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Two new estimates for eigenvalues of Dirac operators

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Abstract. We establish lower and upper eigenvalue estimates for Dirac operators in different settings, a new Kirchberg type estimate for the first eigenvalue of the Dirac operator on a compact Kähler spin manifold in terms of the energy momentum tensor, and an upper bound for the smallest eigenvalues of the twisted Dirac operator on Legendrian submanifolds of Sasakian manifolds. The sharpness of those estimates is also discussed.

1. Introduction and main results

1.1. Previous work. Spectrum estimates for the Dirac operator D on a closed spin manifold have been developed for a long time. In 1980, Friedrich [Fri1] proved that on a compact Riemannian spin manifold (M^n, g) of positive scalar curvature S, the first eigenvalue λ of D satisfies

$$\lambda^2 \geq \frac{n}{4(n-1)} S_0, \tag{1.1}$$

where S_0 is the minimum of S on M, and equality holds in (1.1) if and only if there exists a real Killing spinor ψ, i.e.

$$\nabla_X \psi = -\frac{\lambda_1}{n} X \cdot \psi \tag{1.2}$$

for all vector fields X on M. Moreover Friedrich proved that the existence of a nontrivial Killing spinor implies that (M^n, g) is Einstein. Here λ_1 is the eigenvalue of D with the smallest absolute value. For Kähler manifolds of complex dimension $m > 1$ Hijazi [Hij1] proved that equality in (1.1) cannot be attained since Kähler manifolds cannot have Killing spinors. On the other hand, in 1986 Kirchberg [Kir1] proved that each eigenvalue λ of the Dirac operator on a compact Kähler spin manifold (M, g, J) of complex dimension

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m with positive scalar curvature S satisfies

$$\lambda^2 \geq \begin{cases} \dfrac{m+1}{4m} S_0 & \text{if } m \text{ is odd,} \\ \dfrac{m}{4(m-1)} S_0 & \text{if } m \text{ is even.} \end{cases} \tag{1.3}$$

When $m = 2l + 1$, Kirchberg [Kir4] also showed that equality in the first relation of (1.3) is characterized by the existence of a Kählerian Killing spinor $\psi = \psi_l + \psi_{l+1} \in \Gamma(\Sigma_l M \oplus \Sigma_{l+1} M)$, i.e.

$$\begin{cases} \nabla_X \psi_l = -\dfrac{\lambda_1}{m+1} X^- \cdot \psi_{l+1}, \\ \nabla_X \psi_{l+1} = -\dfrac{\lambda_1}{m+1} X^+ \cdot \psi_l, \end{cases} \tag{1.4}$$

for any vector field X, where $X^{\pm} = \frac{1}{2}(X \mp iJX)$, and that (M, g, J) is always Einstein in this case. Unlike the case of odd complex dimensions, for $m = 2l$ Gauduchon [Gau] proved that equality in the second relation of (1.3) holds if and only if there exists a nontrivial spinor $\psi \in \Gamma(\Sigma_{l+1} M)$ satisfying $D^2 \psi = D_+ D \psi = \lambda^2 \psi$ and

$$\nabla_X \psi = -\frac{1}{m} X^+ \cdot D\psi, \tag{1.5}$$

for any vector field X (see (2.2) for the definition of D_+).

On the other hand, inspired by the idea of Vafa and Witten [Wit], Bär [Bar1] applied the min-max principle to establish upper eigenvalue estimates for the twisted Dirac operators on submanifolds of spin manifolds with Killing spinors. Then Ginoux [Gin2] established eigenvalue estimates for the Dirac operator on Lagrangian submanifolds of Kählerian manifolds.

These results reveal that one can relate analytical properties of the Dirac operator to richer geometric structures of the underlying Riemannian spin manifolds. It is interesting to give lower bounds for the Dirac spectrum depending on additional geometric quantities. In this direction Friedrich and Kirchberg [FK1, FK2, Kir2] generalized (1.1) in terms of the Ricci tensor, curvature tensor and Weyl tensor in, respectively. In special geometric situations, better estimates are obtained in [Hij2, Hij3, HZ]. The famous energy-momentum tensor Q_ψ comes from variation of the Dirac operator, and it not only appears in the Dirac–Einstein equations but also is the second fundamental form of an immersion into a spin manifold with a parallel spinor. With a modified Levi-Civita connection on compact Riemannian spin manifolds, Hijazi [Hij4] proved that the eigenvalues of the Dirac operator could be related to the norm of the energy-momentum tensor Q_ψ associated to an eigenspinor field ψ, i.e.,

$$\lambda^2 \geq \inf_M \left(\tfrac{1}{4} S + |Q_\psi|^2\right), \tag{1.6}$$

and if equality holds there exists a nontrivial spinor ψ such that

$$\nabla_X \psi = -A_\psi(X) \cdot \psi \tag{1.7}$$

for any vector field X, where A_ψ is the field of symmetric endomorphisms associated to the symmetric bilinear tensors Q_ψ. The equality (1.7) can be used to study isometric immersion of hypersurfaces into a manifold, for instance, isometric immersion of a surface into the 3-dimensional Euclidean space [Fri2] and isometric immersion of a semi-Riemannian hypersurface into model spaces of constant sectional curvature [BGM]. It is thus natural to ask if there exists an analogous estimate on compact Kähler spin manifolds.

Sasakian manifolds [Bla] can be thought of as the odd-dimensional analogues of Kähler manifolds. In [FKi, Kim], the authors improved Friedrich's estimates (1.1) with the help of new connection deformation techniques adapted to the Sasakian structure. It is natural to ask whether one can build bridges between upper eigenvalue estimates for the Dirac operator and the Sasakian structure on a Sasakian manifold. Our second result will give an affirmative answer to this question.

1.2. Main results. The following is our first result.

THEOREM 1.1. *Let (M^{2m}, J, g) be a compact Kähler spin manifold with scalar curvature S, and ψ an eigenspinor of type $(r-1, r)$ associated to a nonzero eigenvalue λ of D, where $r \in \{1, \dots, m\}$. Then*

$$\lambda^2 \geq \begin{cases} \dfrac{r}{4r-2} \inf\limits_{M_{\psi_{r-1}}} (S + 4|Q_{\psi_{r-1}}|^2), \\ \dfrac{m-r+1}{2(2m-2r+1)} \inf\limits_{M_{\psi_r}} (S + 4|Q_{\psi_r}|^2), \end{cases} \tag{1.8}$$

where $M_\psi = \{x \in M \mid \psi(x) \neq 0\}$.

