{-ijt}$$
$$= \lim_{\lambda\to1^+}\Big(\sum_{j=0}^{\infty}W_+^m W_+^{-(m-\alpha)}(h_n^\alpha * f_\varepsilon)(j)(\lambda^{-1}e^{-it})^j$$
$$+\sum_{j=-\infty}^{-1}W_+^m W_+^{-(m-\alpha)}(h_n^\alpha * f_\varepsilon)(j)(\lambda e^{-it})^j\Big)$$
$$= \sum_{l=0}^{m}(-1)^l\binom{m}{l}e^{itl}\lim_{\lambda\to1^+}\Big(\sum_{v=l}^{\infty}W_+^{-(m-\alpha)}(h_n^\alpha * f_\varepsilon)(v)(\lambda^{-1}e^{-it})^v$$
$$+\sum_{v=-\infty}^{l-1}W_+^{-(m-\alpha)}(h_n^\alpha * f_\varepsilon)(v)(\lambda e^{-it})^v\Big)$$

$$= (1-e^{it})^m \lim_{\lambda\to 1^+}\Big(\sum_{u=l}^{\infty}\sum_{v=l}^{u} k^{m-\alpha}(u-v)(\lambda^{-1}e^{-it})^v(h_n^\alpha * f_\varepsilon)(u)$$
$$+\sum_{u=-\infty}^{l-1}\sum_{v=-\infty}^{u} k^{m-\alpha}(u-v)(\lambda e^{-it})^v(h_n^\alpha * f_\varepsilon)(u)$$
$$+\sum_{u=l}^{\infty}\sum_{v=-\infty}^{l-1} k^{m-\alpha}(u-v)(\lambda e^{-it})^v(h_n^\alpha * f_\varepsilon)(u)\Big).$$

Now, using

$$\lim_{\lambda\to 1^+}\sum_{j=0}^{\infty} k^{m-\alpha}(j)(\lambda e^{-it})^{-j} = \frac{1}{(1-e^{it})^{m-\alpha}}, \qquad t\neq 2\pi\mathbb{Z},\ 0<m-\alpha<1$$

(see [AL$^+$1, Section 4]), we find for $t\neq 2\pi\mathbb{Z}$ that

$$\sum_{j=-\infty}^{\infty} W_+^\alpha(h_n^\alpha * f_\varepsilon)(-j)e^{ijt}$$
$$= (1-e^{it})^m\Big(\sum_{u=l}^{\infty}(h_n^\alpha * f_\varepsilon)(u)\lim_{\lambda\to 1^+}\Big(\sum_{v=l}^{u}+\sum_{v=-\infty}^{l-1}\Big)k^{m-\alpha}(u-v)(\lambda e^{-it})^v$$
$$+\sum_{u=-\infty}^{l-1}(h_n^\alpha * f_\varepsilon)(u)\lim_{\lambda\to 1^+}\sum_{v=-\infty}^{u}k^{m-\alpha}(u-v)(\lambda e^{-it})^v\Big)$$
$$= (1-e^{it})^\alpha \sum_{u=-\infty}^{\infty}(h_n^\alpha * f_\varepsilon)(u)e^{-itu} = (1-e^{it})^\alpha \mathfrak{f}_\varepsilon(-t)\sum_{j=0}^{n}k^\alpha(n-j)e^{-ijt},$$

If we define $\Delta^{-\alpha}\mathcal{T}(n)=0$ for $n<0$, note that the operator-valued sequence $(\lambda^{-(j+1)}\Delta^{-\alpha}\mathcal{T}(j))_{j\in\mathbb{Z}}$ for $|\lambda|>1$ is summable. Then Parseval's identity implies that

$$\sum_{j=0}^{\infty}W_+^\alpha(h_n^\alpha * f_\varepsilon)(j)\Delta^{-\alpha}\mathcal{T}(j) = \lim_{\lambda\to 1^+}\sum_{j=0}^{\infty}W_+^\alpha(h_n^\alpha * f_\varepsilon)(j)\lambda^{-(j+1)}\Delta^{-\alpha}\mathcal{T}(j)$$
$$= \frac{1}{2\pi}\int_0^{2\pi}\mathfrak{f}_\varepsilon(-t)\Big(\sum_{j=0}^{n}k^\alpha(n-j)e^{-ijt}\Big)e^{-it}(e^{-it}-T)^{-1}\,dt$$
$$= \sum_{j=0}^{n}k^\alpha(n-j)\widehat{G}(j),$$

where $G(t) = e^{-it}\mathfrak{f}_\varepsilon(-t)(e^{-it}-T)^{-1}$. Applying the Riemann–Lebesgue Lemma we deduce that for all $\delta>0$ there exists an $n_0\in\mathbb{N}$ such that

