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Restricted Steinhaus sets in the plane

by

Devon Henkis, **Steve Jackson** and **Jeff Lobe** (Denton, TX)

Abstract. For each prime p we show the existence of a partial Steinhaus set in the plane for the prime p which can be obtained from the points $\{(i/p, j/p) : 0 \le i, j < p\}$ by translating by integer amounts only in the horizontal direction. We raise several questions concerning these sets.

1. Introduction. A *Steinhaus set* in the plane is a set $S \subseteq \mathbb{R}^2$ such that for every isometric copy $\pi(\mathbb{Z}^2)$ of the integer lattice $\mathbb{Z}^2$ we have $|S \cap \pi(\mathbb{Z}^2)| = 1$. Answering a question of Steinhaus from the 50's, it was shown in [JM] (assuming the axiom of choice AC) that Steinhaus sets in the plane do indeed exist. It was also shown that they cannot have the Baire property, and thus AC is necessary for their construction. Many problems concerning Steinhaus sets remain open. For example, it is still unknown if a Steinhaus set in the plane can be measurable. As pointed out in [JM], the notion of Steinhaus set generalizes naturally in several directions. We can replace $\mathbb{R}^2$ by $\mathbb{R}^n$, replace $\mathbb{Z}^2$ by any set $A \subseteq \mathbb{R}^n$ (not necessarily a lattice, although that is an interesting special case), or restrict the isometries π to special subgroups. One can also require that S have certain additional properties, such as being a bounded set, a measurable set, or avoid certain other sets. All of these generalizations seem interesting and non-trivial.

We briefly mention a few of the results and questions along these lines. An argument of D. Goldstein given in [JM] shows that for $n \ge 4$ there is no Steinhaus set in $\mathbb{R}^n$ for the standard lattice $\mathbb{Z}^n$. For $n = 3$ the question is apparently still open. Although it is unknown whether a Steinhaus set in $\mathbb{R}^2$ for the standard lattice can be measurable (and in fact this is open for all lattices in $\mathbb{R}^2$), results of [CM], [MY], generalizing results of [K], show that for many lattices in $\mathbb{R}^3$ there is no measurable Steinhaus set. As pointed

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out in [MY], these arguments do not work for any lattices in $\mathbb{R}^2$, nor for the standard lattice $\mathbb{Z}^3$ in $\mathbb{R}^3$. It is known (Beck and Croft independently, [B], [C]) that there cannot be a bounded and measurable Steinhaus set in the plane for the standard lattice. In another direction, we replace the lattice by a finite set A. Thus, we are asking, for a given finite set $A \subseteq \mathbb{R}^2$, if there is a set $S \subseteq \mathbb{R}^2$ such that $|S \cap \pi(A)| = 1$ for all isometries π of $\mathbb{R}^2$. In [GM] the term *Jackson set* is used to denote a finite set $A \subseteq \mathbb{R}^2$ such that there does not exist a corresponding Steinhaus set for A. An easy argument shows that every set A in the plane of size 2 or 3 is a Jackson set (one-point sets trivially are not Jackson sets as $S = \mathbb{R}^2$ serves as the corresponding Steinhaus set). In [GM] certain classes of 4-point sets in the plane were shown to be Jackson sets, and in [X] it was proved that every 4-point set in the plane is a Jackson set. Recently, the authors [HJL] have extended these results to 5- and 7-point sets. The natural conjecture is that every finite set $A \subseteq \mathbb{R}^2$ with $|A| > 1$ is a Jackson set (the corresponding conjecture for finite sets in $\mathbb{R}^n$ also seems plausible).

In this paper we look in a different direction. We introduce the notion of a partial 2D-1D Steinhaus set. Our main theorem, Theorem 2.4, will give the existence of these for each prime p.

2. The 2D-1D Steinhaus problem. We first introduce the notion of a partial Steinhaus set. Let ρ denote the standard Euclidean metric on $\mathbb{R}^n$ (the n will be understood from context). By an *n-dimensional lattice distance* we mean a non-zero number of the form $\sqrt{a_1^2 + \cdots + a_n^2}$, where $a_1, \ldots, a_n \in \mathbb{Z}$ (so for $n \geq 4$ all $\sqrt{k}$ for k a positive integer are lattice distances).

DEFINITION 2.1. A *partial Steinhaus set* in $\mathbb{R}^n$ is a set $S \subseteq \mathbb{R}^n$ such that for any two distinct points $z_1, z_2 \in S$ the number $\rho(z_1, z_2)$ is not an n-dimensional lattice distance.

