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COMBINATORIAL IDENTITIES AND TRIGONOMETRIC INEQUALITIES

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Abstract. The aim of this paper is threefold: (i) We offer short and elementary new proofs for

$$(*)\qquad \sum_{k=0}^{n} 2^{n-k}\binom{n}{k}\binom{m}{k}=\sum_{k=0}^{n}\binom{n}{k}\binom{m+k}{k},$$

$$(**)\qquad \sum_{k=0}^{n}\binom{\alpha+k-1}{k}(z+1)^k=\alpha\binom{\alpha+n}{n}\sum_{k=0}^{n}\binom{n}{k}\frac{z^k}{\alpha+k}.$$

The first identity was published by Brereton et al. in 2011 and the second one extends a result provided by the same authors. (ii) We present q-analogues of $(*)$ and $(**)$. (iii) We use $(**)$ to derive identities and inequalities for trigonometric polynomials. Among other results, we show that

$$\sin(t)+\sum_{k=2}^{n} c(c+1)\cdots(c+k-2)\frac{\sin(kt)}{k!}>0 \qquad (c\in\mathbb{R})$$

for all $n\in\mathbb{N}$ and $t\in(0,\pi)$ if and only if $c\in[-1,1]$. This provides a new extension of the classical Fejér–Jackson inequality.

1. Introduction and statement of results

(I) Our work has been inspired by an interesting article which was published by Brereton et al. [3] in 2011. The six authors presented the following combinatorial identities.

THEOREM 1. *For all $m,n\in\mathbb{N}_0=\mathbb{N}\cup\{0\}$ we have*

$$(1.1)\qquad \sum_{k=0}^{n} 2^{n-k}\binom{n}{k}\binom{m}{k}=\sum_{k=0}^{n}\binom{n}{k}\binom{m+k}{k}.$$

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THEOREM 2. *For all $m \in \mathbb{N}$ and $n \in \mathbb{N}_0$ we have*

$$\sum_{k=0}^{n} \binom{m+n}{k} = m\binom{m+n}{n} \sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{2^{n-k}}{m+k}. \tag{1.2}$$

A generalization of (1.1) can be found in [9, (3.18)]. The sums in (1.1) are closely related to the *Delannoy numbers*

$$D^*(m,n) = \sum_{k=0}^{n} 2^k \binom{n}{k}\binom{m}{k} = \sum_{k=0}^{n} \binom{m+n-k}{m}\binom{m}{k}.$$

Indeed, $D^*(n,n)$ is equal to the expressions in (1.1) with $m = n$. The Delannoy number counts the number of lattice paths from $(0,0)$ to (m,n) allowing only east $(1,0)$, north $(0,1)$ and northeast $(1,1)$ steps. See [2] for additional information on this subject.

Brereton and his co-authors offered three different proofs for (1.1) and (1.2). They used combinatorial arguments, the computer-assisted method of Wilf–Zeilberger and the generatingfunctionology technique. One of the goals of the present paper is to provide short and elementary new proofs for (1.1) and the following extension of (1.2).

THEOREM 3. *For all $n \in \mathbb{N}_0$, $\alpha \in \mathbb{R} \setminus \{0,-1,\dots,-n\}$ and $z \in \mathbb{C}$ we have*

$$\sum_{k=0}^{n} \binom{\alpha+k-1}{k}(z+1)^k = \alpha\binom{\alpha+n}{n} \sum_{k=0}^{n} \binom{n}{k} \frac{z^k}{\alpha+k}. \tag{1.3}$$

If we set $\alpha = m$, $z = -1/2$ in (1.3), multiply both sides by 2^n and apply the identity

$$\sum_{k=0}^{n} \binom{m+n}{k} = \sum_{k=0}^{n} \binom{m+k-1}{k} 2^{n-k}$$

(which can be proved easily by induction on n), we obtain (1.2).

Applications of (1.3) lead to the combinatorial identities

$$\sum_{k=j}^{n-r} \binom{\alpha+k+r-1}{k+r}\binom{k}{j} \prod_{\nu=0}^{r-1} \frac{k+r-\nu}{j+r-\nu} = \frac{\alpha}{\alpha+j+r}\binom{\alpha+n}{n}\binom{n}{j+r} \tag{1.4}$$

and

$$\sum_{k=j}^{n-r} \frac{(-1)^{k-j}}{\alpha+k+r}\binom{n}{k+r}\binom{k}{j} \prod_{\nu=0}^{r-1} \frac{k+r-\nu}{j+r-\nu} = \frac{\binom{\alpha+j+r-1}{j+r}}{\alpha\binom{\alpha+n}{n}} \tag{1.5}$$

which hold for $j \in \{0,1,\dots,n-r\}$. In fact, if we differentiate both sides of (1.3) r times with respect to z and compare the coefficients, we obtain (1.4). The same method leads to (1.5) if we set $z = w - 1$ in (1.3).

