FUNDAMENTA
MATHEMATICAE
236 (2017)

Remarks on compactifications of pseudofinite groups

by

Anand Pillay (Notre Dame, IN)

Abstract. We discuss the Bohr compactification of a pseudofinite group, motivated by a question of Boris Zilber. We point out that (i) the Bohr compactification of an ultraproduct of finite simple groups is trivial, and (ii) the "definable" Bohr compactification of any pseudofinite group G, relative to an ambient nonstandard model of set theory in which G is definable, is commutative-by-profinite.

1. Introduction. By a *pseudofinite group* G we mean a model of the theory of finite groups, in the group language. An example of a pseudofinite group is an ultraproduct of a family G_i for $i \in \mathbb{N}$ of finite groups, and every pseudofinite group is elementarily equivalent to such an ultraproduct. By a *compact simple Lie group* we mean a compact Lie group of positive dimension which is noncommutative and has no proper nontrivial normal closed subgroups, other than possibly coming from a finite centre. In [9, Section 5.3, Problem 2] Zilber asks the following question, motivated apparently by physics:

Question 1.1. Is there an ultraproduct G of a family $(G_i)_{i\in\mathbb{N}}$ of finite groups and a surjective homomorphism from G to a compact simple Lie group?

It is natural to ask the slightly "weaker" question:

Question 1.2. Is there a *pseudofinite group* G and a surjective homomorphism from G to a compact simple Lie group?

In [9, Section 3], Zilber introduces a formalism of "structural approximation" and Question 1.1 is, assuming CH, supposed to be the same as his question of whether a compact simple Lie group can be "structurally

2010 *Mathematics Subject Classification*: Primary 03C20; Secondary 22C05.
Key words and phrases: pseudofinite group, Bohr compactification, definable.
Received 22 February 2016; revised 6 July 2016.
Published online 26 August 2016.

DOI: 10.4064/fm275-7-2016

approximated" by a sequence $(G_i,\ i<\omega)$ of finite groups. We will not really engage with Zilber's notion of structural approximation, but instead use the usual formalism of group compactifications of (discrete) groups (see [1, Chapter 2]) as well as its model-theoretic treatment in [5].

DEFINITION 1.3. By a *group compactification* of a (discrete) group G we mean a compact (Hausdorff) group C and a homomorphism from G into C with dense image.

In this paper we will always understand compactifications to mean group compactifications. So an apparently even weaker question is:

QUESTION 1.4. Is there a pseudofinite group with a compactification which is a compact simple Lie group?

REMARK 1.5. Questions 1.2 and 1.4 are equivalent, in that a positive answer to one gives a positive answer to the other. Moreover, assuming some set theory such as CH, each is equivalent to Question 1.1.

Proof. Clearly a positive answer to Question 1.2 gives a positive answer to Question 1.4. Conversely, suppose that G is pseudofinite and f is a homomorphism from G into a compact simple Lie group C such that $f(C)$ is dense in G. Let M_0 be the structure with universe G, and relations the group operation on G as well as all subsets of G. Let M_0^* be a sufficiently saturated elementary extension of M and G^* the corresponding extension of G. By the proof of [5, Proposition 3.4], f extends to a surjective homomorphism f^* from G^* to C, so as G^* is also a pseudofinite group we get a positive answer to Question 1.2.

Now for the "moreover" clause. As an ultraproduct of finite groups is pseudofinite, a positive answer to Question 1.1 gives a positive answer to Question 1.2. Now suppose that G is a pseudofinite group with a surjective homomorphism f from G to a compact simple Lie group. By the argument in the first part of the proof we may assume that G is ω_1-saturated, in the group language. Under CH we find an elementary substructure H of G which has cardinality $\aleph_1$, is ($\aleph_1$-) saturated and such that $f|H : H \to C$ is surjective. As H is pseudofinite, H is elementarily equivalent to an ultraproduct of finite groups. Again assuming CH such an ultraproduct has to be saturated of cardinality $\aleph_1$, hence isomorphic to H, and we obtain a positive answer to Question 1.1. ∎

The point of the above discussion is to show that Question 1.1 is really about compactifications in the classical sense. Among compactifications of a group G there will be a universal one, namely a compactification $f : G \to C$ such that for every compactification $h : G \to D$ there is a unique continuous surjection $g : C \to D$ such that $h = g \circ f$. This universal compactification is called the *Bohr compactification* of G and denoted bG.