The significance of this definition lies in the fact that if S is not a partial Steinhaus set then S cannot be extended to a Steinhaus set (for the standard lattice $\mathbb{Z}^n$). Thus, being a partial Steinhaus set is necessary for extendibility to a Steinhaus set, but it is not sufficient, as an example from [JM] shows. We also note that in the definition of a partial Steinhaus set, if $S \subseteq \mathbb{Q}$, we can replace the statement that $\rho(z_1, z_2)$ is not a lattice distance with the statement that $\rho^2(z_1, z_2) \notin \mathbb{Z}$. This is because of the well-known result that an integer is the sum of two (or three) squares of rationals iff it is the sum of two (or three) squares of integers (cf. [P, Theorem 4.13] for an elementary proof due to Aubry). In the construction of the Steinhaus set in $\mathbb{R}^2$ of [JM], an important step was to construct a partial Steinhaus set which meets every rational translation of $\mathbb{Z}^2$ (and thus meets it in exactly one point). As a first step, it was shown that for each prime p there is a partial Steinhaus set meeting every lattice of the form $\mathbb{Z}^2 + (i/p, j/p)$ for all $i, j \in \mathbb{Z}$ (or equivalently $i, j \in \mathbb{Z}_p$).

In this paper we consider the existence of partial Steinhaus sets where we restrict the motions of the points to one-dimensional movements. We refer to such partial Steinhaus sets in the plane as 2D-1D partial Steinhaus sets. The precise definition follows.

DEFINITION 2.2. A *2D-1D partial Steinhaus set* in $\mathbb{R}^2$ is a partial Steinhaus set $S \subseteq \mathbb{R}^2$ such that for all translations $L = \mathbb{Z}^2 + (a, b)$ of the standard lattice, if $L \cap S \neq \emptyset$, say $L \cap S = \{z\}$ where $z = (x, y)$, then $0 \leq y < 1$.

For this paper, since we are only attempting to meet lattices which are translations of the standard lattice, the above definition could be shortened to say S is a partial Steinhaus set with $S \subseteq \{(x, y) \colon 0 \leq y < 1\}$ (a general 2D-1D partial Steinhaus set according to Definition 2.2 need not be in the strip $\{(x, y) \colon 0 \leq y < 1\}$.)

Thus, for a 2D-1D partial Steinhaus S, for every lattice of the form $\mathbb{Z}^2 + (a, b)$ where $(a, b) \in [0, 1)^2$, if the lattice $L = \mathbb{Z}^2 + (a, b)$ meets S then it meets it at a point of the form $(a, b) + (\ell, 0)$ where $\ell \in \mathbb{Z}$. In other words, in forming a 2D-1D partial Steinhaus set we are only allowed to move the points in the unit square horizontally. This is a significant restriction on the movements, and it is not immediately clear if such sets exist (which meet various families of lattices), even when they are known to exist for the corresponding unrestricted version of the problem. For the case of all rational translates of the standard lattice, we do not know the answer and we state this as a problem:

QUESTION 2.3. Does there exist a partial 2D-1D Steinhaus set meeting all rational translations of the standard lattice $\mathbb{Z}^2$?

The main result of this paper is that for each prime p there is a partial 2D-1D Steinhaus set for the prime p. More precisely:

THEOREM 2.4. *For each prime p, there is a partial 2D-1D Steinhaus set which meets every lattice of the form $\mathbb{Z}^2 + (i/p, j/p)$ for $0 \leq i, j < p$.*

We can naturally generalize the notion of a partial 2D-1D set by restricting the direction of movements of the points to some fixed direction, not necessarily horizontal. We consider this generalization in §6.

3. A combinatorial puzzle. In this section we show that existence of a partial 2D-1D Steinhaus set for the prime p is equivalent to a certain combinatorial problem which we call the "puzzle." In §4 we will further reformulate the puzzle and it is this "reformulated puzzle" that we actually solve.

To begin, consider starting points $(i_1/p, j_1/p)$, $(i_2/p, j_2/p)$ inside the unit square, so $0 \le i_1, j_1, i_2, j_2 < p$. We move the point $(i/p, j/p)$ by the amount $(\ell(i,j), 0)$, where $\ell(i,j) \in \mathbb{Z}$. Let

$$z_1 = \left(\frac{i_1}{p}, \frac{j_1}{p}\right) + (\ell_1, 0), \qquad z_2 = \left(\frac{i_2}{p}, \frac{j_2}{p}\right) + (\ell_2, 0),$$

where $\ell_1 = \ell(i_1, j_1)$ and $\ell_2 = \ell(i_2, j_2)$.

The condition that $\rho^2(z_1, z_2) \notin \mathbb{Z}$ becomes

$$\begin{aligned}\rho^2(z_1, z_2) = \left(\frac{i_1 - i_2}{p} + (\ell_1 - \ell_2)\right)^2 + \left(\frac{j_1 - j_2}{p}\right)^2 \notin \mathbb{Z} \\ \Leftrightarrow \ \frac{(i_1 - i_2)^2 + (j_1 - j_2)^2}{p^2} + \frac{2(i_1 - i_2)(\ell_1 - \ell_2)}{p} \notin \mathbb{Z} \\ \Leftrightarrow \ \frac{(i_1 - i_2)^2 + (j_1 - j_2)^2}{p} + 2(i_1 - i_2)(\ell_1 - \ell_2) \notin p\mathbb{Z}.\end{aligned}$$

This automatically holds unless $p \equiv 1 \pmod 4$, and in that case we must have $j_1 - j_2 \equiv \pm\lambda(i_1 - i_2) \pmod p$ where $\pm\lambda$ are the two square roots of $-1 \pmod p$.