In order to verify (1.1) and (1.3) we use induction. A key role in our proofs is played by Pascal's rule

$$\binom{\alpha}{k}+\binom{\alpha}{k-1}=\binom{\alpha+1}{k}. \tag{1.6}$$

(II) The *q-binomial coefficient*, which is also known as the *Gaussian binomial coefficient*, is defined for $\alpha\in\mathbb{R}$ and $n\in\mathbb{N}_0$ by

$$\begin{bmatrix}\alpha\\0\end{bmatrix}_q=1,\qquad \begin{bmatrix}\alpha\\n\end{bmatrix}_q=\prod_{k=0}^{n-1}\frac{1-q^{\alpha-k}}{1-q^{k+1}}\quad (n\in\mathbb{N}).$$

Here, $q\in\mathbb{R}$ with $q\in(0,1)$. Moreover, if $-n\in\mathbb{N}$, then we set $\left[\begin{smallmatrix}\alpha\\n\end{smallmatrix}\right]_q=0$.

The connection between $\left[\begin{smallmatrix}\alpha\\n\end{smallmatrix}\right]_q$ and the ordinary binomial coefficient $\binom{\alpha}{n}$ is given by the limit relation

$$\lim_{q\to 1}\begin{bmatrix}\alpha\\n\end{bmatrix}_q=\binom{\alpha}{n}. \tag{1.7}$$

The q-binomial coefficients have interesting applications in various branches, like the theory of partitions and the theory of projective spaces. A collection of the most important properties of these coefficients can be found, for instance, in [5, Chapter 6].

In many articles it was shown that certain identities involving binomial coefficients have q-analogues. For example, the following extensions of the binomial theorem and the Chu–Vandermonde identity are valid:

$$\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_q q^{\binom{k}{2}}x^k=(-x;q)_n, \tag{1.8}$$

where

$$(a;q)_n=\prod_{k=0}^{n-1}(1-aq^k)$$

and

$$\begin{bmatrix}\alpha+\beta\\k\end{bmatrix}_q=\sum_{j=0}^{k}q^{(\alpha-j)(k-j)}\begin{bmatrix}\alpha\\j\end{bmatrix}_q\begin{bmatrix}\beta\\k-j\end{bmatrix}_q=\sum_{j=0}^{k}q^{j(\beta-k+j)}\begin{bmatrix}\alpha\\j\end{bmatrix}_q\begin{bmatrix}\beta\\k-j\end{bmatrix}_q \tag{1.9}$$

(see [1, Theorem 3.3, eq. (3.3.10)]).

The next two theorems yield q-analogues of (1.1) and (1.3).

THEOREM 4. *For all $n\in\mathbb{N}_0$ and $\alpha\in\mathbb{R}$ we have*

$$\sum_{k=0}^{n}q^{(3k-1)k/2}(-q^k;q)_{n-k}\begin{bmatrix}n\\k\end{bmatrix}_q\begin{bmatrix}\alpha\\k\end{bmatrix}_q=\sum_{k=0}^{n}q^{\binom{k}{2}}\begin{bmatrix}n\\k\end{bmatrix}_q\begin{bmatrix}\alpha+k\\k\end{bmatrix}_q. \tag{1.10}$$

THEOREM 5. *For all $n \in \mathbb{N}_0$, $\alpha \in \mathbb{R} \setminus \{0, -1, \dots, -n\}$ and $z \in \mathbb{C}$ we have*

$$\sum_{k=0}^{n} q^k \begin{bmatrix} \alpha+k-1 \\ k \end{bmatrix}_q (-z;q)_k = [\alpha]_q \begin{bmatrix} \alpha+n \\ n \end{bmatrix}_q \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q \frac{q^{\binom{k+1}{2}} z^k}{[\alpha+k]_q}, \tag{1.11}$$

where

$$[a]_q = \begin{bmatrix} a \\ 1 \end{bmatrix}_q = \frac{1-q^a}{1-q}.$$

Using the limit relations (1.7) and $\lim_{q\to1}[a]_q = a$ as well as $(-z;1)_k = (z+1)^k$ shows that if $q \to 1$, then (1.10) and (1.11) lead to (1.1) and (1.3), respectively. Moreover, we find that (1.1) is valid even if m is a real number.

(III) An application of Theorem 3 gives new identities for trigonometric polynomials. Let $\alpha \in \mathbb{R} \setminus \{0, -1, \dots, -n\}$. We set $z = e^{it}$ $(t \in \mathbb{R})$ in (1.3) and compare the real and imaginary parts on both sides. This leads to

$$\sum_{k=0}^{n} \binom{\alpha+k-1}{k} \sum_{j=0}^{k} \binom{k}{j} \cos(jt) = \alpha \binom{\alpha+n}{n} \sum_{k=0}^{n} \binom{n}{k} \frac{\cos(kt)}{\alpha+k} \tag{1.12}$$

and

$$\sum_{k=1}^{n} \binom{\alpha+k-1}{k} \sum_{j=1}^{k} \binom{k}{j} \sin(jt) = \alpha \binom{\alpha+n}{n} \sum_{k=1}^{n} \binom{n}{k} \frac{\sin(kt)}{\alpha+k}. \tag{1.13}$$

