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GENERAL TOPOLOGY

Every Filter is Homeomorphic to Its Square

by

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Summary. We show that every filter $\mathcal{F}$ on ω, viewed as a subspace of 2^ω, is homeomorphic to $\mathcal{F}^2$. This generalizes a theorem of van Engelen, who proved that this holds for Borel filters.

1. Introduction. In [vE3], van Engelen obtained a purely topological characterization of filters, among the zero-dimensional Borel spaces ([1]). In particular, he obtained the following result (see [vE3, Lemma 3.1]).

THEOREM 1 (van Engelen). *If $\mathcal{F}$ is a Borel filter then $\mathcal{F}$ is homeomorphic to $\mathcal{F}^2$.*

The main ingredients of his proof are the fact that every filter $\mathcal{F}$ is Wadge equivalent to $\mathcal{F}^2$ (which is easy to see using the operation of intersection), a theorem of Steel [St], and some of his previous work [vE1]. It is natural to ask whether the assumption that $\mathcal{F}$ is Borel is really necessary in Theorem 1. Our main result (Theorem 6) shows that this is not the case, and the proof only uses elementary methods.

2. Notation. Throughout this paper, Ω will denote a countably infinite set. A *filter* on Ω is a collection $\mathcal{F}$ of subsets of Ω that satisfies the following conditions. We write $X \subseteq^* Y$ to mean that $X \setminus Y$ is finite, and $X =^* Y$ to mean that $X \subseteq^* Y$ and $Y \subseteq^* X$.

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([1]) Actually, van Engelen stated his results for ideals. Using the homeomorphism $c : 2^\omega \to 2^\omega$ defined by $c(X)(n) = 1 - X(n)$ for $X \in 2^\omega$ and $n \in \omega$, one sees that his results also hold for filters.

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(1) $\emptyset \notin \mathcal{F}$ and $\Omega \in \mathcal{F}$.
(2) If $X \in \mathcal{F}$ and $X =^* Y \subseteq \Omega$ then $Y \in \mathcal{F}$.
(3) If $X \in \mathcal{F}$ and $X \subseteq Y \subseteq \Omega$ then $Y \in \mathcal{F}$.
(4) If $X, Y \in \mathcal{F}$ then $X \cap Y \in \mathcal{F}$.

All filters are assumed to be on ω unless we explicitly say otherwise. We will say that a filter is *principal* if there exists $\Omega \subseteq \omega$ such that $\mathcal{F} = \{X \subseteq \omega : \Omega \subseteq^* X\}$ (2). Define $\mathsf{Fin}(\Omega) = \{X \subseteq \Omega : X \text{ is finite}\}$ and $\mathsf{Cof}(\Omega) = \{X \subseteq \Omega : \Omega \setminus X \text{ is finite}\}$.

We will freely identify any collection $\mathcal{X}$ consisting of subsets of Ω with the subspace of 2^Ω consisting of the characteristic functions of elements of $\mathcal{X}$. In particular, every filter on Ω will inherit the subspace topology from 2^Ω.

Given a function f and a subset S of the domain of f, let $f[S] = \{f(X) : X \in S\}$ denote the image of S under f.

By *space* we will always mean separable metrizable topological space. A space is *crowded* if it is non-empty and it has no isolated points. Given spaces $\mathcal{X}$ and $\mathcal{Y}$, we will write $\mathcal{X} \approx \mathcal{Y}$ to mean that $\mathcal{X}$ and $\mathcal{Y}$ are homeomorphic. We will be using freely the following well-known characterizations of $\mathbb{Q}$ and 2^ω (see [vM, Theorems 1.9.6 and Theorem 1.5.5]).

- If $\mathcal{X}$ is a crowded countable space then $\mathcal{X} \approx \mathbb{Q}$.
- If $\mathcal{X}$ is a crowded compact zero-dimensional space then $\mathcal{X} \approx 2^\omega$.

We will also assume that the reader is familiar with the basic theory of topologically complete spaces (see for example [vM, Section A.6]).