This is a new Kirchberg type estimate for the first eigenvalue of the Dirac operator on compact Kähler spin manifolds in terms of the energy-momentum tensor. The proof is mainly based on the Weitzenböck formula for a modified Kähler twistor operator [Kir3, Pi].

COROLLARY 1.2. *Let (M^{2m}, J, g) be a compact Kähler spin manifold with scalar curvature S. Then any eigenvalue $\lambda \neq 0$ of the Dirac operator satisfies*

$$\lambda^2 \geq \begin{cases} \dfrac{m+1}{4m} \inf\limits_M (S + 4|Q_{\psi_{r-1}}|^2) & \text{if } m \text{ is odd}, \\ \dfrac{m}{4(m-1)} \inf\limits_M (S + 4|Q_{\psi_r}|^2) & \text{if } m \text{ is even}, \end{cases} \tag{1.9}$$

where $\psi = \psi_{r-1} + \psi_r$ is an eigenspinor of type $(r-1, r)$ associated to λ with

$r \in \{(m+1)/2, (m+1)/2+1, \dots, m\}$ if m is odd, and $r \in \{m/2+1, m/2+2, \dots, m\}$ if m is even. Furthermore, assume that some eigenvalue $\lambda^* \neq 0$ of the Dirac operator is such that equality holds in (1.9). *Then*

(i) *for odd m we have $r = (m+1)/2$ and*

$$S + 4|Q_{\psi_{(m-1)/2}}|^2 \equiv \inf_M (S + 4|Q_{\psi_{(m-1)/2}}|^2),$$

$$\nabla_X \psi_{(m-1)/2} + \frac{\lambda^*}{m+1} X^- \cdot \psi_{(m+1)/2} + A_\psi(X) \cdot \psi_{(m-1)/2} = 0 \tag{1.10}$$

for every vector field X;

(ii) *for even m we have $r = (m+2)/2$ and*

$$S + 4|Q_{\psi_{(m+2)/2}}|^2 \equiv \inf_M (S + 4|Q_{\psi_{(m+2)/2}}|^2),$$

$$\nabla_X \psi_{(m+2)/2} + \frac{\lambda^*}{m} X^+ \cdot \psi_{m/2} + A_\psi(X) \cdot \psi_{(m+2)/2} = 0, \tag{1.11}$$

for every vector field X.

Comparing (1.6) and (1.9) we see that (1.9) is a better estimate on compact Kähler spin manifolds. Moreover, for odd m the equality (1.10) can be partially viewed as a generalization of (1.4); and for even m the equality (1.11) is equivalent to

$$\nabla_X \psi_{(m+2)/2} + \frac{1}{m} X^+ \cdot D\psi_{(m+2)/2} + A_{\psi_{(m+2)/2}}(X) \cdot \psi_{(m+2)/2} = 0,$$

which also can be viewed as an analogue of (1.5).

Our second result is an upper eigenvalue estimate for Legendrian submanifolds in Sasakian manifolds. We give a definition of a Sasakian version of Killing spinors and show that it is equivalent to the definition of a Sasakian Killing spinor in [Kim]. Via combining the formula for the square of the twisted Dirac operator (cf. Lemma 4.1) and an adapted local frame (cf. Lemma 4.2) on the underlying manifold, we obtain:

Theorem 1.3. *Let (L, g) be a closed Legendrian spin manifold of a Sasakian spin manifold $(M^{2m+1}, \phi, \xi, \eta, g)$ with $m \geq 3$ and $m \equiv 1 \bmod 2$, and let H be the mean curvature vector field of L in M. Assume that M admits Sakakian Killing spinors and that the normal bundle of L in M is equipped with the induced spin structure. Set $l := (m-1)/2$ and $N := \dim(\mathcal{SK}(\mu))$, where μ is a given nonzero real number and $\mathcal{SK}(\mu)$ is the space of μ-Sasakian Killing spinors on $(M^{2m+1}, \phi, \xi, \eta, g)$. Then there are at least N eigenvalues λ counted with multiplicity of the twisted Dirac operator D^T satisfying*

$$\lambda^2 \leq \frac{(\mu + (-1)^l)^2}{4} + \frac{m^2}{4\,\mathrm{Vol}(L)} \int_L |H|^2 v_g. \tag{1.12}$$

If the submanifold L is minimal in M, then (1.12) becomes

$$\lambda^2 \leq (\mu + (-1)^l)^2/4. \tag{1.13}$$

In Section 5.2 we shall give an example to show that this estimate is sharp.

The arrangement of this paper is as follows. In Section 2 we collect some necessary preliminaries on spinor bundles and the Dirac operator on Kähler manifolds. The proofs of Theorem 1.1 and Corollary 1.2 will be given in Section 3. In Section 4 we shall discuss Legendrian submanifolds of Sasakian manifolds and give some lemmas. Theorem 1.3 will be proved in Section 5, and we also give some examples and corollaries there.

2. Spinor bundles on Kähler manifolds. Let (M, g, J) be a real $n = 2m$-dimensional Kähler manifold with Riemannian metric g, complex structure J and Kähler form $\Omega = g(J\cdot, \cdot)$ (see [Mor]). Fix a spin structure on M and denote the spinor bundle of M by ΣM (see [LM]). The Clifford contraction $c : TM \otimes \Sigma M \to \Sigma M$ is defined on each fiber by the Clifford multiplication on the spinor representation Σ. We always denote $c(X \otimes \varphi) = X \cdot \varphi$ and $\{e_i\}_{i=1,\ldots,n}$ a local orthonormal frame. The tangent bundle and cotangent bundle are identified by the metric g. Each k-form ω acts as an endomorphism of the spinor bundle, locally given by

$$\omega \cdot \varphi = \sum_{1 \leq i_1 < \cdots < i_k \leq 2m} \omega(e_{i_1}, \ldots, e_{i_k}) e_{i_1} \cdot \ldots \cdot e_{i_k} \cdot \varphi.$$

With respect to this action, the Kähler form is given locally by

$$\Omega = \frac{1}{2} \sum_{j=1}^{n} e_j \cdot J e_j.$$

Under the action of the Kähler form Ω, the spinor bundle splits into the orthogonal decomposition

$$\Sigma M = \bigotimes_{r=0}^{m} \Sigma_r M$$

with respect to the Hermitian scalar product $(\cdot, \cdot)$, where each $\Sigma_r M$ is an eigenbundle of Ω associated to the eigenvalue $\mu_r = i(2r - m)$.