If we set $z = e^{it} - 1$ $(t \in \mathbb{R})$ in (1.3), then we obtain counterparts of (1.12) and (1.13):

$$\sum_{k=0}^{n} \binom{\alpha+k-1}{k} \cos(kt) = \alpha \binom{\alpha+n}{n} \sum_{k=0}^{n} \frac{1}{\alpha+k} \binom{n}{k} \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} \cos(jt)$$

and

$$\sum_{k=1}^{n} \binom{\alpha+k-1}{k} \sin(kt) = \alpha \binom{\alpha+n}{n} \sum_{k=1}^{n} \frac{1}{\alpha+k} \binom{n}{k} \sum_{j=1}^{k} (-1)^{k-j} \binom{k}{j} \sin(jt). \tag{1.14}$$

We define three one-parameter families of trigonometric polynomials:

$$\begin{aligned} A_n(c,t) &= \sum_{k=1}^{n} \frac{(c)_{k-1}}{k!} \sin(kt), \\ A_n^*(c,t) &= \sum_{k=1}^{n} \frac{(c)_{k-1}}{k!} \cos(kt) \qquad (c \in \mathbb{R}), \end{aligned} \tag{1.15}$$

where $(x)_n$ denotes the Pochhammer symbol

$$(x)_0 = 1, \qquad (x)_n = \prod_{j=0}^{n-1}(x+j) \quad (n \in \mathbb{N}),$$

and

$$B_n(\alpha, t) = \sum_{k=1}^{n} \frac{1}{\alpha+k}\binom{n}{k}\sum_{j=1}^{k}(-1)^{k-j}\binom{k}{j}\sin(jt) \qquad (\alpha \in \mathbb{R}\setminus\{-1,\ldots,-n\}). \tag{1.16}$$

The polynomials (1.15) satisfy the system of functional-differential equations

$$\begin{aligned}
\frac{1}{c}\left(\frac{\partial}{\partial t}A_n(c,t) - \cos(t)\right) &= -\sin(t)A_{n-1}(c+1,t) + \cos(t)A^*_{n-1}(c+1,t),\\
\frac{1}{c}\left(\frac{\partial}{\partial t}A^*_n(c,t) + \sin(t)\right) &= -\cos(t)A_{n-1}(c+1,t) - \sin(t)A^*_{n-1}(c+1,t).
\end{aligned}$$

From (1.14) we conclude that the sine polynomials (1.15) and (1.16) are connected by the identity

$$A_n(\alpha+1, t) = \binom{\alpha+n}{n}B_n(\alpha, t). \tag{1.17}$$

Setting $c = 1$ in (1.15) leads to the well-known sine polynomial of Fejér:

$$A_n(1,t) = \sum_{k=1}^{n}\frac{\sin(kt)}{k}.$$

A classical result in the theory of trigonometric polynomials states that

$$A_n(1,t) > 0 \qquad (n \in \mathbb{N},\ 0 < t < \pi). \tag{1.18}$$

This is known as the *Fejér–Jackson inequality*. The validity of (1.18) was conjectured by Fejér in 1910 and proved one year later by Jackson [4]. This result has attracted the attention of many researchers, who offered numerous proofs, extensions, variants, refinements and remarkable applications. For more information we refer to [8, Chapter 4] and the references cited therein.

In view of (1.18) it is natural to ask about all parameters c such that $A_n(c,t)$ is positive for all $n \in \mathbb{N}$ and $t \in (0,\pi)$. We prove the following generalization of the Fejér–Jackson inequality.

Theorem 6. *Let $c \in \mathbb{R}$. Then*

$$\sum_{k=1}^{n}\frac{(c)_{k-1}}{k!}\sin(kt) > 0 \tag{1.19}$$

for all $n \in \mathbb{N}$ and $t \in (0,\pi)$ if and only if $c \in [-1,1]$.

Using Theorem 6 and (1.17) we are able to determine all parameters α such that $B_n(\alpha,t)$ is positive for all $n\in\mathbb{N}$ and $t\in(0,\pi)$.

THEOREM 7. *Let $\alpha\in\mathbb{R}\setminus\{-1,-2,\dots\}$. Then*

$$\sum_{k=1}^{n}\frac{1}{\alpha+k}\binom{n}{k}\sum_{j=1}^{k}(-1)^{k-j}\binom{k}{j}\sin(jt)>0 \tag{1.20}$$

for all $n\in\mathbb{N}$ and $t\in(0,\pi)$ if and only if $\alpha\in(-1,0]$.

An elegant well-known variant of the Fejér–Jackson inequality states that

$$\sum_{\substack{k=1\\ k\,\mathrm{odd}}}^{n}\frac{\sin(kt)}{k}>0 \qquad (n\in\mathbb{N},\ 0<t<\pi). \tag{1.21}$$

Applying Theorem 6 yields an extension of (1.21): if $c\in[-1,1]$, then

$$\begin{aligned}\frac{1}{2}&(A_n(c;t)+A_n(c;\pi-t))\\ &=\sum_{\substack{k=1\\ k\,\mathrm{odd}}}^{n}\frac{(c)_{k-1}}{k!}\sin(kt)>0 \qquad (n\in\mathbb{N},\ 0<t<\pi).\end{aligned} \tag{1.22}$$

Are there still more parameters c such that (1.22) is valid? The next theorem gives an affirmative answer.

