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NEW CONGRUENCES MODULO 5 FOR OVERPARTITIONS

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Abstract. Let $\bar{p}(n)$ denote the number of overpartitions of n. Recently, several congruences for $\bar{p}(n)$ modulo 5 have been proved by Chen and Xia, by Chen, Sun, Wang and Zhang, and by Wang. In this paper, we prove new congruences for $\bar{p}(n)$ modulo 5 by using some theta function identities and the generating function for $\bar{p}(4n)$.

1. Introduction. An *overpartition* of n is a partition of n where the first occurrence of each distinct part may be overlined. Let $\bar{p}(n)$ denote the number of overpartitions of n. For example, $\bar{p}(4) = 14$ as there are 14 overpartitions for 4:

$$4,\ \bar{4},\ 3+1,\ \bar{3}+1,\ 3+\bar{1},\ \bar{3}+\bar{1},\ 2+2,\ \bar{2}+2,\ 2+1+1,$$
$$\bar{2}+1+1,\ 2+\bar{1}+1,\ \bar{2}+\bar{1}+1,\ 1+1+1+1,\ \bar{1}+1+1+1.$$

As usual, set $\bar{p}(0) = 1$ and $\bar{p}(n) = 0$ if $n < 0$. Corteel and Lovejoy [CL] showed that the generating function of $\bar{p}(n)$ is given by

$$\sum_{n=0}^{\infty} \bar{p}(n) q^n = \frac{(q^2;q^2)_\infty}{(q;q)_\infty^2},$$

where

$$(a;q)_\infty = \prod_{n=0}^{\infty} (1 - aq^n).$$

We will also write

$$(a_1, \dots, a_k; q)_\infty = (a_1;q)_\infty \dots (a_k;q)_\infty.$$

A number of congruences for overpartitions have been discovered during the past ten years. Congruences modulo powers of 2 and 3 were proved by Chen, Hou, Sun and Zhang [CHSZ], Fortin, Jacob and Mathieu [FJM], Hirschhorn and Sellers [HS], Kim [K], Lovejoy and Osburn [LO], Mahlburg [M], Wang [W], Xia [X] and Yao and Xia [YX]. Treneer [T] proved that $\bar{p}(5m^3n) \equiv 0 \pmod 5$ for any n that is coprime to m, where m is a prime

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satisfying $m \equiv -1 \pmod 5$. Hirschhorn and Sellers [HS] also conjectured that for $n \geq 0$,

$$\bar{p}(40n+35) \equiv 0 \pmod 5. \tag{1.1}$$

This conjecture was confirmed by Chen and Xia [CX] by using (p,k)-parametrization of theta functions. Recently, using half-integral weight modular forms, Chen, Sun, Wang and Zhang [CSWZ] discovered three infinite families of congruences for $\bar{p}(n)$ modulo 5. Wang [W] provided a new elementary proof for (1.1), and found some new congruences for the overpartition function modulo 5 and 9.

In this paper, we establish the following five new congruences for overpartitions modulo 5 by using the generating function for $\bar{p}(4n)$ and some theta function identities due to Ramanujan.

THEOREM 1.1. *For $n \geq 0$,*

$$\bar{p}(80n+8) \equiv 0 \pmod 5, \tag{1.2}$$

$$\bar{p}(80n+52) \equiv 0 \pmod 5, \tag{1.3}$$

$$\bar{p}(80n+68) \equiv 0 \pmod 5, \tag{1.4}$$

$$\bar{p}(80n+72) \equiv 0 \pmod 5, \tag{1.5}$$

and

$$\sum_{i+j=n} \bar{p}(80i+32)\bar{p}(80j+28) \equiv \sum_{i+j=n} \bar{p}(80i+12)\bar{p}(80j+48) \pmod 5. \tag{1.6}$$

2. Proof of the main results. The following generating function for $\bar{p}(4n)$ was established by Fortin, Jacob and Mathieu [FJM], and Hirschhorn and Sellers [HS]:

$$\sum_{n=0}^{\infty} \bar{p}(4n)q^n = \frac{(q^2;q^2)_\infty^{19}}{(q;q)_\infty^{14}(q^4;q^4)_\infty^6}. \tag{2.1}$$

Thanks to the binomial theorem,

$$(q;q)_\infty^5 \equiv (q^5;q^5)_\infty \pmod 5. \tag{2.2}$$

By (2.2), we can write (2.1) as

$$\sum_{n=0}^{\infty} \bar{p}(4n)q^n \equiv \frac{(q;q)_\infty(q^2;q^2)_\infty^4(q^{10};q^{10})_\infty^3}{(q^4;q^4)_\infty(q^5;q^5)_\infty^3(q^{20};q^{20})_\infty} \pmod 5. \tag{2.3}$$

