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Global attractor of the Cauchy problem for a semilinear degenerate damped hyperbolic equation involving the Grushin operator

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Abstract. The aim of this paper is to prove the existence of the global attractor of the Cauchy problem for a semilinear degenerate damped hyperbolic equation involving the Grushin operator with a locally Lipschitz nonlinearity satisfying a subcritical growth condition.

1. Introduction. The understanding of asymptotic behavior of dynamical systems is one of the most important problems of modern mathematical physics. One way to attack the problem for a dissipative dynamical system is to consider its global attractor. The existence of global attractors has been proved for various nonlinear dissipative parabolic and hyperbolic PDEs that involve elliptic operators (see e.g. [2, 4, 5, 16, 17, 19], and the references therein).

One of the classes of degenerate elliptic equations that has been studied widely in recent years is the class of equations involving an operator of the Grushin type (see [8])

$$G_\alpha u = \Delta_x u + |x|^{2\alpha} \Delta_y u, \quad \alpha \geq 0.$$

Note that $G_0 = \Delta$ is the Laplacian operator, and G_α, when $\alpha > 0$, is not elliptic in domains intersecting the surface $x = 0$. Many aspects of the theory of degenerate elliptic differential operators are presented in monographs [11–13, 20–24].

In this paper we are interested in the global existence and long-time behavior of solutions to the following problem:

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$$(1.1)\qquad u_{tt}+\gamma u_t+u=G_\alpha u+f(X,u),\quad t>0,$$
$$X=(x,y)\in\mathbb{R}^{N_1}\times\mathbb{R}^{N_2}:=\mathbb{R}^N,$$
$$(1.2)\qquad u(X,0)=u_0(X),\quad u_t(X,0)=u_1(X),$$

where γ is a positive constant, $u_0(X)\in S_1^2(\mathbb{R}^N), u_1(X)\in L^2(\mathbb{R}^N)$ and

$$\Delta_x=\sum_{i=1}^{N_1}\frac{\partial^2}{\partial x_i^2},\quad \Delta_y=\sum_{j=1}^{N_2}\frac{\partial^2}{\partial y_j^2},\quad u_t=\frac{\partial u}{\partial t}, u_{tt}=\frac{\partial^2 u}{\partial t^2},\quad |x|^{2\alpha}=\Big(\sum_{i=1}^{N_1}x_i^2\Big)^\alpha.$$

We assume that $f:\mathbb{R}^N\times\mathbb{R}\to\mathbb{R}$ is a continuous function satisfying

$$(1.3)\qquad |f(X,\xi_1)-f(X,\xi_2)|\le C_1|\xi_1-\xi_2|(g(X)+|\xi_1|^\rho+|\xi_2|^\rho)$$
$$\text{with } 0\le\rho\le\frac{2}{N_\alpha-2},\ N_\alpha=N_1+(1+\alpha)N_2>2,$$

$$(1.4)\qquad f(\cdot,0)=h(\cdot)\in L^2(\mathbb{R}^N),$$

$$(1.5)\qquad F(X,\xi)\ge C_2f(X,\xi)\xi+g_1(X)\quad \text{for all } X\in\mathbb{R}^N,\ \xi\in\mathbb{R},$$

$$(1.6)\qquad \int_{\mathbb{R}^N}F(X,u(X))dX\le 0\quad \text{for all } u\in S_1^2(\mathbb{R}^N),$$

where ρ, C_1, C_2 are positive constants, and $g\in L^{N_\alpha}(\mathbb{R}^N)\cap L^{N_\alpha/2}(\mathbb{R}^N)$, $g_1\in L^1(\mathbb{R}^N)$, $F(X,\xi)=\int_0^\xi f(X,\tau)\,d\tau$.

The major techniques used to get a global attractor in the natural energy space $H^1(\mathbb{R}^N)\times L^2(\mathbb{R}^N)$ are: working with a weighted Sobolev space as phase space and the method of "tail estimates". Babin and Vishik [3] have been the first to show the existence of attractors for equations of parabolic type in weighted Sobolev spaces. Some other authors have also employed weighted Sobolev spaces to tackle the wave equation, for example, Karachalios and Stavrakakis [9]. However, when working in weighted spaces we have to impose an additional condition that the initial data and forcing term also belong to the corresponding spaces. In 1999, Wang [25] came up with a new idea of "tail estimates" to prove the asymptotic compactness of the semiflows generated by reaction-diffusion equations. The method is based on an approximation of $\mathbb{R}^N$ by a sufficiently large bounded domain $B(0,R)$ and then showing that there is null convergence of the solutions on $\mathbb{R}^N\setminus B(0,R)$. Khanmamedov [10] applied the same idea to plate equations.