Consider the root λ. Let $\bar{j} \in \mathbb{Z}_p$, and let $i_1, i_2 \in \mathbb{Z}_p$ be such that $i_1 \neq i_2$. Let

$$j_1 = \bar{j} + \lambda i_1 - \eta_1 p, \qquad j_2 = \bar{j} + \lambda i_2 - \eta_2 p$$

where η_1, η_2 are the unique integers such that $j_1, j_2 \in \mathbb{Z}_p$.

Plugging this in, we get

$$\begin{aligned}&\frac{(i_1 - i_2)^2 + \lambda^2 (i_1 - i_2)^2 - 2\lambda(i_1 - i_2)(\eta_1 - \eta_2)p}{p} \\ &\qquad\qquad + 2(i_1 - i_2)(\ell_1 - \ell_2) \not\equiv 0 \pmod p \\ &\Leftrightarrow \ (i_1 - i_2)\left(\frac{1 + \lambda^2}{p}\right) - 2\lambda(\eta_1 - \eta_2) + 2(\ell_1 - \ell_2) \not\equiv 0 \pmod p \\ &\Leftrightarrow \ (i_1 - i_2)\frac{\delta}{2} - \lambda(\eta_1 - \eta_2) + (\ell_1 - \ell_2) \not\equiv 0 \pmod p\end{aligned}$$

where $\delta = (1 + \lambda^2)/p$.

For each $\bar{j} \in \mathbb{Z}_p$, and for $i \in \mathbb{Z}_p$, let

$$\pi_{\bar{j}}^{\lambda}(i) = i\frac{\delta}{2} - \lambda\eta + \ell(i, j) \pmod p$$

where

$$\eta = \frac{\bar{j} + \lambda i - (\bar{j} + \lambda i \pmod p)}{p} \quad \text{and} \quad j = \bar{j} + \lambda i \pmod p.$$

So, the above equation becomes: for each $\bar{j}$ the function $i \mapsto \pi_{\bar{j}}^{\lambda}(i)$ is a permutation of $\{0, 1, \ldots, p-1\}$.

Similarly, we consider the root $-\lambda$. If we replace λ above by $-\lambda$, and use the modified version of the η term given by

$$\eta' = -\frac{\bar{j} - \lambda i - (\bar{j} - \lambda i \pmod p)}{p},$$

and we define

$$\pi_{\bar{j}}^{-\lambda}(i) = i\frac{\delta}{2} - \lambda\eta' + \ell(i, j') \pmod p$$

where $j' = \bar{j} - \lambda i \pmod p$, then also $i \mapsto \pi_{\bar{j}}^{-\lambda}(i)$ is a permutation.

Define $A(i,j) = \ell(i,j) + i\delta/2$. Thus, A is a $p \times p$ matrix with entries in $\{0, 1, \ldots, p-1\}$ and with the property that

$$i \mapsto A(i,j) - \lambda\eta \equiv \pi_{\bar{j}}^{\lambda}(i) \pmod p,$$
$$i \mapsto A(i,j') - \lambda\eta' \equiv \pi_{\bar{j}}^{-\lambda}(i) \pmod p$$

are permutations.

Thus, the 2D-1D partial Steinhaus set problem is equivalent to the following combinatorial problem (the "puzzle"):

Puzzle. Let p be a prime (congruent to 1 mod 4), and let $\lambda \in \mathbb{Z}_p^*$ (λ is typically a square root of $-1 \pmod p$). Is there a $p \times p$ matrix A with entries in $\mathbb{Z}_p$ such that for every $\bar{j} \in \mathbb{Z}_p$ the two sequences $i \mapsto A(i,j) - \lambda\eta \pmod p$, $i \mapsto A(i,j') - \lambda\eta' \pmod p$ are both permutations, where for $i \in \mathbb{Z}_p$, $j = j(i) = \bar{j} + \lambda i \pmod p$ (and likewise $j' = j'(i) = \bar{j} - \lambda i \pmod p$) and where $\eta = \eta(i) = \frac{(\bar{j}+\lambda i)-(\bar{j}+\lambda i \pmod p)}{p}$ and $\eta' = -(\frac{\bar{j}-\lambda i-(\bar{j}-\lambda i \pmod p)}{p})$?

We can interpret the η, η' terms as follows. Starting at the point $(0, \bar{j})$ on the left edge of the square, we move along the line with slope λ (or $-\lambda$). Every time we cross the top or bottom edge of the square, we subtract λ from the future values. We require then that the sequence of $\pi(i)$ values we get is a permutation.

Remark 3.1. By dividing all the entries of A by λ, we see that the puzzle is equivalent to the version where we move along lines of slope $\pm\lambda$ as before, but now every time we cross the boundaries we subtract 1. That is, the top and bottom edges of the square now carry a "charge" of -1.