THEOREM 8. *Let $c\in\mathbb{R}$. Then*

$$\sum_{\substack{k=1\\ k\ odd}}^{n}\frac{(c)_{k-1}}{k!}\sin(kt)>0 \tag{1.23}$$

for all $n\in\mathbb{N}$ and $t\in(0,\pi)$ if and only if $c\in(-3,2)$.

An application of Theorem 8 leads to our final result. It provides a counterpart of Theorem 6.

THEOREM 9. *Let $c\in\mathbb{R}$. Then*

$$\sum_{k=1}^{n}\frac{(c)_{2k-2}}{(2k)!}\sin(kt)>0 \tag{1.24}$$

for all $n\in\mathbb{N}$ and $t\in(0,\pi)$ if and only if $c\in[-3,2]$.

In the next section we collect some lemmas. The proofs of the theorems are given in Sections 3–10.

2. Lemmas. Throughout, we maintain the notations introduced in this section. The first two lemmas play a role in the proofs of Theorems 1 and 3, respectively.

LEMMA 1. *Let*

$$C(m,n)=\sum_{k=0}^{n}2^{n-k}\binom{n}{k}\binom{m}{k},\qquad D(m,n)=\sum_{k=0}^{n}2^{n-k}\binom{n}{k}\binom{m}{k+1},$$
$$F(m,n)=\sum_{k=0}^{n}\binom{n}{k}\binom{m+k}{k},\qquad G(m,n)=\sum_{k=0}^{n}\binom{n}{k}\binom{m+k}{k+1}.$$

For all $m\in\mathbb{N}_0$ and $n\in\mathbb{N}_0$ we have

$$C(m,n)=D(m+1,n)-D(m,n), \tag{2.1}$$
$$D(m+1,n)=C(m,n+1)-C(m,n), \tag{2.2}$$
$$F(m,n)=G(m+1,n)-G(m,n), \tag{2.3}$$
$$G(m+1,n)=F(m,n+1)-F(m,n). \tag{2.4}$$

Proof. From (1.6) we deduce (2.1), (2.3) and (2.4). Applying (1.6) and (2.1) gives

$$C(m,n+1)=2C(m,n)+D(m,n)=C(m,n)+D(m+1,n).$$

This leads to (2.2). ∎

LEMMA 2. *Let*

$$E(\alpha,n,z)=\sum_{k=0}^{n}\binom{\alpha+k}{k}(z+1)^k.$$

For all $n\in\mathbb{N}_0$, $\alpha\in\mathbb{R}$ and $z\in\mathbb{C}$ we have

$$E(\alpha-1,n,z)+zE(\alpha,n,z)=\binom{\alpha+n}{n}(z+1)^{n+1}. \tag{2.5}$$

Proof. We use induction on n. If $n=0$, then both sides of (2.5) are equal to $z+1$. Using the induction hypothesis and (1.6) gives

$$\begin{aligned}
&E(\alpha-1,n+1,z)+zE(\alpha,n+1,z)\\
&=E(\alpha-1,n,z)+zE(\alpha,n,z)+\binom{\alpha+n}{n+1}(z+1)^{n+1}+z\binom{\alpha+n+1}{n+1}(z+1)^{n+1}\\
&=\left\{\binom{\alpha+n}{n}+\binom{\alpha+n}{n+1}+z\binom{\alpha+n+1}{n+1}\right\}(z+1)^{n+1}\\
&=\binom{\alpha+n+1}{n+1}(z+1)\,(z+1)^{n+1}=\binom{\alpha+n+1}{n+1}(z+1)^{n+2}.\ \blacksquare
\end{aligned}$$

The following three lemmas provide inequalities for some classes of sine polynomials.

LEMMA 3. *Let* $a_k \geq 0$ $(k = 1, \dots, n)$ *with* $a_1 > \sum_{k=2}^n k a_k$. *Then, for* $t \in (0, \pi)$,

$$a_1 \sin(t) > \sum_{k=2}^n a_k \sin(kt). \tag{2.6}$$

Proof. Let $t \in (0, \pi)$ and

$$s_n = a_1 - \sum_{k=2}^n k a_k.$$

Then

$$a_1 \sin(t) - \sum_{k=2}^n a_k \sin(kt) = s_n \sin(t) + \sum_{k=2}^n a_k (k \sin(t) - \sin(kt)).$$

Since $s_n > 0$, $a_k \geq 0$ $(k = 2, \dots, n)$ and

$$k \sin(t) \geq |\sin(kt)| \quad (k \in \mathbb{N}),$$

we conclude that (2.6) holds. ∎

REMARK. The following converse is valid. If (2.6) with "$\geq$" instead of "$>$" holds for all $t \in (0, \pi)$, then

$$a_1 \frac{\sin(t)}{t} \geq \sum_{k=2}^n k a_k \frac{\sin(kt)}{kt}.$$

By letting $t \to 0$ we find $a_1 \geq \sum_{k=2}^n k a_k$.