We would like to mention the results for the case $\alpha=0$. The existence of a global attractor in $H^1(\mathbb{R}^3)\times L^2(\mathbb{R}^3)$ for (1.1)–(1.2) was proved by Feireisl [7] for $\xi-f(X,\xi)=g(X,\xi)$ satisfying for $N_0:=N=3$ the growth conditions

$$g \in C^2(\mathbb{R}^4), \quad g(\cdot,0) \in H^1(\mathbb{R}^3), \quad \left|\frac{\partial g}{\partial \xi}(X,0)\right| \le C \quad \text{for all } X \in \mathbb{R}^3,$$
$$\left|\frac{\partial^2 g}{\partial \xi^2}(X,\xi)\right| \le C(1+|\xi|) \quad \text{for all } X \in \mathbb{R}^3, \xi \in \mathbb{R},$$
$$\liminf_{|\xi|\to\infty} \frac{g(X,\xi)}{\xi} \ge 0 \quad \text{uniformly in } X \in \mathbb{R}^3,$$
$$\big(g(X,\xi) - g(X,0)\big)\xi \ge C\xi^2 \quad \text{for all } \xi \in \mathbb{R},\ |X| > r_1,$$

for some $C > 0$.

Recently, Fall [6] used the method of "tail estimates" to show the existence of a global attractor in the natural energy space $H^1(\mathbb{R}^N) \times L^2(\mathbb{R}^N)$ for (1.1)–(1.2) (when $\alpha = 0$) under strictly restrained conditions

$$\xi - f(X,\xi) = \xi + h_1(\xi) - h_2(X), \quad h_2 \in L^2(\mathbb{R}^N),$$
$$h_1 \in C^1(\mathbb{R},\mathbb{R}), \quad h_1(0) = 0, \quad h_1(\xi)\xi \ge cF(\xi) \ge 0, \quad \forall \xi \in \mathbb{R},$$
$$0 \le \limsup_{|\xi|\to\infty} \frac{h_1(\xi)}{\xi} < \infty,$$

where c is a positive constant and $F(\xi) = \int_0^\xi h_1(\tau)\,d\tau$.

In the present paper, by using the analytical techniques of [10] and the method of "tail estimates", we prove that there also exist global attractors of (1.1)–(1.2) in the natural energy space $S_1^2(\mathbb{R}^N) \times L^2(\mathbb{R}^N)$ under conditions (1.3)–(1.6).

The structure of our note is as follows: In Section 2 we give some preliminary results on the existence of global mild solutions. In Section 3 we establish the existence of the global attractor for problem (1.1)–(1.2).

2. Existence and uniqueness of a global mild solution

2.1. Function spaces and operators. We use the space $S_1^2(\mathbb{R}^N)$ defined as the completion of $C_0^\infty(\mathbb{R}^N)$ in the norm

$$\|u\|_{S_1^2(\mathbb{R}^N)} = \Big\{ \int_{\mathbb{R}^N} (|u|^2 + |\nabla_\alpha u|^2)\,dX \Big\}^{1/2},$$

where

$$\nabla_\alpha u := \left(\frac{\partial u}{\partial x_1}, \ldots, \frac{\partial u}{\partial x_{N_1}}, |x|^\alpha \frac{\partial u}{\partial y_1}, \ldots, |x|^\alpha \frac{\partial u}{\partial y_{N_2}}\right),$$
$$|\nabla_\alpha u| := \left(\sum_{i=1}^{N_1} \left|\frac{\partial u}{\partial x_i}\right|^2 + |x|^{2\alpha} \sum_{j=1}^{N_2} \left|\frac{\partial u}{\partial y_j}\right|^2\right)^{1/2}.$$

Then $S_1^2(\mathbb{R}^N)$ is a Hilbert space with the inner product

$$(u,v)_{S_1^2(\mathbb{R}^N)} = (u,v)_{L^2(\mathbb{R}^N)} + (\nabla_\alpha u, \nabla_\alpha v)_{L^2(\mathbb{R}^N)}.$$

The following embedding inequality was proved in [1]:

$$\Big(\int_{\mathbb{R}^N} |u|^p\,dX\Big)^{1/p} \le C(p)\|u\|_{S_1^2(\mathbb{R}^N)},$$

where $2 \le p \le 2_\alpha^* = 2N_\alpha/(N_\alpha - 2)$, $C(p) > 0$.

We set

$$U = \begin{pmatrix} u \\ v \end{pmatrix}, \quad A = \begin{pmatrix} 0 & I \\ G_\alpha - I & 0 \end{pmatrix},$$

$$f^*(U)(X) = \begin{pmatrix} 0 \\ -\gamma v(X) + f(X, u(X)) \end{pmatrix}, \quad U_0 = \begin{pmatrix} u_0 \\ u_1 \end{pmatrix},$$

where I is the identity operator on $S_1^2(\mathbb{R}^N)$. Then problem (1.1)–(1.2) can be formulated as an abstract evolutionary equation

$$\frac{dU}{dt} = AU + f^*(U), \tag{2.1}$$

$$U(0) = U_0. \tag{2.2}$$

We set $H = S_1^2(\mathbb{R}^N) \times L^2(\mathbb{R}^N)$. We regard H as a Hilbert space with the inner product

$$(U, \overline{U})_H = \left(\begin{pmatrix} u \\ v \end{pmatrix}, \begin{pmatrix} \overline{u} \\ \overline{v} \end{pmatrix}\right) = (u, \overline{u})_{S_1^2(\mathbb{R}^N)} + (v, \overline{v})_{L^2(\mathbb{R}^N)}.$$

The domain $D(A)$ of A is given by

$$D(A) = \left\{\begin{pmatrix} u \\ v \end{pmatrix} : u, v \in S_1^2(\mathbb{R}^N),\ G_\alpha u - u \in L^2(\mathbb{R}^N))\right\}.$$