Example 3.2. For $p = 5$ consider the matrix

$$L = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 3 & 3 & 2 & 0 \\ 1 & 4 & 0 & 1 & 3 \\ 2 & 3 & 3 & 2 & 3 \\ 3 & 2 & 0 & 3 & 2 \end{bmatrix}.$$

This matrix gives the $L(i,j) = \ell(i,j)$ values for a partial Steinhaus set for $p = 5$. Using the formula $A(i,j) = L(i,j) + i\delta/2$, where now $\delta = (1+\lambda^2)/p = (1+2^2)/5 = 1$, we get

$$A = \begin{bmatrix} 0 & 3 & 1 & 4 & 2 \\ 0 & 1 & 4 & 1 & 2 \\ 1 & 2 & 1 & 0 & 0 \\ 2 & 1 & 4 & 1 & 0 \\ 3 & 0 & 1 & 2 & 4 \end{bmatrix}.$$

We see that the A matrix is a solution to the puzzle (for $\lambda = 2$).

4. The puzzle reformulated. Recall that for any $\bar{j}$, the following two functions must be permutations in order for the ℓ movements to form a 2D-1D partial Steinhaus set:

$$\pi_{\bar{j}}^{\lambda}(i) = i\frac{\delta}{2} - \lambda\eta + \ell(i,j) \pmod p$$

where $\eta = \frac{\bar{j}+\lambda i - (\bar{j}+\lambda i \pmod p)}{p}$ and $j = \bar{j} + \lambda i \pmod p$, and

$$\pi_{\bar{j}}^{-\lambda}(i) = i\frac{\delta}{2} - \lambda\eta' + \ell(i,j') \pmod p$$

where $\eta' = -\frac{\bar{j}-\lambda i - (\bar{j}-\lambda i \pmod p)}{p}$.

We now coordinatize the matrix of ℓ values by using j_1, j_2 values, where j_1 is the value of $\bar{j}$ for the point (i,j) and the root λ, and similarly j_2 is the $\bar{j}$ value for the root $-\lambda$. Thus we have

$$j = j_1 + i\lambda \pmod p, \qquad j = j_2 - i\lambda \pmod p,$$

so that

$$i = \frac{j_2 - j_1}{2\lambda} \pmod p, \qquad j = \frac{j_1 + j_2}{2} \pmod p.$$

Let us define the "vertical charge" matrix as follows. The vertical charge represents the contribution from the $-\lambda\eta$ term in the above formula for $\pi_{\bar{j}}^{\lambda}(i)$ where we use $\bar{j} = j_1$. We define

$$\begin{aligned} V(j_1,j_2) &= (-\lambda)\eta(i,j) \\ &= (-\lambda)\frac{j_1 + \lambda i - (j_1 + \lambda i \pmod p)}{p} \\ &= (-\lambda)\frac{j_1 + \lambda\left(\frac{j_2-j_1}{2\lambda} \pmod p\right) - \left(\frac{j_1+j_2}{2} \pmod p\right)}{p}. \end{aligned}$$

Likewise, we define the "horizontal charge" matrix to correspond to the $-\lambda\eta'$ term in the above formula for $\pi_{\bar{j}}^{-\lambda}(i)$, where now we replace $\bar{j}$ by j_2:

$$H(j_1,j_2)=(-\lambda)\eta'(i,j)$$
$$=(-\lambda)\Big(-\frac{j_2-\lambda i-(j_2-\lambda i\ (\mathrm{mod}\ p))}{p}\Big)$$
$$=(-\lambda)\Big(-\frac{j_2-\lambda(\frac{j_2-j_1}{2\lambda}\ (\mathrm{mod}\ p))-(\frac{j_1+j_2}{2}\ (\mathrm{mod}\ p))}{p}\Big)$$

Now we define the B matrix by using the values $\pi_{\bar{j}}^{\lambda}(i)$, using the (j_1,j_2) representation of these points. So, we define

$$B(j_1,j_2)=\pi_{\bar{j}}^{\lambda}(i)=i\frac{\delta}{2}-\lambda\eta+\ell(i,j)\ (\mathrm{mod}\ p) \tag{4.1}$$
$$=\frac{j_2-j_1}{2\lambda}\,\frac{\delta}{2}+V(j_1,j_2)+\ell\Big(\frac{j_2-j_1}{2\lambda}\ (\mathrm{mod}\ p),\frac{j_1+j_2}{2}\ (\mathrm{mod}\ p)\Big).$$

So, our first condition that the $\pi_{\bar{j}}^{\lambda}(i)$ values form a permutation becomes the requirement that the columns (fixed value of j_1) of the B matrix must be permutations. The second condition that the values $\pi_{\bar{j}}^{-\lambda}(i)$ form a permutation then becomes the requirement that for each fixed value of j_2, the sequence

$$i\mapsto\pi_{\bar{j}}^{-\lambda}(i)=B(j_1,j_2)+(H(j_1,j_2)-V(j_1,j_2))$$

is a permutation. Let us define the "charge matrix" by

$$Q'(j_1,j_2)=H(j_1,j_2)-V(j_1,j_2)$$
$$=(-\lambda)\frac{-(j_1+j_2)+2\big(\frac{j_1+j_2}{2}\ (\mathrm{mod}\ p)\big)}{p}.$$

So, our second requirement becomes that for each row of the B matrix (i.e., fixed value of j_2), that row when added to the corresponding row of the charge matrix Q' results in a permutation.