For $r \in \{0, 1, \ldots, m\}$, denote by c_r the restriction of the Clifford contraction c to $TM \otimes \Sigma_r M$. Then

$$c = c_r^- + c_r^+ : \Gamma(TM \otimes \Sigma_r M) \to \Gamma(TM \otimes \Sigma_{r-1} M) \oplus \Gamma(TM \otimes \Sigma_{r+1} M),$$

where c_r^- and c_r^+, taking values in $\Sigma_{r-1} M$ and $\Sigma_{r+1} M$ respectively, are given by

$$c_r^+(X \otimes \varphi) = X^+ \cdot \varphi \quad \text{and} \quad c_r^-(X \otimes \varphi) = X^- \cdot \varphi$$

with $X^\pm = \frac{1}{2}(X \mp iJX)$ as before.

The *Dirac operator* is defined as the composition

$$\Gamma(\Sigma M) \xrightarrow{\nabla} \Gamma(T^*M \otimes \Sigma M) \xrightarrow{c} \Gamma(\Sigma M),$$

i.e., $D = c \circ \nabla$. Locally, it is given by $D = \sum_{i=1}^{n} e_i \cdot \nabla_{e_i}$. It is well known that the Dirac operator satisfies the Schrödinger–Lichnerowicz formula

$$D^2 = \nabla^*\nabla + \tfrac{1}{4}S, \tag{2.1}$$

where $\nabla^*\nabla$ is the Laplacian operator on the spinor bundle and S is the scalar curvature of M. When restricted to $\Sigma_r M$, the Dirac operator has a natural decomposition

$$D = D_+ + D_- : \Gamma(\Sigma_r M) \to \Gamma(\Sigma_{r+1} M) \oplus \Gamma(\Sigma_{r-1} M),$$

where D_+ and D_- are defined by

$$D_+ := c_r^+ \circ \nabla \quad \text{and} \quad D_- := c_r^- \circ \nabla, \tag{2.2}$$

and satisfy the relations

$$(D_+)^2 = 0, \quad (D_-)^2 = 0, \quad D_+D_- + D_-D_+ = D^2.$$

On the spinor bundle there is a canonical $\mathbb{C}$-anti-linear real or quaternionic structure $\jmath : \Sigma M \to \Sigma M$ such that

$$\jmath^2 = (-1)^{m(m+1)/2}, \quad \jmath : \Sigma_r M \to \Sigma_{m-r} M, \tag{2.3}$$

$$\jmath(Z \cdot \varphi) = \bar{Z} \cdot \jmath(\varphi) \quad \forall Z \in \Gamma(TM^{\mathbb{C}}), \tag{2.4}$$

$$(\jmath\varphi, \jmath\phi) = (\varphi, \phi), \quad D \circ \jmath = \jmath \circ D, \quad D_\pm \circ \jmath = \jmath \circ D_\mp. \tag{2.5}$$

On each subbundle $\Sigma_r M$ $(r = 0, \dots, m)$, the *Kähler twisted operator* for any $X \in \Gamma(TM)$ and any $\varphi \in \Sigma_r M$ is defined by

$$(T_r)_X\varphi = \nabla_X\varphi + \frac{1}{2(r+1)} X^- \cdot D_+\varphi + \frac{1}{2(m-r+1)} X^+ \cdot D_-\varphi \tag{2.6}$$

(see [Kir3, Pi]). As in [Pi], a straightforward calculation leads to

Lemma 2.1. *For any $\varphi \in \Gamma(\Sigma_r M)$,*

$$|T_r\varphi|^2 = |\nabla\varphi|^2 - \frac{1}{2(r+1)}|D_+\varphi|^2 - \frac{1}{2(m-r+1)}|D_-\varphi|^2. \tag{2.7}$$

For the eigenvalues of the Dirac operator on a compact Kähler spin manifold, Kirchberg [Kir3] proved

Lemma 2.2. *Let M be a compact Kähler spin manifold of complex dimension m. Then for any eigenvalue $\lambda \neq 0$ of the Dirac operator, there exists an eigenspinor ψ associated with λ such that $\psi = \psi_{r-1} + \psi_r$ for some $r \in \{1, \dots, m\}$, and the components $\psi_{r-1} \in \Gamma(\Sigma_{r-1})$ and $\psi_r \in \Gamma(\Sigma_r)$ have* *the properties:*

$$\|\psi_{r-1}\| = \|\psi_r\|, \tag{2.8}$$

$$D_+\psi_r = 0, \quad D_+\psi_{r-1} = \lambda\psi_r, \tag{2.9}$$

$$D_-\psi_{r-1} = 0, \quad D_-\psi_r = \lambda\psi_{r-1}. \tag{2.10}$$

For simplicity, we call such an eigenspinor an *eigenspinor of type* $(r, r-1)$.

REMARK 2.3. Some of the components $\psi_r \in \Gamma(\Sigma_r)$ may be trivial. Since there exists an anti-linear parallel map $\jmath$ on ΣM commuting with the Clifford multiplication, $\overline{\psi} = \jmath\psi_r + \jmath\psi_{r-1}$ is an eigenspinor of type $(m-r, m-r+1)$ by (2.5). Keeping this in mind, in the following we always assume that

$$\begin{aligned} &r \in \{(m+1)/2, (m+1)/2+1, \dots, m\} && \text{if } m \text{ is odd},\\ &r \in \{m/2+1, m/2+2, \dots, m\} && \text{if } m \text{ is even}. \end{aligned}$$