LEMMA 2.1. *The adjoint A^* of A is given by*

$$A^* = -\begin{pmatrix} 0 & I \\ G_\alpha - I & 0 \end{pmatrix}$$

with

$$D(A^*) = \left\{\begin{pmatrix} \chi \\ \psi \end{pmatrix} : \chi, \psi \in S_1^2(\mathbb{R}^N),\ G_\alpha \chi - \chi \in L^2(\mathbb{R}^N)\right\}.$$

Proof. The proof is similar to the one of [14, Lemma 1]. We therefore omit the details. ■

2.2. Global solutions

LEMMA 2.2. *Suppose that $f(X, \xi)$ satisfies conditions* (1.3)–(1.4). *Then:*

(a) *The Nemytskiĭ map*

$$\widehat{f} : S_1^2(\mathbb{R}^N) \to L^2(\mathbb{R}^N), \quad u \mapsto \widehat{f}(u)(X) := f(X, u(X)),$$

is Lipschitzian on every bounded subset of $S_1^2(\mathbb{R}^N)$.

(b) *The map*

$$f^* : H \to H, \quad U \mapsto f^*(U) := \begin{pmatrix} 0 \\ -\gamma v + f(X,u) \end{pmatrix},$$

is Lipschitzian on every bounded set of H.

Proof. (a) From (1.3) and (1.4) it follows that

$$|f(X,\xi)|^2 \le C\big(g^2(X)|\xi|^2 + |\xi|^{2(\rho+1)} + |h(X)|^2\big).$$

Hence

$$\begin{aligned}\int_{\mathbb{R}^N} |f(X,u)|^2\,dX &\le C\Big\{\int_{\Omega} \big(g^2(X)|u|^2 + |u|^{2(1+\rho)}\big)\,dX + \int_{\mathbb{R}^N} |h(X)|^2\,dX\Big\} \\ &\le C\big(\|g\|^2_{L^{N\alpha}(\mathbb{R}^N)}\|u\|^2_{L^{2^*_\alpha}(\mathbb{R}^N)} + \|u\|^{2(1+\rho)}_{L^{2(1+\rho)}(\mathbb{R}^N)} + \|h\|^2_{L^2(\mathbb{R}^N)}\big) < \infty.\end{aligned}$$

Since $S^2_1(\mathbb{R}^N)$ is continuously embedded into $L^{2^*_\alpha}(\mathbb{R}^N)$, we conclude that f is a map from $S^2_1(\mathbb{R}^N)$ to $L^2(\mathbb{R}^N)$.

Now, let $u,v \in S^2_1(\mathbb{R}^N)$, $R > 0$ and $\|u\|_{S^2_1(\mathbb{R}^N)}, \|v\|_{S^2_1(\mathbb{R}^N)} \le R$. We have

$$\begin{aligned}\int_{\mathbb{R}^N} |f(X,u) - f(X,v)|^2\,dX &\le C\int_{\mathbb{R}^N} |u-v|^2\big(g^2(X) + |u|^{2\rho} + |v|^{2\rho}\big)\,dX \\ \le C\int_{\mathbb{R}^N} g^2(X)|u-v|^2\,dX &+ C\int_{\mathbb{R}^N} |u-v|^2|u|^{2\rho}\,dX + C\int_{\mathbb{R}^N} |u-v|^2|v|^{2\rho}\,dX.\end{aligned}$$

Applying Hölder's inequality, we obtain

$$\begin{aligned}\int_{\mathbb{R}^N} g^2(X)|u-v|^2\,dX &\le \|g\|^2_{L^{N\alpha}(\mathbb{R}^N)}\|u-v\|^2_{L^{2^*_\alpha}(\mathbb{R}^N)}, \\ \int_{\mathbb{R}^N} |u-v|^2|u|^{2\rho}\,dX &\le \|u\|^{2\rho}_{L^{2(\rho+1)}(\mathbb{R}^N)}\|u-v\|^2_{L^{2(\rho+1)}(\mathbb{R}^N)}, \\ \int_{\mathbb{R}^N} |u-v|^2|v|^{2\rho}\,dX &\le \|v\|^{2\rho}_{L^{2(\rho+1)}(\mathbb{R}^N)}\|u-v\|^2_{L^{2(\rho+1)}(\mathbb{R}^N)}.\end{aligned}$$

Since $S^2_1(\mathbb{R}^N)$ is continuously embedded into $L^{2^*_\alpha}(\mathbb{R}^N)$ and $1 < 2(\rho+1) \le 2^*_\alpha$, we have

$$\|f(X,u) - f(X,v)\|^2_{L^2(\mathbb{R}^N)} \le C_1\|u-v\|^2_{S^2_1(\mathbb{R}^N)}(1 + \|u\|^{2\rho}_{S^2_1(\mathbb{R}^N)} + \|v\|^{2\rho}_{S^2_1(\mathbb{R}^N)}),$$

or

$$\|f(X,u) - f(X,v)\|_{L^2(\mathbb{R}^N)} \le C(R)\|u-v\|_{S^2_1(\mathbb{R}^N)}.$$

(b) Let $R > 0$, $U, \overline{U} \in H$ and $\|U\|_H, \|\overline{U}\|_H \le R$. We have

$$f^*(U) - f^*(\overline{U}) = \begin{pmatrix} 0 \\ \gamma\overline{v} - \gamma v + f(X,u) - f(X,\overline{u}) \end{pmatrix}.$$