We can simplify a little by dividing through by $-\lambda$. So, let us define

$$Q(j_1,j_2)=\frac{-(j_1+j_2)+2\big(\frac{j_1+j_2}{2}\ (\mathrm{mod}\ p)\big)}{p}. \tag{4.2}$$

Since dividing a permutation by $-\lambda$ does change it being a permutation, we can finally state our reformulation of the problem as follows.

Reformulated Problem. Find a $p\times p$ matrix $B(j_1,j_2)$ with the following properties:

(1) Each column, $j_2\mapsto B(j_1,j_2)$, of B is a permutation.
(2) Each $j_1\mapsto B(j_1,j_2)+Q(j_1,j_2)$ is a permutation. That is, each row of B when added to the corresponding row of Q gives a permutation.

For example, for $p = 5$ the Q matrix is

$$Q = \begin{bmatrix} 0 & -1 & 0 & -1 & 0 \\ 1 & 0 & -1 & 0 & -1 \\ 0 & 1 & 0 & -1 & 0 \\ 1 & 0 & 1 & 0 & -1 \\ 0 & 1 & 0 & 1 & 0 \end{bmatrix}.$$

The patterns noticed in the above matrix for $p = 5$ are in fact general:

LEMMA 4.1. *The Q matrix has the following properties:*

(1) *If $j_1 + j_2$ is even, then $Q(j_1, j_2) = 0$.*
(2) *If $j_1 + j_2$ is odd and $j_1 + j_2 < p - 1$, then $Q(j_1, j_2) = 1$.*
(3) *If $j_1 + j_2$ is odd and $j_1 + j_2 > p - 1$, then $Q(j_1, j_2) = -1$.*

Proof. The properties are immediate from (4.2). ■

5. A solution to the puzzle. We now present a solution to the reformulated version of the puzzle, that is, using the above charge matrix Q. The solution we present is valid for any $p \equiv 1 \pmod 4$, p being here any odd integer (not necessarily a prime).

THEOREM 5.1. *Let p be a positive integer with $p \equiv 1 \pmod 4$. Then there is a $p \times p$ matrix A with entries in $\mathbb{Z}_p$ satisfying:*

(1) *Each column $j \mapsto A(i, j)$ of A is a permutation.*
(2) *Each row with the charges added, $i \mapsto A(i, j) + Q(i, j)$, is a permutation.*

Proof. We first give the definition of the $A(i, j)$ matrix which we then show works. Let $a \in \mathbb{Z}_p^*$ be such that $\gcd(a, p) = \gcd(a + 1, p) = 1$. To help follow the proof, we give the $A(i, j)$ matrix for the case $p = 9$ and $a = 1$. The middle column, where $j = (p - 1)/2$, will be defined differently from the other columns and is identified in the following table.

2	2	0	0	3	7	7	5	5
1	1	8	8	1	6	6	4	4
0	0	7	7	8	5	5	3	3
8	8	6	6	6	4	4	2	2
7	7	5	5	4	3	3	1	1
6	6	4	4	2	2	2	0	0
5	5	3	3	0	1	1	8	8
4	4	2	2	7	0	0	7	7
3	3	1	1	5	8	8	6	6

To begin, we fill in the bottom row, where $j = 0$. We begin with the entry to the immediate left of the middle position and put a 1 in this entry and also in the position to the left of this one. We continue moving to the left and write two entries at a time, each time writing the value which is 2 more than the value in the previous group of two. Since $p \equiv 1 \pmod 4$, the entries to the left of the middle position, and also to the right of the middle position, break up into an integer number of groups of two. Note that the first two entries $A(0, 0)$ and $A(1, 0)$ are both equal to $1 + \left(\frac{p-1}{4} - 1\right) \cdot 2 = \frac{p-3}{2}$. We then cycle around to the rightmost entry in the bottom row and for this entry and also the one to the left, write the value which is 3 more that the last entry written (instead of 2 more). Thus, we have $A(p - 2, 0) = A(p - 1, 0) = \frac{p+3}{2}$. This fills in the last row except for the middle position. Still ignoring the middle column, we fill the remaining entries by adding a to go from one row to the row immediately above it. Thus, for $j \neq \frac{p-1}{2}$ we have $A(i, j) = A(i, 0) + ja$.