Replacing q by $-q$ in (2.3), applying the fact that

$$(-q;-q)_\infty = \frac{(q^2;q^2)_\infty^3}{(q;q)_\infty(q^4;q^4)_\infty}$$

and then using (2.2), we get

$$\sum_{n=0}^{\infty}(-1)^n\bar{p}(4n)q^n \equiv \frac{(q^2;q^2)_\infty^7(q^5;q^5)_\infty^3(q^{20};q^{20})_\infty^2}{(q;q)_\infty(q^4;q^4)_\infty^2(q^{10};q^{10})_\infty^6} \equiv \frac{(q^2;q^2)_\infty^2}{(q;q)_\infty}(q^4;q^4)_\infty^3\frac{(q^5;q^5)_\infty^3(q^{20};q^{20})_\infty}{(q^{10};q^{10})_\infty^5} \pmod 5. \tag{2.4}$$

It follows from [B, Corollary (ii), p. 49] that

$$\frac{(q^2;q^2)_\infty^2}{(q;q)_\infty} = (-q^{10},-q^{15},q^{25};q^{25})_\infty + q(-q^5,-q^{20},q^{25};q^{25})_\infty + q^3\frac{(q^{50};q^{50})_\infty^2}{(q^{25};q^{25})_\infty}. \tag{2.5}$$

We have the well-known result of Jacobi [A, p. 176] which states that

$$(q;q)_\infty^3 = \sum_{n=0}^{\infty}(-1)^n(2n+1)q^{n(n+1)/2}.$$

This can be written as

$$(q;q)_\infty^3 \equiv (q^{10},q^{15},q^{25};q^{25})_\infty - 3q(q^5,q^{20},q^{25};q^{25})_\infty \pmod 5. \tag{2.6}$$

Substituting (2.5) and (2.6) into (2.4) and using (2.2), we get

$$\begin{aligned}\sum_{n=0}^{\infty}(-1)^n\bar{p}(4n)q^n \tag{2.7}\\ &\equiv \left((-q^{10},-q^{15},q^{25};q^{25})_\infty + q(-q^5,-q^{20},q^{25};q^{25})_\infty + q^3\frac{(q^{50};q^{50})_\infty^2}{(q^{25};q^{25})_\infty}\right)\\ &\quad\times\left((q^{40},q^{60},q^{100};q^{100})_\infty - 3q^4(q^{20},q^{80},q^{100};q^{100})_\infty\right)\frac{(q^5;q^5)_\infty^3(q^{20};q^{20})_\infty}{(q^{10};q^{10})_\infty^5}\\ &\equiv A_0(q)+qA_1(q)+q^3A_3(q)+q^4A_4(q)+q^5A_5(q)+q^7A_7(q) \pmod 5,\end{aligned}$$

where

$$A_0(q) = \frac{(q^5;q^5)_\infty^3(q^{20};q^{20})_\infty(-q^{10},-q^{15},q^{25};q^{25})_\infty(q^{40},q^{60},q^{100};q^{100})_\infty}{(q^{10};q^{10})_\infty^5},$$

$$A_1(q) = \frac{(q^5;q^5)_\infty^3(q^{20};q^{20})_\infty(-q^5,-q^{20},q^{25};q^{25})_\infty(q^{40},q^{60},q^{100};q^{100})_\infty}{(q^{10};q^{10})_\infty^5},$$

$$A_3(q) = \frac{(q^{20};q^{20})_\infty(q^{50};q^{50})_\infty(q^{40},q^{60},q^{100};q^{100})_\infty}{(q^5;q^5)_\infty^2},$$

$$A_4(q) = 2\frac{(q^5;q^5)_\infty^3(q^{20};q^{20})_\infty(-q^{10},-q^{15},q^{25};q^{25})_\infty(q^{20},q^{80},q^{100};q^{100})_\infty}{(q^{10};q^{10})_\infty^5},$$

$$A_5(q) = 2\frac{(q^5;q^5)_\infty^3(q^{20};q^{20})_\infty(-q^5,-q^{20},q^{25};q^{25})_\infty(q^{20},q^{80},q^{100};q^{100})_\infty}{(q^{10};q^{10})_\infty^5},$$

$$A_7(q) = 2\frac{(q^{20};q^{20})_\infty(q^{50};q^{50})_\infty(q^{20},q^{80},q^{100};q^{100})_\infty}{(q^5;q^5)_\infty^2}.$$

If we extract the terms of the form q^{5n+2} (resp. q^{5n+3}) in (2.7), divide by q^2 (resp. q^3) and replace q^5 by q, we see that

$$\sum_{n=0}^{\infty}(-1)^n\bar{p}(4(5n+2))q^n \equiv 2q\frac{(q^4;q^4)_\infty(q^{10};q^{10})_\infty(q^4,q^{16},q^{20};q^{20})_\infty}{(q;q)_\infty^2} \pmod 5 \tag{2.8}$$

and

$$\sum_{n=0}^{\infty}(-1)^{n+1}\bar{p}(4(5n+3))q^n \equiv \frac{(q^4;q^4)_\infty(q^{10};q^{10})_\infty(q^8,q^{12},q^{20};q^{20})_\infty}{(q;q)_\infty^2} \pmod 5. \tag{2.9}$$