The next two lemmas offer generalizations of the Fejér–Jackson inequality. The first one is a celebrated result of Vietoris [11] (see also [6] and [7]).

LEMMA 4. *If the real numbers* a_k $(k = 1, \dots, n)$ *satisfy*

$$a_1 \geq \dots \geq a_n > 0 \quad \textit{and} \quad (2k-1) a_{2k-1} \geq 2k a_{2k} \quad (k = 1, \dots, [n/2]),$$

then

$$\sum_{k=1}^n a_k \sin(kt) > 0 \quad (0 < t < \pi).$$

The final lemma is due to Turán [10] (see also [8, Section 4.2.1]).

LEMMA 5. *If* a_k $(k = 1, \dots, n)$ *are real numbers such that*

$$\sum_{k=1}^n a_k \sin((2k-1)t) \geq 0 \quad (0 \leq t \leq \pi),$$

then

$$\sum_{k=1}^n a_k \frac{\sin(kt)}{k} > 0 \quad (0 < t < \pi),$$

unless $a_1 = \dots = a_n = 0$.

3. Proof of Theorem 1. We use induction on n to prove $C(m,n) = F(m,n)$. For $n = 0$ we obtain

$$C(m,0) = F(m,0) = 1.$$

Now, we suppose that

$$C(m,n) = F(m,n) \quad \text{for } m \geq 0.$$

Then applying (2.1)–(2.4) and the induction hypothesis yields

$$\begin{aligned} C(m,n+1) &= D(m+1,n) + C(m,n) = D(m,n) + 2C(m,n) \\ &= \sum_{k=0}^{m-1} [D(k+1,n) - D(k,n)] + 2C(m,n) = \sum_{k=0}^{m-1} C(k,n) + 2C(m,n) \\ &= \sum_{k=0}^{m-1} F(k,n) + 2F(m,n) = \sum_{k=0}^{m-1} [G(k+1,n) - G(k,n)] + 2F(m,n) \\ &= G(m,n) + 2F(m,n) = G(m+1,n) + F(m,n) = F(m,n+1). \end{aligned}$$

4. Proof of Theorem 3. Let

$$H(\alpha,n,z) = \frac{1}{\alpha\binom{\alpha+n}{n}} \sum_{k=0}^{n} \binom{\alpha+k-1}{k}(z+1)^k, \qquad I(\alpha,n,z) = \sum_{k=0}^{n} \binom{n}{k} \frac{z^k}{\alpha+k}.$$

We prove $H(\alpha,n,z) = I(\alpha,n,z)$ by induction on n. Setting $n = 0$ yields

$$H(\alpha,0,z) = I(\alpha,0,z) = 1/\alpha.$$

We apply (1.6) and the induction hypothesis to obtain

$$\begin{aligned} I(\alpha,n+1,z) &= I(\alpha,n,z) + zI(\alpha+1,n,z) \\ &= H(\alpha,n,z) + zH(\alpha+1,n,z). \end{aligned} \tag{4.1}$$

Using Lemma 2 leads to

$$\begin{aligned} &H(\alpha,n,z) + zH(\alpha+1,n,z) - H(\alpha,n+1,z) \\ &= \frac{1}{(\alpha+n+1)\binom{\alpha+n}{n}} \left(E(\alpha-1,n,z) + zE(\alpha,n,z) - \binom{\alpha+n}{n}(z+1)^{n+1} \right) = 0. \end{aligned} \tag{4.2}$$

Finally, combining (4.1) and (4.2) gives $H(\alpha,n+1,z) = I(\alpha,n+1,z)$.

5. Proof of Theorem 4. Using the identities

$$(-q^k;q)_{n-k} = q^{n(n-k)-\binom{n-k+1}{2}}(-q^{1-n};q)_{n-k}, \qquad \begin{bmatrix} n \\ k \end{bmatrix}_q = \begin{bmatrix} n \\ n-k \end{bmatrix}_q$$

and

$$\begin{bmatrix} \alpha \\ m \end{bmatrix}_q \begin{bmatrix} m \\ j \end{bmatrix}_q = \begin{bmatrix} \alpha \\ j \end{bmatrix}_q \begin{bmatrix} \alpha-j \\ m-j \end{bmatrix}_q, \tag{5.1}$$