Given a collection $\mathcal{X}$ consisting of subsets of ω and $\Omega \subseteq \omega$, define

$$\mathcal{X} \restriction \Omega = \{X \cap \Omega : X \in \mathcal{X}\}.$$

Notice that $\mathcal{F} \restriction \Omega = \{X \in \mathcal{F} : X \subseteq \Omega\}$ whenever $\mathcal{F}$ is a filter and $\Omega \in \mathcal{F}$.

We conclude this section by remarking that many authors (including van Engelen [vE3]) give a more general notion of filter than the one we gave above. The most general notion possible seems to be the following. Define a *prefilter* on Ω to be a collection $\mathcal{F}$ of subsets of Ω that satisfies conditions (3) and (4). The next proposition, which can be safely assumed to be folklore, shows that our definition does not result in any substantial loss of generality.

PROPOSITION 2. *Let $\mathcal{G}$ be an infinite prefilter on ω. Then either $\mathcal{G} \approx 2^\omega$ or $\mathcal{G} \approx \mathcal{F}$ for some filter $\mathcal{F}$.*

Proof. Let $\Omega = \omega \setminus \bigcap \mathcal{G}$, and observe that Ω is infinite because $\mathcal{G}$ is infinite. Notice that $\mathcal{G} \restriction \Omega$ is a prefilter on Ω. First assume that $\emptyset \in \mathcal{G} \restriction \Omega$. This means that $\bigcap \mathcal{G} = \omega \setminus \Omega \in \mathcal{G}$, hence $\mathcal{G} = \{X \subseteq \omega : \bigcap \mathcal{G} \subseteq X\} \approx 2^\omega$.

(2) This is not quite the standard definition. Notice however that, according to our definitions, a filter is principal if and only if it is generated by a single element.

Now assume that $\emptyset \notin \mathcal{G}\restriction\Omega$. We claim that $\mathcal{G}\restriction\Omega$ is in fact a filter on Ω. In order to prove this claim, it only remains to show that condition (2) is satisfied. Notice that it will be enough to show that $\mathsf{Cof}(\Omega) \subseteq \mathcal{G}\restriction\Omega$. So let $F \in \mathsf{Fin}(\Omega)$. Since $\Omega = \omega \setminus \bigcap \mathcal{G}$ and $\mathcal{G}$ satisfies condition (4), there must be $X \in \mathcal{G}$ such that $X \subseteq \omega \setminus F$. It follows that $\omega \setminus F \in \mathcal{G}$, hence $\Omega \setminus F \in \mathcal{G}\restriction\Omega$. Finally, it is straightforward to check that $\mathcal{G} \approx \mathcal{G}\restriction\Omega$. ∎

3. Preliminary results. The following three lemmas will be needed in the proof of Theorem 6.

LEMMA 3. *Assume that $\mathcal{F}$ is a non-principal filter and $\Omega \in \mathcal{F}$. Then $\mathcal{F}\restriction\Omega \approx \mathcal{F}$.*

Proof. Fix $\Omega^* \subseteq \Omega$ such that $\Omega^* \in \mathcal{F}$ and $\Omega \setminus \Omega^*$ is infinite. This is possible because $\mathcal{F}$ is non-principal. Fix a bijection $\sigma : \omega \setminus \Omega^* \to \Omega \setminus \Omega^*$ and let $\tau : \Omega^* \to \Omega^*$ be the identity. Set $\pi = \sigma \cup \tau$ and notice that $\pi : \omega \to \Omega$ is a bijection. Therefore, the function $h : 2^\omega \to 2^\Omega$ defined by setting $h(X) = \pi[X]$ is a homeomorphism. Furthermore, using the fact that $\Omega^* \in \mathcal{F}$, it is straightforward to check that $h[\mathcal{F}] = \mathcal{F}\restriction\Omega$. This shows that $\mathcal{F} \approx \mathcal{F}\restriction\Omega$. ∎

LEMMA 4. *Assume that $\mathcal{F}$ is a non-principal filter. Then $\mathcal{F} \times 2^\omega \approx \mathcal{F}$.*