3. Kirchberg type spectrum estimates

3.1. Proof of Theorem 1.1. For any spinor ψ and any tangent vector fields X and Y we define, in the classical way, the symmetric bilinear tensor Q_ψ on the complement of the zero set of ψ by

$$Q_\psi(X,Y) = \tfrac{1}{2}\Re(X\cdot\nabla_Y\psi + Y\cdot\nabla_X\psi, \psi/|\psi|^2), \tag{3.1}$$

where $\Re$ is the real part of the scalar product. If $\psi = 0$, we make a convention that $Q_\psi = 0$. For any real parameter $t \in \mathbb{R}$, Consider the differential operator (modified *Kähler twistor operator* [Pi]), $\mathcal{P}^t : \Gamma(\Sigma M) \to \Gamma(TM \otimes \Sigma M)$, which is locally defined by $\mathcal{P}^t\psi = \sum_{i=1}^n e_i \otimes \mathcal{P}^t_{e_i}\psi$ with

$$\begin{aligned} \mathcal{P}^t_X\psi := \nabla_X\psi + \frac{1}{2(r+1)}X^-\cdot D_+\psi + \frac{1}{2(m-r+1)}X^+\cdot D_-\psi\\ + t\sum_{i=1}^n Q_\psi(X,e_j)e_j\cdot\psi. \end{aligned} \tag{3.2}$$

LEMMA 3.1. *Let* $\psi \in \Gamma(\Sigma_r M)$, $r \in \{0,1,\dots,m\}$. *Then*

$$\begin{aligned} \int_M |\mathcal{P}^t\psi|^2 = \int_M |D\psi|^2 - \frac{S}{4}|\psi|^2 - a_r|D_+\psi|^2 - b_r|D_-\psi|^2 + t^2|Q_\psi|^2|\psi|^2\\ - 2t\int_M |Q_\psi|^2|\psi|^2, \end{aligned} \tag{3.3}$$

where $a_r = \frac{1}{2(r+1)}$ *and* $b_r = \frac{1}{2(m-r+1)}$.

Proof. Let $Q_{\psi,i,j} = Q_\psi(e_i, e_j)$. From (2.7) we directly calculate

$$
\begin{aligned}
|\mathcal{P}^t\psi|^2 &= |T_r\psi|^2 + t^2 \sum_{i,j,k} Q_{\psi,i,j} Q_{\psi,i,k}(e_j\cdot\psi, e_k\cdot\psi) \\
&\quad + 2t\sum_{i,j} Q_{\psi,i,j}\Re((T_r)_{e_i}\psi, e_j\cdot\psi) \\
&= |T_r\psi|^2 + t^2\sum_{i,j} Q^2_{\psi,i,j}|\psi|^2 + 2t\sum_{i,j} Q_{\psi,i,j}\Re(\nabla_{e_i}\psi, e_j\cdot\psi) \\
&\quad + 2a_r t\sum_{i,j} Q_{\psi,i,j}\Re(e_i^-\cdot D_+\psi, e_j^+\cdot\psi + e_j^-\cdot\psi) \\
&\quad + 2b_r t\sum_{i,j} Q_{\psi,i,j}\Re(e_i^+\cdot D_-\psi, e_j^+\cdot\psi + e_j^-\cdot\psi) \\
&= |T_r\psi|^2 + t^2|Q_\psi|^2|\psi|^2 - 2t|Q_\psi|^2|\psi|^2 \\
&= |\nabla\psi|^2 - a_r|D_+\psi|^2 - b_r|D_-\psi|^2 + t^2|Q_\psi|^2|\psi|^2 - 2t|Q_\psi|^2|\psi|^2.
\end{aligned}
$$

Using (2.1) and integrating both sides of (3.4) over M we obtain (3.3).

Now we can complete the proof of Theorem 1.1 as follows. Let $\psi = \psi_{r-1} + \psi_r$ be an eigenspinor as in Lemma 2.2. Applying (3.3) to ψ_{r-1} and ψ_r respectively, we deduce that

$$\int_M \big[\lambda^2(1-a_{r-1}) - \tfrac{1}{4}S + t^2|Q_{\psi_{r-1}}|^2 - 2t|Q_{\psi_{r-1}}|^2\big]|\psi_{r-1}|^2 \ge 0 \tag{3.4}$$

and

$$\int_M \big[\lambda^2(1-b_r) - \tfrac{1}{4}S + t^2|Q_{\psi_r}|^2 - 2t|Q_{\psi_r}|^2\big]|\psi_r|^2 \ge 0. \tag{3.5}$$

Clearly, (3.4) implies that

$$\int_M \lambda^2(1-a_{r-1})|\psi_{r-1}|^2 \ge \int_M \big(S/4 + [1-(t-1)^2]|Q_{\psi_{r-1}}|^2\big)|\psi_{r-1}|^2 \tag{3.6}$$

$$\ge \int_M \inf_M \big(S/4 + [1-(t-1)^2]|Q_{\psi_{r-1}}|^2\big)|\psi_{r-1}|^2. \tag{3.7}$$

It follows that

$$\lambda^2(1-a_{r-1}) \ge \inf_M \big(S/4 + [1-(t-1)^2]|Q_{\psi_{r-1}}|^2\big), \quad \forall t\in\mathbb{R}. \tag{3.8}$$

Similarly, we obtain

$$\lambda^2(1-b_r) \ge \inf_M \big(S/4 + [1-(t-1)^2]|Q_{\psi_r}|^2\big), \quad \forall t\in\mathbb{R}. \tag{3.9}$$

By computing the maximum of the right sides with respect to the parameter t, Theorem 1.1 is proved. ∎

3.2. Proof of Corollary 1.2. For any eigenspinor $\psi = \psi_{r-1} + \psi_r$ of type $(r-1, r)$ the inequality (1.9) holds, where r is as in Remark 2.3. If m is odd, we find that since the function $f(x) = x/(4x-2)$ is strictly decreasing on $[1, \infty)$, we have $f(r) \geq f((m+1)/2)$ for $(m+1)/2 \leq r \leq m$. It follows that

$$\frac{r}{4r-2} \inf_M (S + 4|Q_{\psi_{r-1}}|^2) \geq \frac{m+1}{4m} \inf_M (S + 4|Q_{\psi_{r-1}}|^2) \tag{3.10}$$

for $(m+1)/2 \leq r \leq m$. If m is even, we consider the strictly increasing function $g(x) = \frac{m-x+1}{2(2m-2x+1)}$ on $[1, \infty)$ and find that $g(r) \geq g((m+2)/2)$ for $(m+2)/2 \leq r \leq m$. Then

$$\frac{m-r+1}{2(2m-2r+1)} \inf_M (S + 4|Q_{\psi_r}|^2) \geq \frac{m}{4(m-1)} \inf_M (S + 4|Q_{\psi_r}|^2) \tag{3.11}$$

for $(m+2)/2 \leq r \leq m$. Altogether we arrive at the first part of Corollary 1.2.