as well as (1.8) and (1.9), gives

$$\begin{aligned}\sum_{k=0}^{n} q^{(3k-1)k/2}(-q^k;q)_{n-k}\begin{bmatrix} n\\ k\end{bmatrix}_q\begin{bmatrix} \alpha\\ k\end{bmatrix}_q &= q^{\binom{n}{2}}\sum_{k=0}^{n} q^{k^2}(-q^{1-n};q)_{n-k}\begin{bmatrix} n\\ k\end{bmatrix}_q\begin{bmatrix} \alpha\\ k\end{bmatrix}_q\\ &= q^{\binom{n}{2}}\sum_{k=0}^{n} q^{k^2}\begin{bmatrix} n\\ n-k\end{bmatrix}_q\begin{bmatrix} \alpha\\ k\end{bmatrix}_q\sum_{j=0}^{n-k}\begin{bmatrix} n-k\\ j\end{bmatrix}_q q^{\binom{j}{2}+j(1-n)}\\ &= q^{\binom{n}{2}}\sum_{j=0}^{n} q^{\binom{j}{2}+j(1-n)}\begin{bmatrix} n\\ j\end{bmatrix}_q\sum_{k=0}^{n-j} q^{k^2}\begin{bmatrix} n-j\\ n-k-j\end{bmatrix}_q\begin{bmatrix} \alpha\\ k\end{bmatrix}_q\\ &= \sum_{j=0}^{n} q^{\binom{n-j}{2}}\begin{bmatrix} n\\ j\end{bmatrix}_q\begin{bmatrix} \alpha+n-j\\ n-j\end{bmatrix}_q = \sum_{k=0}^{n} q^{\binom{k}{2}}\begin{bmatrix} n\\ k\end{bmatrix}_q\begin{bmatrix} \alpha+k\\ k\end{bmatrix}_q.\end{aligned}$$

6. Proof of Theorem 5. We have

$$\begin{bmatrix} \alpha+k-1\\ k\end{bmatrix}_q = (-1)^k q^{\alpha k+\binom{k}{2}}\begin{bmatrix} -\alpha\\ k\end{bmatrix}_q. \tag{6.1}$$

The special case $\alpha = 1$ leads to

$$(-1)^k q^{\binom{k+1}{2}}\begin{bmatrix} -1\\ k\end{bmatrix}_q = 1. \tag{6.2}$$

Applying (6.1), (1.8), (5.1) and (6.2) yields

$$\begin{aligned}&\sum_{k=0}^{n} q^k\begin{bmatrix} \alpha+k-1\\ k\end{bmatrix}_q(-z;q)_k \\ &= \sum_{k=0}^{n}(-1)^k q^{(\alpha+1)k+\binom{k}{2}}\begin{bmatrix} -\alpha\\ k\end{bmatrix}_q\sum_{j=0}^{k}\begin{bmatrix} k\\ j\end{bmatrix}_q q^{\binom{j}{2}}z^j\\ &= \sum_{j=0}^{n} q^{\binom{j}{2}}z^j\begin{bmatrix} -\alpha\\ j\end{bmatrix}_q\sum_{k=j}^{n}(-1)^k q^{(\alpha+1)k+\binom{k}{2}}\begin{bmatrix} -\alpha-j\\ k-j\end{bmatrix}_q\\ &= (-1)^n\sum_{j=0}^{n} q^{\binom{j}{2}}z^j\begin{bmatrix} -\alpha\\ j\end{bmatrix}_q\sum_{k=j}^{n} q^{(\alpha+1)k+\binom{k}{2}+\binom{n-k+1}{2}}\begin{bmatrix} -1\\ n-k\end{bmatrix}_q\begin{bmatrix} -\alpha-j\\ k-j\end{bmatrix}_q.\end{aligned} \tag{6.3}$$

Next, we make use of (1.9). We obtain

$$\begin{aligned}&\sum_{k=j}^{n} q^{(\alpha+1)k+\binom{k}{2}+\binom{n-k+1}{2}}\begin{bmatrix} -1\\ n-k\end{bmatrix}_q\begin{bmatrix} -\alpha-j\\ k-j\end{bmatrix}_q\\ &= q^{\alpha n+\binom{n+1}{2}}\sum_{k=j}^{n} q^{(-\alpha-k)(n-k)}\begin{bmatrix} -1\\ n-k\end{bmatrix}_q\begin{bmatrix} -\alpha-j\\ k-j\end{bmatrix}_q = q^{\alpha n+\binom{n+1}{2}}\begin{bmatrix} -\alpha-j-1\\ n-j\end{bmatrix}_q.\end{aligned} \tag{6.4}$$

Combining (6.3) and (6.4) gives

$$\sum_{k=0}^{n} q^k \begin{bmatrix} \alpha+k-1 \\ k \end{bmatrix}_q (-z;q)_k = (-1)^n q^{\alpha n + \binom{n+1}{2}} \sum_{j=0}^{n} q^{\binom{j}{2}} z^j \begin{bmatrix} -\alpha \\ j \end{bmatrix}_q \begin{bmatrix} -\alpha-j-1 \\ n-j \end{bmatrix}_q. \tag{6.5}$$