Note that if we start from the entry to the left of the middle position in the bottom row and, moving to the left and wrapping around, read off the entries with the Q matrix added, we get (up to reordering the two numbers within each grouped pair) the partial permutation $1, 2, \ldots, \frac{p-1}{2}, \frac{p+3}{2}, \frac{p+5}{2}, \ldots, p-1, 0$. Note that the value to the immediate right of the middle position is indeed $1 + 2\left[\left(\frac{p-1}{4} - 1\right) + 1 + \left(\frac{p-1}{4} - 1\right)\right] + 1 = p - 1$. Note also that the value that is skipped in this partial permutation is $\frac{p+1}{2}$. We write this value in the middle position of the bottom row, and this completes the definition of $A(i, 0)$, and also therefore for $A(i, j)$ except when $i = \frac{p-1}{2}$ and $j > 0$. For $i = \frac{p-1}{2}$ and $j > 0$ we define $A(i, j)$ by adding $a + 1$ to the entry below it. Thus, we define

$$A\left(\frac{p-1}{2}, j\right) = \frac{p+1}{2} + j(a + 1).$$

We write the formula for the entries $A(i, j)$ in general. From the algorithm given we easily see that the entries for $i \neq \frac{p-1}{2}$ are given by

$$A(i, j) = ja + \begin{cases} \frac{p-3}{2} - i & \text{if } i < \frac{p-1}{2} \text{ and } i \text{ is even,} \\ \frac{p-3}{2} - (i - 1) & \text{if } i < \frac{p-1}{2} \text{ and } i \text{ is odd,} \\ \frac{p-3}{2} - (i - 2) & \text{if } i > \frac{p-1}{2} \text{ and } i \text{ is even,} \\ \frac{p-3}{2} - (i - 1) & \text{if } i > \frac{p-1}{2} \text{ and } i \text{ is odd.} \end{cases}$$

It is immediate that all columns $j \mapsto A(i, j)$, except for the middle one, are permutations since $\gcd(a, p) = 1$. Since $\gcd(a+1, p) = 1$, the middle column is also a permutation. It is also clear from the above description that the bottom row with the charges added, $i \mapsto A(i, 0) + Q(i, 0)$, is a permutation. Finally, it is clear for $j > 0$ that, except for the middle position in the jth row, the sum of the row and its corresponding Q values gives a partial permutation. That is, $i \mapsto A(i, j) + Q(i, j)$, $i \neq \frac{p-1}{2}$, is a partial permutation.

To finish, it is enough to show that for each $j > 0$ the value in the middle position of the jth row with the charge added, namely $A\big(\frac{p-1}{2}, j\big) + Q\big(\frac{p-1}{2}, j\big) = \frac{p+1}{2} + j(a+1) + Q\big(\frac{p-1}{2}, j\big)$, is the unique value missing in the partial permutation $i \mapsto A(i,j) + Q(i,j)$ $\big(i \neq \frac{p-1}{2}\big)$.

Consider the jth row of the A matrix, where $j \neq \frac{p-1}{2}$. The positions in this row, except for the middle column, are grouped into pairs in a natural way, namely we group from left to right skipping the middle column. More precisely, for $i < \frac{p-1}{4}$ we group together the $(2i, j)$ entry and the $(2i+1, j)$ entry. For $\frac{p-1}{4} \leq i < \frac{p-1}{2}$ we group together the $(2i+1, j)$ entry and the $(2i+2, j)$ entry. Note that for each grouped pair, exactly one of the entries has a non-zero value for the charge $Q(i,j)$. Note also that if we start to the right $(i = p-1)$ and read to the left, the values we see are constant in each grouped pair, and the values in these pairs increment (mod p) by 2 each time we move to the left. Finally, the value in the rightmost pair is 3 more than the value in the leftmost pair.

Consider the diagonal entry $(p-1-j, j)$ (diagonal here refers to the (i,j) such that $i + j = p-1$). For $j \neq \frac{p-1}{2}$ this diagonal entry is the left/right entry of its grouped pair as follows:

- If $j < \frac{p-1}{2}$ and j is even, then **right**.
- If $j < \frac{p-1}{2}$ and j is odd, then **left**.
- If $j > \frac{p-1}{2}$ and j is even, then **left**.
- If $j > \frac{p-1}{2}$ and j is odd, then **right**.

Let $k = A(p-1-j, j)$ be the A entry in this diagonal position. First assume that the $(p-1-j, j)$ entry is the right entry in its pair. In this case, if we start from this position and read to the left along this row (skipping the middle column), adding the $Q(i,j)$ entries to the values, we get consecutive values starting with k. After reaching the leftmost entry, we wrap around to the rightmost entry and continue reading to the left. The A values in the rightmost pair are 3 more than those of the leftmost pair. Also, the Q matrix will subtract 1 from one of the entries in the pairs which we get to after wrapping around (as opposed to adding 1 to one of the entries for the pairs before the wrap around). So, after adding the Q values, we continue to generate consecutive values (we stop at the entry to the immediate right of the diagonal position). Thus, skipping the middle column, we see the values $k, k+1, \ldots, k-2$. So, the value that is skipped in the partial permutation $i \mapsto A(i,j) + Q(i,j)$ $\big(i \neq \frac{p-1}{2}\big)$ is $k-1$. Similarly, if the $(p-1-j, j)$ entry is the left entry in its pair, we have a similar argument, except we now start from the diagonal position and read to the right (skipping the middle column), wrapping around to the left after we reach the rightmost entry. After adding the $Q(i,j)$ values, we see a consecutive sequence $k, k-1, \ldots$

of length $p-1$. Thus the value that is skipped in the partial permutation $i \mapsto A(i,j)+Q(i,j)$ $\bigl(i \neq \frac{p-1}{2}\bigr)$ is $k+1$.