Hence

$$\begin{aligned}
\|f^*(U)-f^*(\overline{U})\|_H^2 &= \|\gamma\overline{v}-\gamma v+f(X,u)-f(X,\overline{u})\|_{L^2(\mathbb{R}^N)}^2\\
&\le 2\|\gamma\overline{v}-\gamma v\|_{L^2(\mathbb{R}^N)}^2+2\|f(X,u)-f(X,\overline{u})\|_{L^2(\mathbb{R}^N)}^2\\
&\le 2\gamma^2\|\overline{v}-v\|_{L^2(\mathbb{R}^N)}^2+2C\|u-\overline{u}\|_{S_1^2(\mathbb{R}^N)}^2(1+\|u\|_{S_1^2(\mathbb{R}^N)}^{2\rho}+\|\overline{u}\|_{S_1^2(\mathbb{R}^N)}^{2\rho})\\
&\le C_1(R)\|U-\overline{U}\|_H^2. \ \blacksquare
\end{aligned}$$

Lemma 2.1 and [15, Theorem 10.8 (p. 41)] imply that A generates a C_0-semigroup e^{At} on H.

Definition 2.3 (see [18]). Let $T>0$. A (strongly) continuous mapping $U:[0,T)\to H$ is said to be a *mild solution* of problem (2.1)–(2.2) if it solves the integral equation

$$U(X,t)=e^{At}U_0+\int_0^t e^{A(t-s)}f^*(U(s))\,ds, \quad t\in[0,T).$$

If U is (strongly) differentiable almost everywhere in $[0,T)$ with U_t and AU in $L^1_{\mathrm{loc}}([0,T),H)$, and satisfies the differential equation

$$\frac{dU}{dt}\overset{\text{a.e.}}{=}AU+f^*(U) \quad \text{on } (0,T), \quad \text{and} \quad U(0)=U_0.$$

then U is called a *strong solution* of problem (2.1)–(2.2).

Using Lemma 2.2 and [18, Theorems 46.1 (p. 235) and 46.2 (p. 236)] it is not difficult to establish

Proposition 2.4. *Assume that $f(X,u)$ satisfies conditions* (1.3)–(1.6). *Then for any $R>0$ and $U_0\in H$ such that $\|U_0\|_H\le R$, there exists $T=T(R)>0$ small enough such that problem* (2.1)–(2.2) *has a unique mild solution $U\in C([0,T);H)$. Moreover, if $U_0\in D(A)$ then U is a strong solution for* (2.1)–(2.2).

From (1.3) and (1.4) it follows that

$$|F(X,\xi)|\le C\big(g(X)|\xi|^2+|\xi|^{2+\rho}+|f(X,0)|\,|\xi|\big).$$

Hence

$$\begin{aligned}
\int_{\mathbb{R}^N}|F(X,u)|\,dX &\le C\int_{\mathbb{R}^N}\big(g(X)|u|^2+|u|^{2+\rho}+|h(X)|\,|u|\big)\,dX\\
&\le C\big(\|g\|_{L^{N_\alpha/2}(\mathbb{R}^N)}\|u\|_{L^{2^*_\alpha}(\mathbb{R}^N)}^2+\|u\|_{L^{2+\rho}(\mathbb{R}^N)}^{2+\rho}+\|h\|_{L^2(\mathbb{R}^N)}\|u\|_{L^2(\mathbb{R}^N)}\big)<\infty
\end{aligned}$$

for all $u\in S_1^2(\mathbb{R}^N)$.

LEMMA 2.5. *Assume that $f(X,u)$ satisfies conditions* (1.3)–(1.6). *Then any solution $u(t)$ of problem* (1.1)–(1.2) *satisfies*

$$\|u\|^2_{S^2_1(\mathbb{R}^N)} + \|u_t\|^2_{L^2(\mathbb{R}^N)} \le M, \quad t \ge T_1, \tag{2.3}$$

where M is a constant depending only on γ, $g_1(X)$ and T_1 depending on the data $\gamma, g_1(X), R$ when $\|u_0\|^2_{S^2_1(\mathbb{R}^N)} + \|u_1\|^2_{L^2(\mathbb{R}^N)} \le R$.

Proof. Let $U(t)$ be the solution of (2.1)–(2.2) with the initial condition U_0. Then $\overline{u} = u_t + \delta u$ satisfies the equation

$$\overline{u}_t + (\gamma - \delta)\overline{u} + (\delta^2 - \gamma\delta + 1)u = G_\alpha u + f(X,u). \tag{2.4}$$

Set

$$\mathcal{A}(\overline{u}, u) = \|\overline{u}\|^2_{L^2(\mathbb{R}^N)} + (\delta^2 - \delta\gamma + 1)\|u\|^2_{L^2(\mathbb{R}^N)} + \|\nabla_\alpha u\|^2_{L^2(\mathbb{R}^N)}.$$