It follows from [B, Entry 25, p. 40] that

$$\frac{1}{(q;q)_\infty^2} = \frac{(q^8;q^8)_\infty^5}{(q^2;q^2)_\infty^5(q^{16};q^{16})_\infty^2} + 2q\frac{(q^4;q^4)_\infty^2(q^{16};q^{16})_\infty^2}{(q^2;q^2)_\infty^5(q^8;q^8)_\infty}. \tag{2.10}$$

Substituting (2.10) into (2.8) and (2.9), and then applying (2.2), we get

$$\begin{aligned}\sum_{n=0}^{\infty}(-1)^n\bar{p}(20n+8)q^n &\equiv 2q(q^4;q^4)_\infty(q^{10};q^{10})_\infty(q^4,q^{16},q^{20};q^{20})_\infty \\ &\quad\times\left(\frac{(q^8;q^8)_\infty^5}{(q^2;q^2)_\infty^5(q^{16};q^{16})_\infty^2} + 2q\frac{(q^4;q^4)_\infty^2(q^{16};q^{16})_\infty^2}{(q^2;q^2)_\infty^5(q^8;q^8)_\infty}\right) \\ &\equiv 2q\frac{(q^4;q^4)_\infty(q^8;q^8)_\infty^5(q^4,q^{16},q^{20};q^{20})_\infty}{(q^{16};q^{16})_\infty^2} \\ &\quad + 4q^2\frac{(q^4;q^4)_\infty^3(q^{16};q^{16})_\infty^2(q^4,q^{16},q^{20};q^{20})_\infty}{(q^8;q^8)_\infty} \pmod 5\end{aligned} \tag{2.11}$$

and

$$\begin{aligned}\sum_{n=0}^{\infty}(-1)^{n+1}\bar{p}(20n+12)q^n &\equiv (q^4;q^4)_\infty(q^{10};q^{10})_\infty(q^8,q^{12},q^{20};q^{20})_\infty \\ &\quad\times\left(\frac{(q^8;q^8)_\infty^5}{(q^2;q^2)_\infty^5(q^{16};q^{16})_\infty^2} + 2q\frac{(q^4;q^4)_\infty^2(q^{16};q^{16})_\infty^2}{(q^2;q^2)_\infty^5(q^8;q^8)_\infty}\right) \\ &\equiv \frac{(q^4;q^4)_\infty(q^8;q^8)_\infty^5(q^8,q^{12},q^{20};q^{20})_\infty}{(q^{16};q^{16})_\infty^2} \\ &\quad + 2q\frac{(q^4;q^4)_\infty^3(q^{16};q^{16})_\infty^2(q^8,q^{12},q^{20};q^{20})_\infty}{(q^8;q^8)_\infty} \pmod 5.\end{aligned} \tag{2.12}$$

Congruences (1.2) and (1.4) follow from (2.11), and congruences (1.3) and (1.5) follow from (2.12). Moreover, congruences (2.11) and (2.12) imply that

$$\sum_{n=0}^{\infty} \bar{p}(80n+12)q^n \equiv 4\frac{(q;q)_\infty (q^2;q^2)_\infty^5 (q^2,q^3,q^5;q^5)_\infty}{(q^4;q^4)_\infty^2} \pmod 5, \tag{2.13}$$

$$\sum_{n=0}^{\infty} \bar{p}(80n+28)q^n \equiv 3\frac{(q;q)_\infty (q^2;q^2)_\infty^5 (q,q^4,q^5;q^5)_\infty}{(q^4;q^4)_\infty^2} \pmod 5, \tag{2.14}$$

$$\sum_{n=0}^{\infty} \bar{p}(40n+32)q^n \equiv 2\frac{(q;q)_\infty^3 (q^4;q^4)_\infty^2 (q^2,q^3,q^5;q^5)_\infty}{(q^2;q^2)_\infty} \pmod 5, \tag{2.15}$$

$$\sum_{n=0}^{\infty} \bar{p}(80n+48)q^n \equiv 4\frac{(q;q)_\infty^3 (q^4;q^4)_\infty^2 (q,q^4,q^5;q^5)_\infty}{(q^2;q^2)_\infty} \pmod 5. \tag{2.16}$$

In view of (2.13)–(2.16), we have

$$\sum_{n=0}^{\infty} \sum_{i+j=n} \bar{p}(80i+12)\bar{p}(80j+48)q^n \equiv \sum_{n=0}^{\infty} \sum_{i+j=n} \bar{p}(80i+28)\bar{p}(80j+32)q^n \pmod 5,$$

which yields (1.6). This completes the proof.

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