To summarize, if the diagonal entry $(p-1-j,j)$ is the right entry of its pair, then $A(p-1-j,j)-1$ is the value skipped in the partial permutation along the jth row (skipping the middle column), and if $(p-1-j,j)$ is the left entry of its pair, then $A(p-1-j,j)+1$ is the value skipped.

Let $s(j)$ be the value skipped in the partial permutation for the jth row. From the above and the formulas for A we find that (using $i=p-1-j$)

$$s(j) = j\cdot a + \begin{cases} \frac{p-3}{2} - (p-2-j) & \text{if } j < \frac{p-1}{2} \text{ and } j \text{ is even},\\ \frac{p-3}{2} - (p-3-j) & \text{if } j < \frac{p-1}{2} \text{ and } j \text{ is odd},\\ \frac{p-3}{2} - (p-2-j) & \text{if } j > \frac{p-1}{2} \text{ and } j \text{ is even},\\ \frac{p-3}{2} - (p-1-j) & \text{if } j > \frac{p-1}{2} \text{ and } i \text{ is odd}.\end{cases}$$

On the other hand,

$$A\Bigl(\frac{p-1}{2},j\Bigr)+Q\Bigl(\frac{p-1}{2},j\Bigr) = \frac{p+1}{2}+j(a+1) + \begin{cases} 0 & \text{if } j < \frac{p-1}{2} \text{ and } j \text{ is even},\\ 1 & \text{if } j < \frac{p-1}{2} \text{ and } j \text{ is odd},\\ 0 & \text{if } j > \frac{p-1}{2} \text{ and } j \text{ is even},\\ -1 & \text{if } j > \frac{p-1}{2} \text{ and } j \text{ is odd}.\end{cases}$$

We see that $s(j) \equiv A\bigl(\frac{p-1}{2},j\bigr)+Q\bigl(\frac{p-1}{2},j\bigr) \pmod p$.

Finally, if $j=\frac{p-1}{2}$ we see that $A\bigl(\frac{p-1}{2},\frac{p-1}{2}\bigr) = \frac{p+1}{2}+\frac{p-1}{2}(a+1) \equiv a\frac{p-1}{2} \pmod p$. Also, $Q\bigl(\frac{p-1}{2},\frac{p-1}{2}\bigr)=0$. But in this case if we start at the entry to the immediate left of the diagonal entry and read left, adding the charges, and wrap around to the right edge after the leftmost entry, we get a partial permutation. The value to the immediate left of the central position is $A\bigl(\frac{p-3}{2},\frac{p-1}{2}\bigr) = \frac{p-3}{2}-\bigl(\frac{p-3}{2}-1\bigr)+\frac{p-1}{2}a = a\frac{p-1}{2}+1$. So, $s(j)=a\frac{p-1}{2}$. So, $s(j)=A\bigl(\frac{p-1}{2},j\bigr)+Q\bigl(\frac{p-1}{2},j\bigr)$ again holds. This completes the proof of Theorem 5.1. ∎

Example 5.2. To illustrate the algorithm just described in the proof of Theorem 5.1, we present the output for $p=5$. The following computations were done on *Maple*. First, the solution to the puzzle according to the algorithm gives the following A matrix:

0	0	1	3	3
4	4	4	2	2
3	3	2	1	1
2	2	0	0	0
1	1	3	4	4

From the above A matrix, we first multiply the entries by $-\lambda$ to undo the scaling described at the end of Section 4. Let $B = (-\lambda)A$. From (4.1) we can then solve for the array of horizontal movements, the ℓ values in (4.1). Doing this gives the following array of ℓ values:

4	2	1	0	4
1	0	4	1	1
1	2	1	0	4
1	0	0	2	1
3	2	1	0	4

We can apply the above horizontal movements to generate the partial Steinhaus set. Figure 1 shows the resulting partial Steinhaus set.

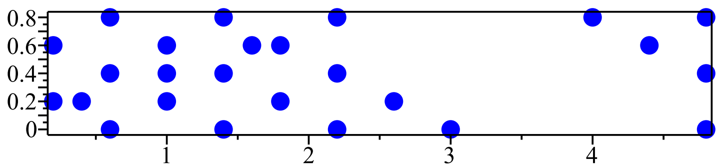

Fig. 1. A restricted Steinhaus set for $p = 5$

6. Partial 2D-1D Steinhaus set in the $(1, \lambda)$ direction. In the definition of 2D-1D partial Steinhaus set (Definition 2.2) the movements were restricted to be in the horizontal direction. That is, each point (x, y) was translated by an integer multiple of $(1, 0)$ to produce the point $(x, y)+\ell\cdot(1, 0)$ in the partial Steinhaus set. One can consider the version of the problem where we use translations in other directions.