Proof. Fix a $\Omega \in \mathcal{F} \setminus \mathsf{Cof}(\omega)$. This is possible because $\mathcal{F}$ is non-principal. Let $h : 2^\Omega \times 2^{\omega\setminus\Omega} \to 2^\omega$ be the function defined by setting $h(F, X) = F \cup X$. It is clear that h is a homeomorphism. Furthermore, using the fact that $\Omega \in \mathcal{F}$, one sees that $h[\mathcal{F}\restriction\Omega \times 2^{\omega\setminus\Omega}] = \mathcal{F}$. Therefore $\mathcal{F}\restriction\Omega \times 2^{\omega\setminus\Omega} \approx \mathcal{F}$. An application of Lemma 3 concludes the proof. ∎

LEMMA 5. *Assume that $\mathcal{F}$ is a principal filter. Then $\mathcal{F}^2 \approx \mathcal{F}$.*

Proof. It will be enough to show that $\mathcal{F} \approx \mathbb{Q}$ or $\mathcal{F} \approx \mathbb{Q} \times 2^\omega$. Fix $\Omega \subseteq \omega$ such that $\mathcal{F} = \{X \subseteq \omega : \Omega \subseteq^* X\}$. If $\Omega \in \mathsf{Cof}(\omega)$, then $\mathcal{F} = \mathsf{Cof}(\omega) \approx \mathbb{Q}$. So assume that $\Omega \notin \mathsf{Cof}(\omega)$. The proof of Lemma 4 shows that $\mathcal{F} \approx \mathcal{F}\restriction\Omega \times 2^{\omega\setminus\Omega}$. Since $\mathcal{F}\restriction\Omega = \mathsf{Cof}(\Omega) \approx \mathbb{Q}$, it follows that $\mathcal{F} \approx \mathbb{Q} \times 2^\omega$. ∎

4. The main result. We begin by introducing some useful notation. Given $S \subseteq \omega$ such that $\omega \setminus S$ is infinite, let $\phi_S : \omega \setminus S \to \omega$ denote the unique bijection such that $m < n$ implies $\phi_S(m) < \phi_S(n)$ for all $m, n \in \omega \setminus S$. Given an infinite $\Omega \subseteq \omega$, define

$$\mathcal{D}(\Omega) = \{(X, Y) \in 2^\Omega \times 2^\Omega : X \cap Y = \emptyset\}.$$

It is easy to check that $\mathcal{D}(\Omega)$ is a closed crowded subspace of $2^\Omega \times 2^\Omega$, which implies $\mathcal{D}(\Omega) \approx 2^\omega$.

THEOREM 6. *If $\mathcal{F}$ is a filter then $\mathcal{F}^2 \approx \mathcal{F}$.*

Proof. Let $\mathcal{F}$ be a filter. If $\mathcal{F}$ is principal, then the desired conclusion follows from Lemma 5. So assume that $\mathcal{F}$ is non-principal, and fix $\Omega \in \mathcal{F} \setminus \mathsf{Cof}(\omega)$.

Let $h : 2^\Omega \times 2^\Omega \times \mathcal{D}(\omega \setminus \Omega) \to 2^\Omega \times \mathcal{D}(\omega)$ be the function defined by

$$h(F, G, X, Y) = (F \cap G, \phi_{F\cap G}[(F \setminus G) \cup X], \phi_{F\cap G}[(G \setminus F) \cup Y]),$$

and observe that h is continuous.

Let $g : 2^\Omega \times \mathcal{D}(\omega) \to 2^\Omega \times 2^\Omega \times \mathcal{D}(\omega \setminus \Omega)$ be the function defined by

$$g(H, Z, W) = (H \cup (\phi_H^{-1}[Z] \cap \Omega), H \cup (\phi_H^{-1}[W] \cap \Omega), \phi_H^{-1}[Z] \cap (\omega \setminus \Omega), \phi_H^{-1}[W] \cap (\omega \setminus \Omega)),$$

and observe that g is continuous. It is straightforward to verify that g is the inverse function of h. Therefore h is a homeomorphism.