We choose $\delta \in (0,1)$ sufficiently small that

$$\gamma - 2\delta > 0, \quad \delta^2 - \delta\gamma + 1 > 0,$$

and $C_1, C_2 > 0$ such that

$$\begin{aligned} C_1\big(\|u\|^2_{L^2(\mathbb{R}^N)} + \|u_t\|^2_{L^2(\mathbb{R}^N)} + \|\nabla_\alpha u\|^2_{L^2(\mathbb{R}^N)}\big) \\ \le \mathcal{A}(\overline{u}, u) \le C_2\big(\|u\|^2_{L^2(\mathbb{R}^N)} + \|u_t\|^2_{L^2(\mathbb{R}^N)} + \|\nabla_\alpha u\|^2_{L^2(\mathbb{R}^N)}\big). \end{aligned} \tag{2.5}$$

Multiplying (2.4) by $\overline{u}$ and integrating over $\mathbb{R}^N$, we obtain

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}(\mathcal{A}(\overline{u}, u)) &= -(\gamma - \delta)\|\overline{u}\|^2_{L^2(\mathbb{R}^N)} - \delta(\delta^2 - \delta\gamma + 1)\|u\|^2_{L^2(\mathbb{R}^N)} \\ &\quad - \delta\|\nabla_\alpha u\|^2_{L^2(\mathbb{R}^N)} + \int_{\mathbb{R}^N} f(X,u)\overline{u}\, dX \\ &= (2\delta - \gamma)\|\overline{u}\|^2_{L^2(\mathbb{R}^N)} + \int_{\mathbb{R}^N} f(X,u)u_t\, dX \\ &\quad + \delta \int_{\mathbb{R}^N} f(X,u)u\, dX - \delta\mathcal{A}(\overline{u}, u). \end{aligned}$$

From (1.5), we have

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\Big(\mathcal{A}(\overline{u}, u) - 2\int_{\mathbb{R}^N} F(X,u)\, dX\Big) &\le -\delta\mathcal{A}(\overline{u}, u) + \delta\int_{\mathbb{R}^N} f(X,u)u\, dX \\ &\le -\delta\mathcal{A}(\overline{u}, u) + \frac{\delta}{C_2}\int_{\mathbb{R}^N} \big(F(X,u) - g_1(X)\big)\, dX. \end{aligned}$$

From (1.6), we deduce that

$$\frac{1}{2}\frac{d}{dt}\Big(\mathcal{A}(\overline{u},u)-2\int_{\mathbb{R}^N}F(X,u)\,dX\Big) \le -\mu\Big(\mathcal{A}(\overline{u},u)-2\int_{\mathbb{R}^N}F(X,u)\,dX\Big)+C_3,$$

where

$$\mu=\min\Big\{\delta,\frac{\delta}{2C_2}\Big\}>0,\qquad C_3=-\frac{\delta}{C_2}\int_{\mathbb{R}^N}g_1(X)\,dX.$$

Applying the Gronwall inequality we get

$$\mathcal{A}(\overline{u},u)-2\int_{\mathbb{R}^N}F(X,u)\,dX \le e^{-2\mu t}\Big(\mathcal{A}(\overline{u}_0,u_0)-2\int_{\mathbb{R}^N}F(X,u_0)\,dX-C_3/\mu\Big)+C_3/\mu,\quad \forall t\ge 0.$$

Thus

$$\|u\|^2_{S_1^2(\mathbb{R}^N)}+\|u_t\|^2_{L^2(\mathbb{R}^N)}\le Ce^{-2\mu t}\mathcal{T}(u_0,u_1)+C_5/\mu, \tag{2.6}$$

where $\mathcal{T}(u_0,u_1)=\|u_0\|^2_{S_1^2(\mathbb{R}^N)}+\|u_1\|^2_{L^2(\mathbb{R}^N)}+C_4$, which yields

$$\|u\|^2_{S_1^2(\mathbb{R}^N)}+\|u_t\|^2_{L^2(\mathbb{R}^N)}\le 2|C_5|/\mu\quad\text{for all } t\ge T_1,$$

where

$$T_1:=\begin{cases}\dfrac{1}{2\mu}\ln\dfrac{C\mu\mathcal{T}(u_0,u_1)}{|C_5|} & \text{if } \dfrac{C\mu\mathcal{T}(u_0,u_1)}{|C_5|}>1,\\[2ex] 0 & \text{if } \dfrac{C\mu\mathcal{T}(u_0,u_1)}{|C_5|}\le 1,\end{cases}$$

and (2.3) follows with $M=2|C_5|/\mu$. ∎

THEOREM 2.6. *Assume that $f(X,u)$ satisfies conditions* (1.3)–(1.6) *and $U_0\in H$. Then problem* (1.1)–(1.2) *has a unique global solution $U\in C([0,\infty);H)$. Moreover, for each fixed t the map $U_0\mapsto S(t)U_0:=U(t)$ is continuous on H.*

Proof. The uniqueness of the local solution was obtained in Proposition 2.4. We will show that the local solution can be extended globally in time. Suppose that $U(t)$ is defined on the maximal interval $[0,T_{\max})$. By (2.6), we have

$$\|U\|_H\le C\quad\text{for all } 0\le t<T_{\max}.$$

As in [14, proof of Theorem 2] we show that $T_{\max}=\infty$. It is easy to prove that the map $U_0\mapsto S(t)U_0:=U(t)$ is continuous on H. We omit the details. ∎

3. Existence of a global attractor in $S_1^2(\mathbb{R}^N) \times L^2(\mathbb{R}^N)$. In view of Theorem 2.6, we can define a continuous semigroup $S(t) : H \to H$ by

$$S(t)U_0 := U(t),$$

where $U(t)$ is the unique global mild solution of (1.1)–(1.2) with initial datum U_0.