Definition 6.1. Given $(u, v) \in \mathbb{Z}^2 - \{(0, 0)\}$, we say $S \subseteq \mathbb{R}^2$ is a *partial 2D-1D Steinhaus set in the (u, v) direction* if S is a partial Steinhaus set, and each translation L of the standard integer lattice meets S in a point of the form $(x, y) + \ell(u, v)$ for $\ell \in \mathbb{Z}$, where (x, y) is in the unit square $[0, 1)^2$. We say S is a partial 2D-1D Steinhaus set in the direction (u, v) for the prime p if we just require lattices of the form $L = \mathbb{Z}^2 + (i/p, j/p)$ meet S as above.

The next theorem shows that there are directions for which 2D-1D partial Steinhaus sets do not exist.

Theorem 6.2. *Let p be prime and let λ be a root (i.e. $1 + \lambda^2 \equiv 0 \pmod p$). Then there is no partial 2D-1D Steinhaus set for the $(1, \lambda)$ direction for the prime p.*

Proof. Fix $a, b \in \mathbb{Z}_p^*$ such that $a^2 + b^2 = p^2$. Note that $b \equiv \pm\lambda a \pmod p$. First assume $b \equiv \lambda a \pmod p$. Let $x_1 = (0, 0)$ and $x_2 = (a/p, b/p)$. Then

$$\rho^2(x_1, x_2) = \left\| \left(\frac{a}{p}, \frac{b}{p} \right) \right\|^2 = \frac{a^2 + b^2}{p^2} \in \mathbb{Z}.$$

Let $z_1 = \ell_1(1, \lambda)$ and $z_2 = (a/p, b/p) + \ell_2(1, \lambda)$ for some $\ell_1, \ell_2 \in \mathbb{Z}$ be the corresponding points in the partial Steinhaus set. Then $\rho^2(z_1, z_2) \in \mathbb{Z}$ according to the following:

$$\begin{aligned}
\rho^2(z_1, z_2) &= \left\| \left(\frac{a}{p} + (\ell_1 - \ell_2), \frac{b}{p} + (\ell_1 - \ell_2)\lambda \right) \right\|^2 \\
&= \frac{a^2 + b^2}{p^2} + 2\frac{a}{p}(\ell_1 - \ell_2) + 2\frac{b}{p}(\ell_1 - \ell_2)\lambda + (\ell_1 - \ell_2)^2 + (\ell_1 - \ell_2)^2\lambda^2 \\
&= 2\frac{a}{p}(\ell_1 - \ell_2) + 2\frac{b}{p}(\ell_1 - \ell_2)\lambda \bmod \mathbb{Z} \\
&= 2(\ell_1 - \ell_2)\frac{1}{p}(a + a\lambda^2) \bmod \mathbb{Z} \\
&= 0 \bmod \mathbb{Z}.
\end{aligned}$$

If $b \equiv -\lambda a$ then note that $a \equiv \lambda b \pmod p$, then use $x_2 = (b/p, a/p)$ and the argument is similar. ∎

QUESTION 6.3. For which directions (u, v) and primes p do partial 2D-1D Steinhaus sets in the direction (u, v) for the prime p exist?

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References

[B] J. Beck, *On a lattice point problem of H. Steinhaus*, Studia Sci. Math. Hungar. 24 (1989), 263–268.

[CM] W. K. Chan and R. D. Mauldin, *Steinhaus tiling problem and integral quadratic forms*, Proc. Amer. Math. Soc. 135 (2007), 337–342.

[C] H. T. Croft, *Three lattice point problems of Steinhaus*, Quart. J. Math. Oxford Ser. 33 (1982), 71–83.

[GM] S. Gao and A. Miller, *Steinhaus sets and Jackson sets*, in: Contemp. Math. 425, Amer. Math. Soc., 2007, 127–145.

[HJL] D. Henkis, S. Jackson, and J. Lobe, *The finite Steinhaus problem*, Quart. J. Math. Oxford Ser. 67 (2016), 551–564.

[JM] S. Jackson and R. D. Mauldin. *On a lattice problem of H. Steinhaus*, J. Amer. Math. Soc. 15 (2002), 817–856.

[K] M. N. Kolountzakis and T. Wolff, *On the Steinhaus tiling problem*, Mathematika 46 (1999), 253–280.

[MY] R. D. Mauldin and A. Q. Yingst, *Comments about the Steinhaus tiling problem*, Proc. Amer. Math. Soc. 131 (2003), 2071–2079.

[P] P. Pollack, *A Second Course in Elementary Number Theory*, Amer. Math. Soc., 2009.

[X] M. Z. Xuan, *On Steinhaus sets, orbit trees, and universal properties of various subgroups in the permutation group of natural numbers*, PhD thesis, Univ. of North Texas, 2012.

Devon Henkis, Steve Jackson, Jeff Lobe
Department of Mathematics
University of North Texas
Denton, TX 76203, U.S.A.
E-mail: devonhenkis@my.unt.edu
stephen.jackson@unt.edu
jeffreylobe@my.unt.edu