Furthermore, it is easy to realize that

$$h[\mathcal{F}{\restriction}\Omega \times \mathcal{F}{\restriction}\Omega \times \mathcal{D}(\omega \setminus \Omega)] \subseteq \mathcal{F}{\restriction}\Omega \times \mathcal{D}(\omega)$$

and

$$g[\mathcal{F}{\restriction}\Omega \times \mathcal{D}(\omega)] \subseteq \mathcal{F}{\restriction}\Omega \times \mathcal{F}{\restriction}\Omega \times \mathcal{D}(\omega \setminus \Omega).$$

Since $g = h^{-1}$, it follows that $h[\mathcal{F}{\restriction}\Omega \times \mathcal{F}{\restriction}\Omega \times \mathcal{D}(\omega \setminus \Omega)] = \mathcal{F}{\restriction}\Omega \times \mathcal{D}(\omega)$. Therefore $\mathcal{F}{\restriction}\Omega \times \mathcal{F}{\restriction}\Omega \times \mathcal{D}(\omega \setminus \Omega) \approx \mathcal{F}{\restriction}\Omega \times \mathcal{D}(\omega)$. Finally, using Lemmas 3 and 4, one sees that $\mathcal{F}^2 \approx \mathcal{F}$. ∎

COROLLARY 7. *If $\mathcal{F}$ is a filter then $\mathcal{F}^m \approx \mathcal{F}^n$ for any natural numbers $m, n \geq 1$.*

5. Counterexamples for semifilters. A *semifilter* on Ω is a collection $\mathcal{F}$ of subsets of Ω that satisfies conditions (1), (2) and (3). All semifilters are assumed to be on ω. The following proposition shows that Theorem 6 would not hold if condition (4) were dropped from the definition of filter.

PROPOSITION 8. *There exists a semifilter $\mathcal{T}$ such that $\mathcal{T}^2 \not\approx \mathcal{T}$.*

Proof. Fix infinite sets Ω_1 and Ω_2 such that $\Omega_1 \cup \Omega_2 = \omega$ and $\Omega_1 \cap \Omega_2 = \emptyset$. Define

$$\mathcal{T} = \{X_1 \cup X_2 : X_1 \subseteq \Omega_1,\ X_2 \subseteq \Omega_2,\ \text{and } (X_1 \notin \mathsf{Fin}(\Omega_1) \text{ or } X_2 \in \mathsf{Cof}(\Omega_2))\},$$

and observe that $\mathcal{T}$ is a semifilter. Furthermore, it is clear that $\mathcal{T}$ is the union of its topologically complete subspace $\{X \subseteq \omega : X \cap \Omega_1 \notin \mathsf{Fin}(\Omega_1)\}$ and its countable subspace $\{X_1 \cup X_2 : X_1 \in \mathsf{Fin}(\Omega_1) \text{ and } X_2 \in \mathsf{Cof}(\Omega_2)\}$.

The following two statements are easy to verify.

- $\mathsf{Cof}(\Omega_2)$ is a closed subspace of $\mathcal{T}$ that is homeomorphic to $\mathbb{Q}$.
- $\{X \subseteq \omega : \Omega_1 \subseteq X\}$ is a closed subspace of $\mathcal{T}$ that is homeomorphic to 2^ω.

It follows that $\mathcal{T}^2$ has a closed subspace homeomorphic to $\mathbb{Q} \times 2^\omega$. Since, as is not hard to check, the space $\mathbb{Q} \times 2^\omega$ cannot be written as the union of a topologically complete subspace and a countable subspace, this concludes the proof. ∎

We remark that the semifilter $\mathcal{T}$ in the above proof is actually homeomorphic to the notable space **T** introduced by van Douwen (unpublished, see [vEvM]). See [Me, Proposition 5.4] for more details.

In fact, the main result of [Me] shows that every homogeneous zero-dimensional Borel space that is not locally compact is homeomorphic to a semifilter. Together with [vE2, Proposition 4.1], which states that $\mathcal{X}^2 \not\approx \mathcal{X}$ for almost every homogeneous zero-dimensional Borel space $\mathcal{X}$ of low complexity, this yields many more counterexamples as in Proposition 8.

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