We refer to [7] for background on the structure of compact (Lie) groups, but let us mention a few key facts we will be using: Any (connected) compact group is an inverse limit of (connected) compact Lie groups. The connected component C^0 of a compact group C is the intersection of all open subgroups of finite index (and is of course connected). The quotient C/C^0 is profinite and is the maximal profinite quotient of C. When C is compact Lie, C^0 has finite index in C. A compact Lie group C is defined to be *semisimple* if C is connected and has no positive-dimensional closed abelian normal subgroup. Semisimplicity of the compact Lie group is equivalent to it being an almost direct product of finitely many compact simple Lie groups. Any connected compact Lie group is the direct product of the connected component of its centre and a semisimple compact Lie group.

LEMMA 1.6. *Let G be any group. Then the following are equivalent:*

(i) $(bG)^0$ *is commutative.*
(ii) *There is no compactification C of G such that C is a compact Lie group and C^0 is semisimple.*

Proof. If bG is the inverse limit of a directed system $(L_i)_i$ of compact Lie groups, then clearly $(bG)^0$ is the inverse limit of the L_i^0. So $(bG)^0$ is commutative iff each L_i^0 is commutative iff no L_i^0 has a semisimple image. This suffices. ∎

So the following conjecture is equivalent to a negative answer to Question 1.4 (hence also to Questions 1.1, 1.2), modulo passing to connected components and allowing a finite product of compact simple Lie groups in place of a single one.

CONJECTURE 1.7 (Tentative). If G is a pseudofinite group then $(bG)^0$ is commutative.

We say "tentative" in Conjecture 1.7, because a positive answer to Question 1.1 is considered to be plausible, as Zilber has informed us.

In Section 2 we will prove a *weak* version of Conjecture 1.7. Instead of considering compactifications of the pseudofinite group G as an abstract group, we consider compactifications of G which are *definable* in an ambient nonstandard model of set theory in which G lives as a nonstandard finite group. This notion of definable compactification (with respect to an ambient structure) is explained in the next section. Actually our weak version of Conjecture 1.7 is a special case of one of the main theorems of [2], namely, nilpotence of good connected Lie models of ultra-approximate subgroups. In Section 3 we prove the (strong) Conjecture 1.7 when G is an ultraproduct of finite simple groups. In fact in this case we show that bG is trivial, and moreover G will be absolutely connected in the sense of Gismatullin [4].

2. Definable compactifications. We first briefly recall notions from [5]. By "definable" in a given structure M we mean definable with parameters, unless we say otherwise. Suppose M is a first order structure and G is a group definable in M. By a *definable* (with respect to the structure M) group compactification of G we mean a homomorphism $f : G \to C$ where C is a compact group, $f(G)$ is dense in C and satisfying the additional property that the map f is "definable":

(∗) whenever C_1, C_2 are disjoint closed subsets of C then there is a subset D of G, definable in M, such that $f^{-1}(C_1) \subseteq D$ and $f^{-1}(C_2) \subseteq G \setminus D$.

When all subsets of G happen to be definable in M, which we call the *absolute case*, then condition (∗) is automatically satisfied, and so a definable compactification is just a compactification. But even in the *relative case* where not all subsets of G need be definable in M there is always a universal defiinable (in M) compactification of G which we call *the definable Bohr compactfication* of G, denoted $\mathrm{def}_M\, bG$. A model-theoretic description of the definable Bohr compactification is as follows: Let M^* be a saturated elementary extension of M, G^* the interpretation of the formula defining G in M^*, and $(G^*)^{00}$ the smallest subgroup of G^* which is type-definable over M (that is, defined by some conjunction of formulas with parameters from M) and has "bounded index" in G. Then $G^*/(G^*)^{00}$ equipped with the logic topology and with the natural homomorphism from G coincides with $\mathrm{def}_M\, bG$.

Alternatively one can explicitly obtain $\mathrm{def}_M\, bG$ without specific reference to model theory, by defining it to be the completion of G with respect to the topology on G, whose neighbourhoods of the identity are those subsets V of G which are definable in M, and admit a sequence $\{V = V_0, V_1, V_2, \ldots\}$ of subsets of G definable in M such that (i) $V_{n+1}^2 \subseteq V_n$ for all n, and (ii) each V_m is symmetric and "left generic" (finitely many left translates cover G).

Definition 2.1. By a *nonstandard finite group*, we mean a finite group in the sense of some elementary extension V^* of the standard model V of set theory. That is, G and the graph of its group operation are elements of V^* and $|G| \in \mathbb{N}^*$.

So implicit in the definition above is that a nonstandard finite group G comes together with the ambient structure $V^* = M$ in which it is obviously definable. And $\mathrm{def}_M\, bG$ is the "relative" compactification of G that we are interested in.

Let us note first that an ultraproduct of finite groups "is" a nonstandard finite group in the sense above: Suppose G_i for $i \in \mathbb{N}$ are finite groups, U is an ultrafilter on $\mathbb{N}$ and $G = \prod_i G_i/U$ is the ultraproduct. For each i let V_i be a copy of the standard model of set theory. Let V^* be the ultraproduct $\prod_i V_i/U$; then G is an element of V^*, and of course $|G| \in \mathbb{N}^*$, so we have G living canonically as a finite group in the sense of this elementary extension V^* of the standard model. Note that any first order formula (in the language of set theory) true of each G_i is true of G in V^*.