Denote

$$\mathcal{B} = \{U \in H : \|U\|_H^2 < M\},$$

where M is the constant in (2.3). It follows from (2.3) that $\mathcal{B}$ is an absorbing set for $S(t)$ in H and for every bounded set B in H there exists a constant $T(B)$ depending only on $(\gamma, g_1(X))$ and B such that

$$S(t)B \subseteq \mathcal{B}, \quad t \geq T(B). \tag{3.1}$$

In particular there exists a constant T_0 depending only on $(\gamma, g_1(X))$ and $\mathcal{B}$ such that

$$S(t)\mathcal{B} \subseteq \mathcal{B}, \quad t \geq T_0. \tag{3.2}$$

LEMMA 3.1. *Assume that $f(X,u)$ satisfies conditions* (1.3)–(1.6) *and $U_0 \in \mathcal{B}$. Then for every $\epsilon > 0$, there exist positive constants $T(\epsilon)$ and $K(\epsilon)$ such that the solution $U(t)$ of problem* (2.1)–(2.2) *satisfies*

$$\int_{|X|_\alpha \geq k} (|u|^2 + |u_t|^2 + |\nabla_\alpha u|^2)\, dX \leq \epsilon, \quad t \geq T(\epsilon),\ k \geq K(\epsilon), \tag{3.3}$$

where

$$|X|_\alpha = [|x|^{2(1+\alpha)} + (1+\alpha)^2|y|^2]^{1/(2(1+\alpha))}.$$

Proof. Choose a smooth function ϑ such that $0 \leq \vartheta(s) \leq 1$ for $s \in \mathbb{R}^+$ and

$$\vartheta(s) = 0 \quad \text{for } 0 \leq s \leq 1, \qquad \vartheta(s) = 1 \quad \text{for } s \geq 2.$$

Define $\vartheta_k : \mathbb{R}^N \to \mathbb{R}$ by

$$\vartheta_k(X) = \vartheta\bigg(\frac{|X|_\alpha^{2(1+\alpha)}}{k^{2(1+\alpha)}}\bigg) \qquad \text{for any } k \in \mathbb{R}_*^+.$$

Then

$$\nabla_\alpha \vartheta\bigg(\frac{|X|_\alpha^{2(1+\alpha)}}{k^{2(1+\alpha)}}\bigg) = \frac{1}{k^{2(1+\alpha)}}\vartheta'\bigg(\frac{|X|_\alpha^{2(1+\alpha)}}{k^{2(1+\alpha)}}\bigg)\nabla_\alpha |X|_\alpha^{2(1+\alpha)},$$

where

$$\begin{aligned}&\nabla_\alpha |X|_\alpha^{2(1+\alpha)} \\ &\quad = 2(1+\alpha)\big(x_1|x|^{2\alpha}, \ldots, x_{N_1}|x|^{2\alpha}, (1+\alpha)|x|^\alpha y_1, \ldots, (1+\alpha)|x|^\alpha y_{N_2}\big),\end{aligned}$$

hence

$$|\nabla_\alpha |X|_\alpha^{2(1+\alpha)}| = 2(1+\alpha)|x|^\alpha |X|_\alpha^{1+\alpha}.$$

Notice that there exists a constant $C_\vartheta > 0$ such that $|\vartheta'(s)| \le C_\vartheta$ for $s \in \mathbb{R}^+$.

Multiplying (2.4) by $\vartheta\big(\frac{|X|_\alpha^{2(1+\alpha)}}{k^{2(1+\alpha)}}\big)\overline{u}$ and integrating over $\mathbb{R}^N$, we get

$$\begin{aligned}(3.4)\qquad &\int_{\mathbb{R}^N} \vartheta\Big(\frac{|X|_\alpha^{2(1+\alpha)}}{k^{2(1+\alpha)}}\Big)\overline{u}_t\overline{u}\,dX + (\gamma-\delta)\int_{\mathbb{R}^N} \vartheta\Big(\frac{|X|_\alpha^{2(1+\alpha)}}{k^{2(1+\alpha)}}\Big)|\overline{u}|^2\,dX\\ &\qquad + (\delta^2-\gamma\delta+1)\int_{\mathbb{R}^N} \vartheta\Big(\frac{|X|_\alpha^{2(1+\alpha)}}{k^{2(1+\alpha)}}\Big)\overline{u}u\,dX\\ &= \int_{\mathbb{R}^N} G_\alpha u\vartheta\Big(\frac{|X|_\alpha^{2(1+\alpha)}}{k^{2(1+\alpha)}}\Big)\overline{u}\,dX + \int_{\mathbb{R}^N} f(X,u)\vartheta\Big(\frac{|X|_\alpha^{2(1+\alpha)}}{k^{2(1+\alpha)}}\Big)\overline{u}\,dX.\end{aligned}$$