To prove the remaining limiting case of (1.9), we denote by A_ψ the field of symmetric endomorphisms associated with the field of quadratic forms

$$Q_\psi(X) := \Re(X \cdot \nabla_X \psi, \psi/|\psi|^2),$$

that is, for any tangent vector field X,

$$A_\psi(X) := \sum_{i=1}^{2m} Q_\psi(X, e_i) e_i.$$

Then the trace of A_ψ satisfies $\operatorname{tr} A_\psi = \Re(D\psi, \psi/|\psi|^2)$. Clearly $\operatorname{tr} A_\psi = \lambda$ if ψ is an eigenspinor associated with λ.

Now assume that an eigenvalue $\lambda^* \neq 0$ of the Dirac operator satisfies

$$(\lambda^*)^2 = \begin{cases} \dfrac{m+1}{4m} \inf\limits_M (S + 4|Q_{\psi_{r-1}}|^2) & \text{if } m \text{ is odd,} \\ \dfrac{m}{4(m-1)} \inf\limits_M (S + 4|Q_{\psi_r}|^2) & \text{if } m \text{ is even,} \end{cases}$$

where $\psi = \psi_{r-1} + \psi_r$ is an eigenspinor of type $(r-1, r)$ associated to λ^* with $r \in \{(m+1)/2, (m+1)/2+1, \ldots, m\}$ if m is odd, and $r \in \{m/2+1, m/2+2, \ldots, m\}$ if m is even.

We only handle the case of m odd, i.e., conclusion (i) of Corollary 1.2; the proof of the other case is completely similar. In the present situation, from the proof of Corollary 1.2 we find that $r = (m+1)/2$. Taking $\lambda = \lambda^*$, $t = 1$ and $r = (m+1)/2$ in (3.6) and (3.7) we arrive at

$$S/4 + |Q_{\psi_{(m-1)/2}}|^2 \equiv \inf_M (S/4 + |Q_{\psi_{(m-1)/2}}|^2).$$

Then equality holds in (3.4) for $t = 1$. It follows that

$$\int_M |\mathcal{P}^1 \psi_{(m-1)/2}|^2 = 0,$$

and thus $\mathcal{P}^1\psi_{(m-1)/2} = 0$. Finally, (1.10) immediately follows from this equality, (2.9) and (2.10). ∎

REMARK 3.2. Taking $t = 0$ in (3.4) we recover (1.3). Indeed, in this case (3.10) and (3.11) become

$$\frac{r}{4r-2}\inf_M S \geq \frac{m+1}{4m}\inf_M S \quad \text{and} \quad \frac{m-r+1}{2(2m-2r+1)}\inf_M S \geq \frac{m}{4(m-1)}\inf_M S,$$

respectively. Similarly, the limit cases in [Kir4] and [Gau] can be obtained.

4. Legendrian submanifolds of Sasakian manifolds. We begin with some notation of [BoyG]. A *Sasakian manifold* $(M^{2m+1}, \phi, \xi, \eta, g)$ is an odd-dimensional Riemannian manifold equipped with a type $(1,1)$-tensor ϕ, a unit Killing vector field ξ and a 1-form η such that for any vector fields X, Y,

$$\eta(\xi) = 1, \quad \phi^2(X) = -X + \eta(X)\xi, \quad g(\phi(X), \phi(Y)) = g(X,Y) - \eta(X)\eta(Y),$$
$$(\nabla_X \phi)(Y) = g(X,Y)\xi - \eta(Y)X.$$

A submanifold L of a Sasakian manifold M^{2m+1} is *Legendrian* if $\dim L = m$ and the inclusion map $i : L \hookrightarrow M$ satisfies $i^*\eta = 0$. Let L be equipped with the induced Riemannian metric i^*g. Suppose now that both manifolds have spin structures and denote their spinor bundles by ΣM and ΣL with Hermitian inner product $\langle\cdot,\cdot\rangle$ and $\langle\cdot,\cdot\rangle_L$ respectively. In particular, M and L are both orientable. Therefore there exists an induced spin structure on the normal bundle NL such that the restricted spinor bundle $\Sigma M|_L$ can be identified with $\Sigma L \otimes \Sigma N$ (see [Mil]), where ΣN is the spin bundle of NL with the induced Hermitian inner product $\langle\cdot,\cdot\rangle$. To compare the different spin bundles on the submanifold L, we denote by "$\cdot_L$", "$\cdot_N$" and "$\cdot$" the Clifford multiplications of L, NL and M respectively.

Assume that m is odd. We can require that the above identification $\Sigma M|_L \equiv \Sigma L \otimes \Sigma N$ is unitary by suitably normalizing the associated Hermitian inner products. Then for all $\varphi \in \Sigma M|_L = \Sigma L \otimes \Sigma N$ we have

$$\begin{cases} X \cdot \varphi = \{X \cdot_L \otimes (\mathrm{Id}_{\Sigma^+ N} - \mathrm{Id}_{\Sigma^- N})\}\varphi & \forall X \in TL, \\ Y \cdot \varphi = (\mathrm{Id} \otimes Y \cdot_N)\varphi & \forall Y \in NL, \end{cases}$$

where $\Sigma N = \Sigma^+ N \oplus \Sigma^- N$ is the orthogonal decomposition induced by the complex volume form (cf. [Bar1, GM]). Denote by $\overline{\nabla}$ (resp. ∇) the Levi-Civita connection of (M, g) (resp. (L, i^*g)). They can be lifted to the spinorial connections $\overline{\nabla}$ on ΣM and $\nabla := \nabla^{\Sigma L \otimes \Sigma N}$ on $\Sigma L \otimes \Sigma N$, respectively. Moreover the following Gauss-type formula holds:

$$\overline{\nabla}_X \varphi = \nabla_X \varphi + \frac{1}{2}\sum_{u=1}^{m} E_u \cdot II(X, E_u) \cdot \varphi \tag{4.1}$$

for any vector field X tangent to L and any section φ of $\Sigma M|_L$, where $(E_u)_{1\le u\le m}$ stands for a local positively oriented orthonormal basis of TL, and II is the second fundamental form of L in M. The Dirac–Witten operator $\overline{D}$ (see [Wit]) and the twisted Dirac operator D^T (see [LM]) on the Legendrian submanifold L are locally given by

$$\overline{D} := \sum_{u=1}^{m} E_u \cdot \overline{\nabla}_{E_u} \quad \text{and} \quad D^T := \sum_{u=1}^{m} E_u \cdot \nabla_{E_u},$$

respectively. Note that both operators acting on sections of $\Sigma M|_L$ are elliptic and that D^T is also formally self-adjoint with respect to the L^2-Hermitian inner product $\langle\cdot,\cdot\rangle_{L^2}$. From (4.1), it is easy to see that

$$\overline{D} = D^T - mH/2 \tag{4.2}$$

(cf. [Bar1]), where $H = \frac{1}{m}\operatorname{tr}(II)$ is the mean curvature vector field of the immersion i. Moreover, the following lemma holds:

Lemma 4.1. *For any section φ of $\Sigma M|_L$,*

$$\overline{D}^2\varphi = (D^T)^2\varphi - \frac{m^2|H|^2}{4}\varphi - \frac{m}{2}\sum_{u=1}^{m} E_u \cdot \nabla^N_{E_u} H \cdot \varphi, \tag{4.3}$$

where $\nabla^N H$ denotes the normal covariant derivative of H.

Proof. Using (4.2) we calculate

$$\begin{aligned}
\overline{D}^2\varphi &= \overline{D}D^T\varphi - \frac{m}{2}\overline{D}(H\cdot\varphi)\\
&= (D^T)^2\varphi - \frac{m}{2}H\cdot D^T\varphi - \frac{m}{2}\sum_{u=1}^{m} E_u\cdot\overline{\nabla}_{E_u}(H\cdot\varphi)\\
&= (D^T)^2\varphi - \frac{m}{2}H\cdot\Big(\overline{D} + \frac{m}{2}H\Big)\varphi - \frac{m}{2}\sum_{u=1}^{m} E_u\cdot H\cdot\overline{\nabla}_{E_u}\varphi\\
&\quad - \frac{m}{2}\sum_{u=1}^{m} E_u\cdot\overline{\nabla}_{E_u}H\cdot\varphi\\
&= (D^T)^2\varphi + \frac{m^2}{4}|H|^2\varphi - \frac{m}{2}\sum_{u=1}^{m} E_u\cdot\overline{\nabla}_{E_u}H\cdot\varphi\\
&= (D^T)^2\varphi + \frac{m^2}{4}|H|^2\varphi - \frac{m}{2}\sum_{u=1}^{m} E_u\cdot\sum_{v=1}^{m} g(\overline{\nabla}_{E_u}H, E_v)E_v\cdot\varphi\\
&\quad - \frac{m}{2}\sum_{u=1}^{m} E_u\cdot\nabla^N_{E_u}H\cdot\varphi
\end{aligned}$$

$$= (D^T)^2\varphi + \frac{m^2}{4}|H|^2\varphi + \frac{m}{2}\sum_{u,v=1}^{m} g(H, \overline{\nabla}_{E_u}E_v)E_u \cdot E_v \cdot \varphi$$
$$- \frac{m}{2}\sum_{u=1}^{m} E_u \cdot \nabla^N_{E_u} H \cdot \varphi$$
$$= (D^T)^2\varphi + \frac{m^2}{4}|H|^2\varphi + \frac{m}{2}\sum_{u=1}^{m} g(H, \overline{\nabla}_{E_u}E_u)\varphi$$
$$- \frac{m}{2}\sum_{u=1}^{m} E_u \cdot \nabla^N_{E_u} H \cdot \varphi$$
$$= (D^T)^2\varphi + \frac{m^2}{4}|H|^2\varphi - \frac{m}{2}\sum_{u=1}^{m} g\big(H, II(E_u, E_u)\big)\varphi$$
$$- \frac{m}{2}\sum_{u=1}^{m} E_u \cdot \nabla^N_{E_u} H \cdot \varphi$$
$$= (D^T)^2\varphi + \frac{m^2}{4}|H|^2\varphi - \frac{m^2}{2}|H|^2\varphi - \frac{m}{2}\sum_{u=1}^{m} E_u \cdot \nabla^N_{E_u} H \cdot \varphi$$
$$= (D^T)^2\varphi - \frac{m^2}{4}|H|^2\varphi - \frac{m}{2}\sum_{u=1}^{m} E_u \cdot \nabla^N_{E_u} H \cdot \varphi. \blacksquare$$

LEMMA 4.2. *Let (L, g) be a Legendrian submanifold of a Sasakian manifold (M, ϕ, ξ, η, g). Then for any local orthonormal frame $(E_u)_{1\le u\le m}$ on L, $\{E_1, \ldots, E_m, \phi(E_1), \ldots, \phi(E_m), \xi\}$ is an adapted local orthonormal frame on M.*

Proof. This results directly from the formula

$$d\eta(X, Y) = g(X, \phi(Y))$$

for all vector fields X, Y on M (see [Bla, Theorem 6.3]). ■

Recall that the *fundamental* 2*-form* Φ on M is defined by $\Phi := g\big(\cdot, \phi(\cdot)\big)$. The tangent bundle and cotangent bundle are identified via the metric g. With respect to the local frame above, the fundamental 2-form Φ can be written as

$$\Phi = \sum_{u=1}^{m} \phi(E_u) \wedge E_u.$$

Assume that $(E_u)_{1\le u\le 2m+1}$ is a local orthonormal frame. Under the action of $\Phi = \sum_{u=1}^{m} \phi(E_u) \cdot E_u$ the spinor bundle ΣM of the Sasakian spin manifold $(M^{2m+1}, \phi, \xi, \eta, g)$ splits into the orthogonal direct sum $\Sigma M = \Sigma_0 M \oplus \Sigma_1 M \oplus \cdots \oplus \Sigma_m M$ with

$$\Phi|_{\Sigma_r M} = \sqrt{-1}\,(2r-m)\,\mathrm{Id}, \qquad \dim(\Sigma_r M) = \binom{m}{r}, \tag{4.4}$$

$$\xi|_{\Sigma_r M} = \sqrt{-1}\,(-1)^{m+r}\,\mathrm{Id}, \qquad r \in \{0,1,\ldots,m\}. \tag{4.5}$$

Moreover, a direct computation leads to

LEMMA 4.3. *For every vector field Z tangent to M with $\eta(Z)=0$,*

$$P_+(Z)\cdot \Sigma_r M \subset \Sigma_{r+1}M \quad \text{and} \quad P_-(Z)\cdot \Sigma_r M \subset \Sigma_{r-1}M \tag{4.6}$$

with the convention $\Sigma_r M = M \times \{0\}$ for $r \notin \{0,1,\ldots,m\}$, where $P_\pm(Z) := \frac{1}{2}\big(Z \pm \sqrt{-1}\,\phi(Z)\big)$.