We apply

$$\begin{bmatrix} -\alpha \\ j \end{bmatrix}_q = q^j \frac{[\alpha]_q}{[\alpha+j]_q} \begin{bmatrix} -\alpha-1 \\ j \end{bmatrix}_q$$

as well as (5.1) and (6.1). Then

$$\begin{aligned} \sum_{j=0}^{n} q^{\binom{j}{2}} \begin{bmatrix} -\alpha \\ j \end{bmatrix}_q \begin{bmatrix} -\alpha-j-1 \\ n-j \end{bmatrix}_q z^j &= \sum_{j=0}^{n} q^{\binom{j+1}{2}} \frac{[\alpha]_q}{[\alpha+j]_q} \begin{bmatrix} -\alpha-1 \\ j \end{bmatrix}_q \begin{bmatrix} -\alpha-j-1 \\ n-j \end{bmatrix}_q z^j \\ &= \begin{bmatrix} -\alpha-1 \\ n \end{bmatrix}_q \sum_{j=0}^{n} q^{\binom{j+1}{2}} \frac{[\alpha]_q}{[\alpha+j]_q} \begin{bmatrix} n \\ j \end{bmatrix}_q z^j \\ &= (-1)^n q^{-\alpha n - \binom{n+1}{2}} [\alpha]_q \begin{bmatrix} \alpha+n \\ n \end{bmatrix}_q \sum_{j=0}^{n} \begin{bmatrix} n \\ j \end{bmatrix}_q \frac{q^{\binom{j+1}{2}} z^j}{[\alpha+j]_q}. \end{aligned} \tag{6.6}$$

From (6.5) and (6.6) we conclude that (1.11) holds.

7. Proof of Theorem 6. We use the notation of (1.15). First, we assume that (1.19) is valid for all $n \in \mathbb{N}$ and $t \in (0, \pi)$. Then

$$\frac{A_2(c,t)}{\sin(t)} = 1 + c\cos(t) > 0.$$

Letting $t \to 0$ and $t \to \pi$, respectively, we obtain $-1 \le c \le 1$.

Next, let $c \in [-1, 1]$ and $t \in (0, \pi)$. In order to prove (1.19) we distinguish three cases.

Case 1: $c = -1$. Since

$$(-1)_0 = 1, \quad (-1)_1 = -1 \quad \text{and} \quad (-1)_k = 0 \quad \text{for } k \ge 2,$$

we obtain

$$A_1(-1,t) = \sin(t) > 0$$

and, for $n \ge 2$,

$$A_n(-1,t) = A_2(-1,t) = \sin(t)(1-\cos(t)) > 0.$$

CASE 2: $-1 < c \le 0$. We set

$$b_1 = 1 \quad \text{and} \quad b_k = -\frac{(c)_{k-1}}{k!} \quad (k = 2, \dots, n).$$

Then $b_k \ge 0$ $(k = 1, \dots, n)$ and

$$1 + \sum_{k=2}^{n} \frac{(c)_{k-1}}{(k-1)!} = b_1 - \sum_{k=2}^{n} k b_k = \frac{1}{(n-1)!} \prod_{k=1}^{n-1} (k+c) > 0. \tag{7.1}$$

From Lemma 3 we conclude that (1.19) holds.

CASE 3: $0 < c \le 1$. Let

$$d_k = \frac{(c)_{k-1}}{k!} \qquad (k = 1, \dots, n).$$

Then, for $k \ge 1$,

$$d_k > 0, \quad \frac{d_k}{d_{k+1}} = 1 + \frac{2-c}{c+k-1} > 1, \quad \frac{(2k-1)d_{2k-1}}{2k d_{2k}} = 1 + \frac{1-c}{c+2(k-1)} \ge 1.$$

Applying Lemma 4 yields $A_n(c,t) > 0$.

8. Proof of Theorem 7. We use the notation of (1.16). Let (1.20) be valid for all $n \in \mathbb{N}$ and $t \in (0, \pi)$. From

$$B_1(\alpha, t) = \frac{1}{\alpha+1} \sin(t) > 0$$

we conclude that $\alpha + 1 > 0$. Since

$$B_2(\alpha, t) = \frac{2\sin(t)}{(\alpha+1)(\alpha+2)} (1 + (\alpha+1)\cos(t)) > 0,$$

we get

$$1 + (\alpha+1)\cos(t) > 0.$$

Letting $t \to \pi$ gives $1 - (\alpha+1) \ge 0$. Thus, $\alpha \in (-1, 0]$.

Next, let $\alpha \in (-1, 0]$. We have

$$\binom{\alpha+n}{n} > 0,$$

so that (1.17) and (1.19) with $c = \alpha+1$ show that $B_n(\alpha, t) > 0$ for all $n \in \mathbb{N}$ and $t \in (0, \pi)$.

9. Proof of Theorem 8. We denote the sum in (1.23) by $S_n(c,t)$ and suppose that (1.23) holds for all $n \in \mathbb{N}$ and $t \in (0, \pi)$. Then

$$\frac{S_3(c,t)}{\sin(t)} = 1 + \frac{c(c+1)}{6} (3 - 4\sin^2(t)) > 0.$$

Setting $t = \pi/2$ gives

$$1 - \frac{c(c+1)}{6} = -\frac{1}{6}(c-2)(c+3) > 0.$$

This leads to $c \in (-3, 2)$.

Now, we prove that if $c \in (-3, 2)$, then $S_n(c,t)$ is positive for all $n \in \mathbb{N}$ and $t \in (0, \pi)$. Since $S_1(c,t) = S_2(c,t) = S_n(-1,t) = \sin(t) > 0$, we may suppose that $n \geq 3$ and $c \neq -1$. Let

$$d_k = \frac{(c)_{k-1}}{k!} \quad (k = 1, \ldots, n) \quad \text{and} \quad N = \left[\frac{n+1}{2}\right].$$

We consider three cases.