Of course a nonstandard finite group is a pseudofinite group, and it is also worth remarking that any saturated pseudofinite group will have the structure of a nonstandard finite group. [Hopefully the distinction between nonstandard finite and pseudofinite is clear: for L a fixed finite language, a nonstandard finite L-structure is a finite L-structure in the sense of some nonstandard model of set theory, so comes together with this nonstandard model of set theory. On the other hand, a pseudofinite L-structure is an (infinite) model of the theory of finite L-structures. Any pseudofinite L-structure has an elementary extension which "is" a nonstandard finite L-structure.]

In any case we prove:

THEOREM 2.2. *Let G be a nonstandard finite group in the ambient structure $M = V^*$. Then the connected component of* $\mathrm{def}_M bG$ *is commutative, so Conjecture* 1.7 *holds in this* definable *context.*

Proof. Let M^* be a saturated elementary extension of M, let G^* the the interpretation of G in M^* and let $f : G^* \to \mathrm{def}_M bG$ be the canonical surjective homomorphism. So G^* is a nonstandard finite group in the structure M^*, and is definable over M.

Now $\mathrm{def}_M bG$ is an inverse limit of compact Lie groups L_i and we want to show that L_i^0 is commutative for each i. Fix i and let $L = L_i$, so we have an induced surjective homomorphism $h : G^* \to L$, defined over M. Since L^0 has finite index in L, $h^{-1}(L^0)$ is a definable (over M) subgroup H of G^* of finite index, which is also clearly pseudofinite. So $h : H \to L^0$ is a "good model of H" in the sense of [2, Definition 3.5]. (Note that a pseudofinite group is a special case of a pseudofinite approximate subgroup.) As L^0 is connected, by [2, Theorem 9.6], L^0 is nilpotent. But L^0 is a compact (rather than just locally compact) connected Lie group, so L^0 is commutative. ∎

3. Ultraproducts of finite simple groups. We work back in the "absolute" context, and prove:

THEOREM 3.1. *Suppose G is an ultraproduct of finite simple groups. Then bG is trivial.*

Note that this means the following: if M_0^* is a saturated elementary extension of the structure M_0 which consists of the group G with predicates for *all* subsets, and G^* is the corresponding extension of G, then $G^* = (G^*)^{00}_{M_0}$. Now $(G^*)^{000}$ is defined to be the smallest subgroup of G^* which has bounded index in G^* and is $\mathrm{Aut}(M_0^*/M_0)$-invariant. Our methods will also yield (with this notation):

Proposition 3.2. *$G^* = (G^*)^{000}_{M_0}$. So G is* absolutely connected *in the sense of* [4].

The proof of Theorem 3.1 makes use of results on the normal subgroup structure of ultraproducts of finite simple groups [3] and [8], some of which depend on the work of Liebeck and Shalev [6].

Definition 3.3. Let G be a nonstandard finite group (living in an ambient nonstandard model $M = V^*$ of set theory as in Section 2). For $g \in G$, $\ell_c(g) = \log|g^G|/\log|G|$, where g^G denotes the conjugacy class of g. So $\ell(g)$ is in the unit interval of $\mathbb{R}^*$.

Lemma 3.4. *Let G be an ultraproduct of finite simple groups, considered* (*as above*) *as a nonstandard finite group in the structure $M = V^*$. Let M^* be an elementary extension of M, and G^* the corresponding extension of G.*

(i) *Let $N = \{g \in G^* : \ell_c(g) < 1/n$ for all $n \in \mathbb{N}\}$. Then N is a proper normal subgroup of G^* and G^*/N is simple* (*and noncommutative*) *as an abstract group.*
(ii) *The family of normal subgroups of G^* is linearly ordered by inclusion.*

Proof. Note that G^* is also a nonstandard finite group. When $G = G^*$ then both (i) and (ii) are contained in [8] and [3]. So it is just a question of passing from G to G^*. This follows by inspection of the proofs in the above references, and we say a few words.

(i) is precisely as in [8, Proposition 3.1]: First, as ℓ_c is (by transfer) invariant under conjugation, N is a normal subgroup of G^*. Now if $\ell_c(g) \geq \epsilon$ for some positive (standard) real ϵ, then $\log|G^*|/\log|g^{G^*}| \leq K = 1/\epsilon$. But by [6, Theorem 1.1], there is a constant c such that for every finite simple group H, for all $h \in H$, if $\log|H|/\log|h^H| \leq K$ then $(h^H)^m = H$ for any integer $m \geq cK$. The same is therefore true of the ultraproduct of finite simple groups G in M, so also of G^* in M^*, whereby $(g^{G^*})^m = G^*$. This shows that G/N is simple (and clearly noncommutative).