$\|\widehat{G}(j)\| < \delta$ for all $|j| \geq n_0$. Then

$$\begin{aligned}\frac{1}{k^{\alpha+1}(n)}\Big\|\sum_{j=0}^{n} k^\alpha(n-j)\widehat{G}(j)\Big\| &\leq \frac{1}{k^{\alpha+1}(n)}\Big(\sum_{j=0}^{n-n_0} + \sum_{j=n-n_0+1}^{n}\Big)k^\alpha(j)\|\widehat{G}(n-j)\| \\ &\leq \delta + \sum_{j=n-n_0+1}^{n} \frac{\alpha}{(\alpha+j)}\|\widehat{G}(n-j)\| \\ &\leq \delta + \frac{\|\widehat{G}\|_\infty(n_0-1)}{\alpha+n-n_0+1},\end{aligned}$$

where we have applied the fact that $k^{\alpha+1}(j)$ is increasing as a function of j, and $\|\widehat{G}\|_\infty = \sup_{j\geq 0}\|\widehat{G}(j)\|$. Taking $n \to \infty$ we get the result. ∎

Remark 3.5. Parseval's identity for the product of a scalar-valued function and a vector-valued function, and the Riemann–Lebesgue Lemma for a vector-valued function, can be proved by applying linear functionals, and using the scalar-valued results and the Hahn–Banach Theorem. The first reference for these results is [B]. Analogous results for the continuous case are in [ABHN, Theorem 1.8.1].

Remark 3.6. When T is a power-bounded operator, the proof of Theorem 3.4 gives a short alternative proof of the Katznelson–Tzafriri Theorem [KT, Theorem 5]:

Let $\mathfrak{f}$ in $A_+(\mathbb{T})$ be of spectral synthesis in $A(\mathbb{T})$ with respect to $\sigma(T)\cap\mathbb{T}$, that is, for $\varepsilon > 0$ there exists $\mathfrak{f}_\varepsilon \in A(\mathbb{T})$ such that $\|\mathfrak{f} - \mathfrak{f}_\varepsilon\|_{A(\mathbb{T})} < \varepsilon$ with $\mathfrak{f}_\varepsilon = 0$ in a neighborhood F of $\sigma(T)\cap\mathbb{T}$. We denote by $(\mathcal{T}(n))_{n\in\mathbb{Z}}$ the family of operators given by $\mathcal{T}(n) = T^n$ for $n \in \mathbb{N}_0$ and $\mathcal{T}(n) = 0$ for $n < 0$. Then it is clear that

$$\Big\|\sum_{j=-\infty}^{\infty} \widehat{\mathfrak{f}_\varepsilon}(j)\mathcal{T}(n+j) - T^n\theta(\widehat{\mathfrak{f}})\Big\| < C\varepsilon,$$

since $\|T^n\| \leq C$ for all $n \in \mathbb{N}_0$. Now, using Parseval's identity, we get

$$\begin{aligned}\sum_{j=-\infty}^{\infty} \widehat{\mathfrak{f}_\varepsilon}(j)\mathcal{T}(n+j) &= \lim_{\lambda\to 1^+}\sum_{j=-\infty}^{\infty} \widehat{\mathfrak{f}_\varepsilon}(j)\lambda^{-(n+j+1)}\mathcal{T}(n+j) \\ &= \lim_{\lambda\to 1^+}\frac{1}{2\pi}\int_0^{2\pi} e^{-it(n+1)}\mathfrak{f}_\varepsilon(-t)(\lambda e^{-it} - T)^{-1}\,dt \\ &= \frac{1}{2\pi}\int_0^{2\pi} e^{-it(n+1)}\mathfrak{f}_\varepsilon(-t)(e^{-it} - T)^{-1}\,dt,\end{aligned}$$

which converges to 0 by the Riemann–Lebesgue Lemma, and we conclude the proof.

4. Ergodic applications. Several authors have investigated the connections between the stability of the Cesàro mean differences of size n and $n+1$, that is,

$$\lim_{n\to\infty} \|M_T^\alpha(n+1) - M_T^\alpha(n)\| = 0, \tag{4.1}$$

and spectral conditions for (C,α)-bounded operators $T \in \mathcal{B}(X)$ (see [SZ] and the references therein). We cannot get (4.1) using Theorem 3.4 directly, because this problem is equivalent to finding a sequence $f \in \tau^\alpha(n^\alpha)$ such that

$$\frac{1}{k^{\alpha+1}(n)}(h_n^\alpha * f) = \frac{1}{k^{\alpha+1}(n)} h_n^\alpha - \frac{1}{k^{\alpha+1}(n+1)} h_{n+1}^\alpha$$

for all $n \in \mathbb{N}_0$, which has no solution. However, the following theorem shows how using Theorem 3.4 and other techniques we get the desired result, which is a consequence of [SZ, Theorems 2.2(ii) and 3.1(i)] when $\alpha \in \mathbb{N} = \{1,2,\ldots\}$.