DEFINITION 4.4. Let $(M^{2m+1},\phi,\xi,\eta,g)$ be a Sasakian spin manifold with $m \geq 3$ and $m \equiv 1 \bmod 2$. Let $l := (m-1)/2$. A nontrivial section $\psi = \psi_l + \psi_{l+1}$ of ΣM is called a *μ-Sasakian Killing spinor* with $\psi_l \in \Sigma_{(m-1)/2}M$ and $\psi_{l+1} \in \Sigma_{(m+1)/2}M$ if

$$\begin{cases} \overline{\nabla}_Z \psi_l = -\dfrac{\mu}{m+1} p_-(Z)\psi_{l+1} + \dfrac{1}{2}\phi(Z)\cdot\xi\cdot\psi_l, \\[2mm] \overline{\nabla}_Z \psi_{l+1} = -\dfrac{\mu}{m+1} p_+(Z)\psi_l + \dfrac{1}{2}\phi(Z)\cdot\xi\cdot\psi_{l+1}. \end{cases} \tag{4.7}$$

for all vector fields $Z \in \Gamma(TM)$ with $\eta(Z)=0$, where μ is a certain nonzero real number.

REMARK 4.5. Since $\widetilde{\nabla}_Z\psi := \overline{\nabla}_Z\psi - \frac{1}{2}\phi(Z)\cdot\xi\cdot\psi$ commutes with the fundamental form Φ (cf. [Kim]), we deduce from (4.5) and (4.6) that (4.7) is equivalent to

$$\widetilde{\nabla}_Z\psi = -\frac{\mu}{2(m+1)} Z\cdot\psi - (-1)^{(m+1)/2}\frac{\mu}{2(m+1)}\phi(Z)\cdot\xi\cdot\psi. \tag{4.8}$$

This exactly gives the definition of a Sasakian spin manifold with characteristic number μ in [Kim, Definition 3.4] with the property that if φ satisfies (4.8) then $\xi\cdot\varphi$ is also a Sasakian Killing spinor with characteristic number $-\mu$.

REMARK 4.6. Recall that a Sasakian manifold $(M^{2m+1},\phi,\xi,\eta,g)$ is called *η-Einstein* if the scalar curvature S is constant and the Ricci curvature satisfies

$$\mathrm{Ric} = \left(\frac{S}{2m} - 1\right) g + \left(2m+1-\frac{S}{2m}\right)\eta\otimes\eta.$$

The existence of a μ-Sasakian Killing spinor [Kim, Proposition 3.5] implies that the manifold $(M^{2m+1},\phi,\xi,\eta,g)$ must be η-Einstein with $S = 4m\mu^2/(m+1) - 2m$. Moreover, if M itself is also Einstein and $\mu = (-1)^{l+1} \times (m+1)$, then (4.7) reduces to the Killing spinor equation

$$\overline{\nabla}_X\psi = \frac{(-1)^l}{2} X\cdot\psi, \qquad X \in \Gamma(TM).$$

5. Proof of Theorem 1.3 and some related results

5.1. Proof of Theorem 1.3. Denote by $\mathcal{SK}(\mu)$ the space of μ-Sasakian Killing spinors on $(M^{2m+1}, \phi, \xi, \eta, g)$. A straightforward computation shows that $\mathcal{SK}(\mu_1) \cap \mathcal{SK}(\mu_2) = \{0\}$ if $\mu_1 \neq \mu_2$. For any μ-Sasakian Killing spinor $\psi = \psi_l + \psi_{l+1}$, we estimate the Rayleigh quotient

$$Q((D^T)^2, \psi) := \frac{\int_L \mathrm{Re}\langle (D^T)^2\psi, \psi\rangle v_g}{\int_L |\psi| v_g}. \tag{5.1}$$

Let $(E_u)_{1\le u\le m}$ be a local orthonormal frame on TL. Then $\{E_1, \dots, E_m, \phi(E_1), \dots, \phi(E_m), \xi\}$ is a local orthonormal frame on TM. Note that

$$\sum_{u=1}^m p_-(E_u)\cdot p_+(E_u) = -\frac{\sqrt{-1}}{2}\Phi - \frac{m}{2}, \quad \sum_{u=1}^m p_+(E_u)\cdot p_-(E_u) = \frac{\sqrt{-1}}{2}\Phi - \frac{m}{2}.$$

From (4.4), (4.5) and (4.8), we deduce

$$\begin{aligned} \overline{D}\psi_l &= \sum_{u=1}^m E_u \cdot \overline{\nabla}_{E_u}\psi_l \\ &= -\frac{\mu}{m+1}\sum_{u=1}^m E_u\cdot p_-(E_u)\cdot\psi_{l+1} + \frac{1}{2}\sum_{u=1} E_u\cdot\phi(E_u)\cdot\xi\cdot\psi_l \\ &= -\frac{\mu}{m+1}\sum_{u=1}^m p_+(E_u)\cdot p_-(E_u)\cdot\psi_{l+1} - \frac{(-1)^{m+l}}{2}\sqrt{-1}\,\Phi\cdot\psi_l \\ &= -\frac{\mu}{m+1}\left(\frac{\sqrt{-1}}{2}\Phi\cdot\psi_{l+1} - \frac{m}{2}\psi_{l+1}\right) + \frac{(-1)^l}{2}\psi_l \\ &= \frac{\mu}{2}\psi_{l+1} + \frac{(-1)^l}{2}\psi_l. \end{aligned} \tag{5.2}$$

A similar computation yields

$$\begin{aligned} \overline{D}\psi_{l+1} &= \sum_{u=1}^m E_u \cdot \overline{\nabla}_{E_u}\psi_{l+1} \\ &= -\frac{\mu}{m+1}\sum_{u=1}^m E_u\cdot p_+(E_u)\cdot\psi_{l+1} + \frac{1}{2}\sum_{u=1} E_u\cdot\phi(E_u)\cdot\xi\cdot\psi_{l+1} \\ &= -\frac{\mu}{m+1}\sum_{u=1}^m p_-(E_u)\cdot p_+(E_u)\cdot\psi_{l+1} + \frac{(-1)^{m+l}}{2}\sqrt{-1}\,\Phi\cdot\psi_{l+1} \\ &= -\frac{\mu}{m+1}\left(-\frac{\sqrt{-1}}{2}\Phi\cdot\psi_l - \frac{m}{2}\psi_l\right) + \frac{(-1)^l}{2}\psi_{l+1} \\ &= \frac{\mu}{2}\psi_l + \frac{(-1)^l}{2}\psi_{l+1}. \end{aligned} \tag{5.3}$$