(ii) We separate into cases according to whether G is an ultraproduct of alternating groups, or an ultraproduct of finite simple groups of Lie type. In the first case, G^* is a nonstandard finite alternating group, and for $g \in G^*$ we define $s(g)$ to be the cardinality of the support of g. Now [3, Proposition 2.4] says that for g, h nonidentity elements of G, g is in the normal subgroup generated by h iff $s(g)/s(h)$ is finite (i.e. $< r$ for some standard positive real number r). The nontrivial statement is right to left. This transfers from G to G^* in the following way (or alternatively the proof simply works for G^*): The proof of [3, Lemma 2.7] shows that if $g, h \in G$ and $s(g)/s(h) \leq k$ where $k \geq 2$ is a given integer, then g is a product of $4k$ conjugates of h. This transfers to G^* (for a given integer k), whereby we see that for $g, h \in G^*$, g is in the normal subgroup generated by h iff $s(g)/s(h)$ is finite. This implies that the family of normal subgroups of G^* is linearly ordered.

In the case where G is an ultraproduct of finite simple groups of Lie type, we can adapt the proof of [8, Lemma 3.12] in a similar fashion. ∎

Proof of Theorem 3.1. Let, as before, M_0 be the structure consisting of G, its group structure and predicates for all subsets of G, let M_0^* be a saturated elementary extension of M_0, let G^* be the corresponding extension of G, and denote by N_1 the smallest type-definable (over M_0) subgroup of G^* of bounded index. We must show that $N_1 = G^*$.

By saturation we may identify G^* (as a group extending G) with the interpretation of the formula defining G in an elementary extension M^* of the nonstandard model M of set theory in which G, as a nonstandard finite group, lives.

Now assuming that $N_1 \neq G^*$, N_1 would be a proper normal subgroup of G^*, which is therefore contained in the N from part (i) of Lemma 3.4. As N_1 has bounded index in G^*, N also has bounded index in G^*. Now N is type-definable over M in the structure M^*, so G^*/N is a definable (in M) compactification of G. But G^*/N is simple (noncommutative), which contradicts Theorem 2.2. (Actually the appeal to Theorem 2.2 should not really be necessary as one should be able to see directly that N does not have "bounded index" in G^*.)

Proof of Proposition 3.2. With notation from the proof above, suppose that N_2 were a proper $\mathrm{Aut}(M_0^*/M_0)$-invariant normal subgroup of G^* of bounded index in G^*. Then N would be of bounded index in G^*, again yielding a contradiction.

Remark 3.5. Routine model-theoretic arguments allow us to conclude that if G, as a group, is a model of the theory of finite simple groups, then bG is trivial. Analogously for the conclusion of Proposition 3.2.

Acknowledgments. This paper is basically a write-up of a talk at the model theory meeting in Oaxaca, Mexico, in July 2015, but including solutions to some questions we posed there. Thanks to Pierre Simon and Boris Zilber for helpful comments and suggestions.

This research was partly supported by the NSF (grant no. DMS-1360702).

References

[1] J. Auslander, *Minimal Flows and their Extensions*, North-Holland, 1988.

[2] E. Breuillard, B. Green and T. Tao, *The structure of approximate subgroups*, Publ. Math. IHES 116 (2012), 115–221.

[3] P. Ellis, S. Hachtman, S. Schneider and S. Thomas, *Ultraproducts of finite alternating groups*, RIMS Kokyuroku 1619 (2008), 1–7.

[4] J. Gismatullin, *Absolute connectedness and classical groups*, arXiv:1002.1516 (2010).

[5] J. Gismatullin, D. Penazzi and A. Pillay, *On compactifications and the topological dynamics of definable groups*, Ann. Pure Appl. Logic 165 (2014), 552–562.

[6] M. Liebeck and A. Shalev, *Diameters of finite simple groups: sharp bounds and applications*, Ann. of Math. 154 (2001), 383–406.

[7] J. F. Price, *Lie Groups and Compact Groups*, London Math. Soc. Lecture Note Ser. 25, Cambridge Univ. Press, 1977.

[8] A. Stolz and A. Thom, *On the lattice of normal subgroups in ultraproducts of compact simple groups*, Proc. London Math. Soc. 108 (2014), 73–102.

[9] B. Zilber, *Perfect infinities and finite approximation*, in: Infinity and Truth, World Sci., 2014, 199–223.

Anand Pillay
Department of Mathematics
University of Notre Dame
Notre Dame, IN 46556, U.S.A.
E-mail: apillay@nd.edu