THEOREM 4.1. *Let $\alpha > 0$ and $T \in \mathcal{B}(X)$ be a (C,α)-bounded operator such that $\sigma(T) \cap \mathbb{T} \subseteq \{1\}$. Then*

$$\lim_{n\to\infty} \|M_T^\alpha(n+1) - M_T^\alpha(n)\| = 0.$$

Proof. Observe that if $\sigma(T)\cap\mathbb{T} = \emptyset$, then $r(T) < 1$ by Lemma 1.1, and therefore $\|T^n\| \to 0$ exponentially; in particular T is power-bounded. So, we shall prove the result when $\sigma(T)\cap\mathbb{T} = \{1\}$.

First we suppose that $\alpha \geq 1$. Then using the relation

$$\frac{n+\alpha+1}{n+1} M_T^\alpha(n+1) - M_T^\alpha(n) = \frac{\alpha}{n+1} M_T^{\alpha-1}(n+1), \qquad n \in \mathbb{N}_0,$$

which is easy to get from the definition of Cesàro mean of order α, we can write

$$M_T^\alpha(n+1) - M_T^\alpha(n) = \frac{\alpha}{n+1}(M_T^{\alpha-1}(n+1) - I) + \frac{\alpha}{n+1}(I - M_T^\alpha(n+1)).$$

Using the identity

$$M_T^\alpha(n)(T-I) = \frac{\alpha}{n+1}(M_T^{\alpha-1}(n+1) - I), \qquad n \in \mathbb{N}_0,$$

which can easily be obtained from the definition of Cesàro mean of order α, and applying Theorem 3.4 to the function $\mathfrak{f}(t) = e^{it} - 1$, we see that the first summand goes to zero as $n \to \infty$. On the other hand, the second summand goes to zero since T is (C,α)-bounded.

Now let $0 < \alpha < 1$. Using $k^\alpha = k^{-(1-\alpha)} * k^1$, we write

$$\begin{aligned} M_T^\alpha(n) &= \frac{1}{k^{\alpha+1}(n)} \Delta^{-\alpha}\mathcal{T}(n) \\ &= \frac{1}{k^{\alpha+1}(n)} (k^{-(1-\alpha)} * \Delta^{-1}\mathcal{T})(n). \end{aligned}$$

So we can write

$$
\begin{aligned}
&M_T^\alpha(n+1) - M_T^\alpha(n)\\
&\quad= \frac{k^{-(1-\alpha)}(n+1)}{k^{\alpha+1}(n+1)} I + \sum_{j=0}^{n} k^{-(1-\alpha)}(n-j)\left(\frac{\Delta^{-1}\mathcal{T}(j+1)}{k^{\alpha+1}(n+1)} - \frac{\Delta^{-1}\mathcal{T}(j)}{k^{\alpha+1}(n)}\right)\\
&\quad= \frac{k^{-(1-\alpha)}(n+1)}{k^{\alpha+1}(n+1)} I + \frac{n+1}{(n+\alpha+1)k^{\alpha+1}(n)} \sum_{j=0}^{n} k^{-(1-\alpha)}(n-j) T^{j+1}\\
&\qquad - \frac{\alpha}{(n+\alpha+1)k^{\alpha+1}(n)} \sum_{j=0}^{n} k^{-(1-\alpha)}(n-j)\Delta^{-1}\mathcal{T}(j),
\end{aligned}
$$

where we have used

$$
\frac{\Delta^{-1}\mathcal{T}(j+1)}{k^{\alpha+1}(n+1)} - \frac{\Delta^{-1}\mathcal{T}(j)}{k^{\alpha+1}(n)} = \frac{1}{(n+\alpha+1)k^{\alpha+1}(n)}\big((n+1)T^{j+1} - \alpha\Delta^{-1}\mathcal{T}(j)\big).
$$

If we add and subtract the term

$$
\frac{n+1}{(n+\alpha+1)k^{\alpha+1}(n)} \sum_{j=0}^{n} k^{-(1-\alpha)}(n-j) I = \frac{(k^{-(1-\alpha)} * k^1)(n)}{k^{\alpha+1}(n+1)} I = \frac{k^\alpha(n)}{k^{\alpha+1}(n+1)} I
$$

we obtain

$$
\begin{aligned}
&M_T^\alpha(n+1) - M_T^\alpha(n) = \frac{k^\alpha(n+1)}{k^{\alpha+1}(n+1)} I\\
&\quad + \frac{n+1}{(n+\alpha+1)k^{\alpha+1}(n)} \sum_{j=0}^{n} k^{-(1-\alpha)}(n-j)(T^{j+1} - I) - \frac{\alpha}{n+\alpha+1} M_T^\alpha(n).
\end{aligned}
$$