Thus we obtain

$$
\begin{aligned}
(5.4)\qquad \overline{D}^2\psi_l &= \frac{\mu}{2}\overline{D}\psi_{l+1} + \frac{(-1)^l}{2}\overline{D}\psi_l \\
&= \frac{\mu}{2}\left(\frac{\mu}{2}\psi_l + \frac{(-1)^l}{2}\psi_{l+1}\right) + \frac{(-1)^l}{2}\left(\frac{\mu}{2}\psi_{l+1} + \frac{(-1)^l}{2}\psi_l\right) \\
&= \frac{\mu^2+1}{4}\psi_l + \frac{(-1)^l\mu}{2}\psi_{l+1}
\end{aligned}
$$

and

$$
\begin{aligned}
(5.5)\qquad \overline{D}^2\psi_{l+1} &= \frac{\mu}{2}\overline{D}\psi_l + \frac{(-1)^l}{2}\overline{D}\psi_{l+1} \\
&= \frac{\mu}{2}\left(\frac{\mu}{2}\psi_{l+1} + \frac{(-1)^l}{2}\psi_l\right) + \frac{(-1)^l}{2}\left(\frac{\mu}{2}\psi_l + \frac{(-1)^l}{2}\psi_{l+1}\right) \\
&= \frac{\mu^2+1}{4}\psi_{l+1} + \frac{(-1)^l\mu}{2}\psi_l.
\end{aligned}
$$

Combining (4.3), (5.4) and (5.5), we get

$$
(5.6)\qquad (D^T)^2\psi = \frac{(\mu+(-1)^l)^2}{4}\psi + \frac{m^2|H|^2}{4}\psi + \frac{m}{2}\sum_{u=1}^{m} E_u\cdot\nabla^N_{E_u}H\cdot\psi.
$$

Plugging (4.7) into (4.2) leads to

$$
Q((D^T)^2,\psi) = \frac{(\mu+(-1)^l)^2}{4} + \frac{m^2\int_L |H|^2\langle\psi,\psi\rangle v_g}{\int_L\langle\psi,\psi\rangle v_g}.
$$

Applying the Min-Max principle, we find that there are at least N eigenvalues λ of the twisted Dirac operator D^T. According to Remark 4.5, the one-to-one correspondence between $\mathcal{SK}(\mu)$ and $\mathcal{SK}(-\mu)$ implies $\dim(\mathcal{SK}(\mu)) = \dim(\mathcal{SK}(-\mu))$. Estimating the Rayleigh quotient on the vector space $\mathcal{SK}(\mu)\oplus\mathcal{SK}(-\mu)$ as above, we obtain the desired result. ∎

5.2. Examples and corollaries

EXAMPLE 5.1. A natural example of a Sasakian-Einstein manifold is the odd-dimensional round sphere S^{2m+1} (cf. [Gol]). Let $m \geq 3$ and $m \equiv 1 \mod 2$. By Remark 4.6, the space of μ-Sasakian Killing spinors on S^{2m+1} is the same as the space of Killing spinors with Killing number $(-1)^l/2$, where $\mu := (-1)^{l+1}(m+1)$ and $l := (m-1)/2$, that is,

$$
\mathcal{SK}(\mu) = \left\{\psi\in\Gamma(\Sigma S^{2m+1}) \,\middle|\, \nabla_X^{\Sigma S^{2m+1}}\psi = \frac{(-1)^l}{2}X\cdot\psi,\ \forall X\in\Gamma(TS^{2m+1})\right\}.
$$

Recall that the complex dimension of the space of $-\frac{1}{2}$- (or $\frac{1}{2}$-) Killing spinor on the round sphere S^n is $2^{[n/2]}$ (see [Bar2]). As a consequence of Theorem 1.3 we recover Bär's result [Bar1, Corollary 3.3].

COROLLARY 5.2. *Let L be a $(2l+1)$-dimensional closed Legendrian spin submanifold with $l \geq 1$ in the Sasakian-Einstein spin manifold S^{4l+3}. Let the normal bundle of L in S^{4l+3} carry the induced spin structure, and H be the mean curvature vector field of L in S^{4l+3}. Then the twisted Dirac operator D^T on L has at least 2^{2l+2} eigenvalues λ counted with multiplicity satisfying*

$$\lambda^2 \leq \frac{(2l+1)^2}{4} + \frac{(2l+1)^2}{4\,\mathrm{Vol}(L)} \int_L |H|^2 v_g. \tag{5.7}$$

EXAMPLE 5.3. If L in Corollary 5.2 is the canonical embedding $S^{2l+1} \to S^{4l+3}$, which is minimal, we claim that (5.7) becomes

$$\lambda^2 = (2l+1)^2/4. \tag{5.8}$$

This means that the estimate in (1.13) or (5.7) is sharp. In fact, by Corollary 5.2 there are at least 2^{2l+2} eigenvalues λ counted with multiplicity satisfying

$$\lambda^2 \leq (2l+1)^2/4. \tag{5.9}$$

On the other hand, the normal bundle is trivial with flat connection. By identifying the normal spin bundle ΣN with $S^{2l+1} \times \Sigma_{2l+2}$, we find that the twisted Dirac eigenvalues are the same as those of the Dirac operator. Since the smallest eigenvalues of the Dirac operator on S^n are $\pm n/2$ and each has multiplicity $2^{[n/2]}$ (see [Gin3, Theorem 2.1.3]), $\pm(2l+1)/2$ are the smallest twisted Dirac eigenvalues on S^{2l+1} with multiplicity

$$2^{[(2l+1)/2]} \cdot 2^{[(2l+2)/2]} = 2^{2l+1},$$

where the second factor on the left side is contributed by the dimension of Σ_{2l+2}. Therefore, there are precisely 2^{2l+2} eigenvalues counted with multiplicity satisfying (5.9). This proves the desired claim.

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