But

$$\begin{aligned}-\int_{\mathbb{R}^N} G_\alpha u\vartheta\Big(\frac{|X|_\alpha^{2(1+\alpha)}}{k^{2(1+\alpha)}}\Big)\overline{u}\,dX &= \int_{\mathbb{R}^N} \vartheta\Big(\frac{|X|_\alpha^{2(1+\alpha)}}{k^{2(1+\alpha)}}\Big)\nabla_\alpha u\cdot\nabla_\alpha\overline{u}\,dX\\ &\quad + \frac{1}{k^{2(1+\alpha)}}\int_{\mathbb{R}^N} \vartheta'\Big(\frac{|X|_\alpha^{2(1+\alpha)}}{k^{2(1+\alpha)}}\Big)\overline{u}\nabla_\alpha|X|_\alpha^{2(1+\alpha)}\cdot\nabla_\alpha u\,dX\\ &= \int_{\mathbb{R}^N} \vartheta\Big(\frac{|X|_\alpha^{2(1+\alpha)}}{k^{2(1+\alpha)}}\Big)\nabla_\alpha u\cdot[\delta\nabla_\alpha u+\nabla_\alpha u_t]\,dX\\ &\quad + \frac{1}{k^{2(1+\alpha)}}\int_{\mathbb{R}^N} \vartheta'\Big(\frac{|X|_\alpha^{2(1+\alpha)}}{k^{2(1+\alpha)}}\Big)\overline{u}\nabla_\alpha|X|_\alpha^{2(1+\alpha)}\cdot\nabla_\alpha u\,dX\\ &= \frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^N} \vartheta\Big(\frac{|X|_\alpha^{2(1+\alpha)}}{k^{2(1+\alpha)}}\Big)|\nabla_\alpha u|^2\,dX + \delta\int_{\mathbb{R}^N} \vartheta\Big(\frac{|X|_\alpha^{2(1+\alpha)}}{k^{2(1+\alpha)}}\Big)|\nabla_\alpha u|^2\,dX\\ &\quad + \frac{1}{k^{2(1+\alpha)}}\int_{\mathbb{R}^N} \vartheta'\Big(\frac{|X|_\alpha^{2(1+\alpha)}}{k^{2(1+\alpha)}}\Big)\overline{u}\nabla_\alpha|X|_\alpha^{2(1+\alpha)}\cdot\nabla_\alpha u\,dX,\\ \int_{\mathbb{R}^N} \vartheta\Big(\frac{|X|_\alpha^{2(1+\alpha)}}{k^{2(1+\alpha)}}\Big)\overline{u}_t\overline{u}\,dX &= \frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^N} \vartheta\Big(\frac{|X|_\alpha^{2(1+\alpha)}}{k^{2(1+\alpha)}}\Big)|\overline{u}|^2\,dX,\end{aligned}$$

and

$$\begin{aligned}&(\delta^2-\gamma\delta+1)\int_{\mathbb{R}^N} \vartheta\Big(\frac{|X|_\alpha^{2(1+\alpha)}}{k^{2(1+\alpha)}}\Big)\overline{u}u\,dX\\ &\qquad = (\delta^2-\gamma\delta+1)\int_{\mathbb{R}^N} \vartheta\Big(\frac{|X|_\alpha^{2(1+\alpha)}}{k^{2(1+\alpha)}}\Big)u(u_t+\delta u)\,dX\end{aligned}$$

$$= (\delta^2 - \gamma\delta + 1)\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^N} \vartheta\left(\frac{|X|_\alpha^{2(1+\alpha)}}{k^{2(1+\alpha)}}\right)|u|^2\,dX$$
$$+ \delta(\delta^2 - \gamma\delta + 1)\int_{\mathbb{R}^N} \vartheta\left(\frac{|X|_\alpha^{2(1+\alpha)}}{k^{2(1+\alpha)}}\right)|u|^2\,dX.$$

Set

$$\mathcal{C}(\overline{u}, u) = (\delta^2 - \gamma\delta + 1)|u|^2 + |\nabla_\alpha u|^2 + |\overline{u}|^2.$$

Then (3.4) becomes

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^N} \vartheta\left(\frac{|X|_\alpha^{2(1+\alpha)}}{k^{2(1+\alpha)}}\right)\mathcal{C}(\overline{u}, u)\,dX$$
$$= -\delta\int_{\mathbb{R}^N} \vartheta\left(\frac{|X|_\alpha^{2(1+\alpha)}}{k^{2(1+\alpha)}}\right)\mathcal{C}(\overline{u}, u)\,dX$$
$$- (\gamma - 2\delta)\int_{\mathbb{R}^N} \vartheta\left(\frac{|X|_\alpha^{2(1+\alpha)}}{k^{2(1+\alpha)}}\right)|\overline{u}|^2\,dX + \int_{\mathbb{R}^N} f(X,u)\vartheta\left(\frac{|X|_\alpha^{2(1+\alpha)}}{k^{2(1+\alpha)}}\right)\overline{u}\,dX$$
$$- \frac{1}{k^{2(1+\alpha)}}\int_{\mathbb{R}^N} \vartheta'\left(\frac{|X|_\alpha^{2(1+\alpha)}}{k^{2(1+\alpha)}}\right)\overline{u}\nabla_\alpha|X|_\alpha^{2(1+\alpha)}\cdot\nabla_\alpha u\,dX,$$