Case 1: $c \in (-2, -1) \cup (0, 2)$. We have

$$\begin{aligned} S_n(c,t) &= \sum_{k=1}^{N} d_{2k-1} \sin((2k-1)t) \\ &= \sum_{k=1}^{N} (d_{2k-1} - d_{2k+1}) \frac{\sin^2(kt)}{\sin(t)} + d_{2N+1} \frac{\sin^2(Nt)}{\sin(t)}. \end{aligned}$$

Since

$$d_{2k-1} - d_{2k+1} = \frac{(2-c)(4k-1+c)}{(2k+1)!} \prod_{j=0}^{2k-3} (c+j) > 0 \qquad (k = 1, \ldots, N)$$

and $d_{2N+1} > 0$, we conclude that $S_n(c,t)$ is positive.

Case 2: $c \in (-1, 0]$. We set

$$h_1 = 1, \quad h_{2k-1} = -\frac{(c)_{2k-2}}{(2k-1)!} \quad (k = 2, \ldots, N), \qquad h_{2k} = 0 \quad (k = 1, \ldots, N-1).$$

Then $h_k \geq 0$ $(k = 1, \ldots, 2N-1)$ and

$$S_n(c,t) = h_1 \sin(t) - \sum_{k=2}^{2N-1} h_k \sin(kt).$$

Using $(c)_{2k-1} \leq 0$ $(k = 1, \ldots, N)$ and (7.1) with $n = 2N$ we obtain

$$h_1 - \sum_{k=2}^{2N-1} k h_k = 1 + \sum_{k=2}^{2N} \frac{(c)_{k-1}}{(k-1)!} - \sum_{k=1}^{N} \frac{(c)_{2k-1}}{(2k-1)!} > 0.$$

An application of Lemma 3 implies that $S_n(c,t) > 0$.

Case 3: $c \in (-3, -2]$. We have the representation

$$S_n(c,t) = 4 d_3 \sin(t)(1 - \sin^2(t)) + (1 - d_3)\sin(t) - \sum_{\substack{k=5 \\ k \text{ odd}}}^{n} |d_k| \sin(kt).$$

Using $d_3 > 0$ gives

$$S_n(c,t) \geq (1-d_3)\sin(t) - \sum_{\substack{k=5 \\ k\,\mathrm{odd}}}^{n} |d_k|\sin(kt).$$

We have $1 - d_3 = (2-c)(3+c)/6 > 0$. This implies that $S_3(c,t) > 0$ and $S_4(c,t) > 0$. Let $n \geq 5$. From Lemma 3 we conclude that in order to show that $S_n(c,t)$ is positive it suffices to prove

$$1 - d_3 > \sum_{\substack{k=5 \\ k\,\mathrm{odd}}}^{n} k|d_k|. \tag{9.1}$$

We set $c = -2 - s$ with $s \in [0,1)$. Then we find

$$1 - d_3 - \sum_{\substack{k=5 \\ k\,\mathrm{odd}}}^{n} k|d_k| = \frac{(1-s^2)(2+s)}{24}\big(U(s) - V_N(s)\big) \tag{9.2}$$

with

$$U(s) = \frac{(2-s)(s^2+5s+8)}{(s+1)(s+2)} \quad \text{and} \quad V_N(s) = 24s\sum_{k=3}^{N-1}\frac{1}{(2k)!}\prod_{\nu=4}^{2k-1}(\nu-2-s).$$

Since

$$U(s) > \frac{1\cdot 8}{2\cdot 3} = \frac{4}{3}, \quad V_N(s) < 24\sum_{k=3}^{N-1}\frac{1}{(2k)!}\prod_{\nu=4}^{2k-1}(\nu-2) < \frac{6}{5}\sum_{k=3}^{N-1}\frac{1}{k(k-1)} < \frac{3}{5},$$

it follows from (9.2) that (9.1) is valid.

10. Proof of Theorem 9. We suppose that (1.24) is valid for all $n \in \mathbb{N}$ and $t \in (0,\pi)$. Setting $n = 2$ yields

$$\frac{\sin(t)}{12}\big(6 + c(c+1)\cos(t)\big) > 0.$$

Thus,

$$6 + c(c+1)\cos(t) > 0.$$

If $t \to \pi$, then

$$6 - c(c+1) = (2-c)(3+c) \geq 0.$$

This gives $c \in [-3,2]$.

Conversely, let $c \in [-3,2]$, $n \in \mathbb{N}$ and $t \in [0,\pi]$. From (1.23) with $2n$ and "$\geq$" instead of n and "$>$" we obtain

$$\sum_{k=1}^{n}\frac{(c)_{2k-2}}{(2k-1)!}\sin((2k-1)t) \geq 0.$$

Applying Lemma 5 yields, for $t \in (0, \pi)$,

$$\sum_{k=1}^{n} \frac{(c)_{2k-2}}{(2k-1)!} \frac{\sin(kt)}{k} > 0.$$

This leads to (1.24).

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