The first term on the right hand side goes to zero as $n \to \infty$ by (2.1). Theorem 3.4 implies that the second term goes to zero since

$$
\begin{aligned}
M_T^\alpha(n)(T-I) &= \frac{1}{k^{\alpha+1}(n)} \sum_{j=0}^{n} k^{-(1-\alpha)}(n-j)\Delta^{-1}\mathcal{T}(j)(T-I)\\
&= \frac{1}{k^{\alpha+1}(n)} \sum_{j=0}^{n} k^{-(1-\alpha)}(n-j)(T^{j+1} - I).
\end{aligned}
$$

Finally, the third term goes to zero because T is (C,α)-bounded. ∎

Observe that under the assumption of Theorem 4.1, T is also $(C,[\alpha]+1)$-bounded, so by [SZ, Theorems 2.2(ii) and 3.1(i)] we have $\frac{1}{n}\|M_T^{[\alpha]}(n)\| \to 0$ as $n \to \infty$, and $\|T^n\| = o(n^{[\alpha]+1})$. Our next result extends [SZ, Theorem 2.2(ii)] for $\alpha \ge 1$.

Remark 4.2. Before stating the theorem, note that [SZ, Theorem 3.1(i)] is valid for any $\alpha \ge 1$, by the same proof.

THEOREM 4.3. *Let $\alpha \geq 1$ and $T \in \mathcal{B}(X)$ be a (C,α)-bounded operator such that $\sigma(T) \cap \mathbb{T} \subseteq \{1\}$. Then*

$$\|M_T^{\alpha-1}(n)\| = o(n) \quad \textit{and} \quad \|T^n\| = o(n^\alpha), \qquad \textit{as } n \to \infty.$$

Proof. By Theorem 4.1 and Remark 4.2 we have $\|M_T^{\alpha-1}(n)\| = o(n)$. Now, we suppose that $\alpha > 1$ for convenience (for $\alpha = 1$ the result is already proved). We can write $T^n = (k^{-(\alpha-1)} * \Delta^{-(\alpha-1)}\mathcal{T})(n)$; we have mentioned in the previous section that the sign of $k^{-(\alpha-1)}(n)$ is $(-1)^{[\alpha]}$ for all $n \geq [\alpha]$. For $n \geq [\alpha]$, note that

$$\begin{aligned}
\|T^n\| &\leq \sum_{j=0}^{n} |k^{-(\alpha-1)}(n-j)| \|\Delta^{-(\alpha-1)}\mathcal{T})(j\| \\
&= (-1)^{[\alpha]} \sum_{j=0}^{n} k^{-(\alpha-1)}(n-j) \|\Delta^{-(\alpha-1)}\mathcal{T}(j)\| \\
&\quad + \sum_{j=n-[\alpha]+1}^{n} (|k^{-(\alpha-1)}(n-j)| - (-1)^{[\alpha]} k^{-(\alpha-1)}(n-j)) \|\Delta^{-(\alpha-1)}\mathcal{T}(j)\| \\
&= I + II.
\end{aligned}$$

From $\|M_T^{\alpha-1}(n)\| = o(n)$ we have

$$\|\Delta^{-(\alpha-1)}\mathcal{T}(n)\| \leq Ck^{\alpha+1}(n),$$

so

$$\frac{|I|}{n^\alpha} \leq \frac{Ck^2(n)}{n^\alpha} \to 0, \qquad n \to \infty.$$

Secondly,

$$\begin{aligned}
\frac{|II|}{n^\alpha} &\leq \frac{C_\alpha}{n^\alpha} \sum_{j=n-[\alpha]+1}^{n} \|\Delta^{-(\alpha-1)}\mathcal{T}(j)\| \\
&= \frac{C_\alpha}{n^\alpha} \sum_{u=0}^{[\alpha]-1} k^\alpha(u+n-[\alpha]+1) \|M_T^{\alpha-1}(u+n-[\alpha]+1)\| \to 0, \qquad n \to \infty. \quad \blacksquare
\end{aligned}$$

Acknowledgements. The author thanks Ralph Chill, José E. Galé, Pedro J. Miana, Daniel J. Rodriguez, Francisco J. Ruiz and the anonymous referee for pieces of advice, comments and nice ideas that have contributed to improve the paper. This paper was partially written during a research visit at the Technical University of Dresden under the supervision of Ralph Chill. The author has been partially supported by Project MTM2013-42105-P, DGI-FEDER, of the MCYTS, and Project E-64, D.G. Aragón.

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