and from (1.6) we have

$$\int_{\mathbb{R}^N} f(X,u)\vartheta\left(\frac{|X|_\alpha^{2(1+\alpha)}}{k^{2(1+\alpha)}}\right)\overline{u}\,dX$$
$$= \int_{\mathbb{R}^N} f(X,u)\vartheta\left(\frac{|X|_\alpha^{2(1+\alpha)}}{k^{2(1+\alpha)}}\right)(u_t + \delta u)\,dX$$
$$= \int_{\mathbb{R}^N} f(X,u)\vartheta\left(\frac{|X|_\alpha^{2(1+\alpha)}}{k^{2(1+\alpha)}}\right)u_t\,dX + \delta\int_{\mathbb{R}^N} f(X,u)\vartheta\left(\frac{|X|_\alpha^{2(1+\alpha)}}{k^{2(1+\alpha)}}\right)u\,dX$$
$$= \frac{d}{dt}\int_{\mathbb{R}^N} F(X,u)\vartheta\left(\frac{|X|_\alpha^{2(1+\alpha)}}{k^{2(1+\alpha)}}\right)dX + \delta\int_{\mathbb{R}^N} f(X,u)\vartheta\left(\frac{|X|_\alpha^{2(1+\alpha)}}{k^{2(1+\alpha)}}\right)u\,dX$$
$$\le \frac{d}{dt}\int_{\mathbb{R}^N} F(X,u)\vartheta\left(\frac{|X|_\alpha^{2(1+\alpha)}}{k^{2(1+\alpha)}}\right)dX$$
$$+ \frac{\delta}{C_2}\int_{\mathbb{R}^N} \vartheta\left(\frac{|X|_\alpha^{2(1+\alpha)}}{k^{2(1+\alpha)}}\right)\big(F(X,u) - g_1(X)\big)\,dX.$$

We deduce that

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int\limits_{\mathbb{R}^N} \vartheta\Big(\frac{|X|_\alpha^{2(1+\alpha)}}{k^{2(1+\alpha)}}\Big)\big(\mathcal{C}(\overline{u},u)-2F(X,u)\big)\,dX\\
&\quad\le -\mu\int\limits_{\mathbb{R}^N} \vartheta\Big(\frac{|X|_\alpha^{2(1+\alpha)}}{k^{2(1+\alpha)}}\Big)\big(\mathcal{C}(\overline{u},u)-2F(X,u)\big)\,dX\\
&\qquad -\big(\gamma-2\delta\big)\int\limits_{\mathbb{R}^N} \vartheta\Big(\frac{|X|_\alpha^{2(1+\alpha)}}{k^{2(1+\alpha)}}\Big)|\overline{u}|^2\,dX-\frac{\delta}{C_2}\int\limits_{\mathbb{R}^N} \vartheta\Big(\frac{|X|_\alpha^{2(1+\alpha)}}{k^{2(1+\alpha)}}\Big)g_1(X)\,dX\\
&\qquad -\frac{1}{k^{2(1+\alpha)}}\int\limits_{\mathbb{R}^N} \vartheta'\Big(\frac{|X|_\alpha^{2(1+\alpha)}}{k^{2(1+\alpha)}}\Big)\overline{u}\nabla_\alpha|X|_\alpha^{2(1+\alpha)}\cdot\nabla_\alpha u\,dX\\
&\quad\le -\int\limits_{\mathbb{R}^N} \vartheta\Big(\frac{|X|_\alpha^{2(1+\alpha)}}{k^{2(1+\alpha)}}\Big)\Big[\mu\big(\mathcal{C}(\overline{u},u)-2F(X,u)\big)+\frac{\delta}{C_2}g_1(X)\Big]\,dX\\
&\qquad -\frac{1}{k^{2(1+\alpha)}}\int\limits_{\mathbb{R}^N} \vartheta'\Big(\frac{|X|_\alpha^{2(1+\alpha)}}{k^{2(1+\alpha)}}\Big)\overline{u}\nabla_\alpha|X|_\alpha^{2(1+\alpha)}\cdot\nabla_\alpha u\,dX,
\end{aligned}
$$

where $\mu=\min\{\delta,\delta/(2C_2)\}>0$.

On the other hand, since $g_1\in L^1(\mathbb{R}^N)$, there exists $K_1>0$ such that for all $k\ge K_1$, we have

$$
\begin{aligned}
-\frac{\delta}{C_2}\int\limits_{\mathbb{R}^N} \vartheta\Big(\frac{|X|_\alpha^{2(1+\alpha)}}{k^{2(1+\alpha)}}\Big)g_1(X)\,dX &= -\frac{\delta}{C_2}\int\limits_{|X|_\alpha\ge k} \vartheta\Big(\frac{|X|_\alpha^{2(1+\alpha)}}{k^{2(1+\alpha)}}\Big)g_1(X)\,dX\\
&\le C\int\limits_{|X|_\alpha\ge k}|g_1(X)|\,dX\le\frac{\epsilon}{4}.
\end{aligned}
$$

Applying Hölder's inequality, we obtain

$$
\begin{aligned}
&-\frac{1}{k^{2(1+\alpha)}}\int\limits_{\mathbb{R}^N} \vartheta'\Big(\frac{|X|_\alpha^{2(1+\alpha)}}{k^{2(1+\alpha)}}\Big)\overline{u}\nabla_\alpha|X|_\alpha^{2(1+\alpha)}\cdot\nabla_\alpha u\,dX\\
&\qquad = -\frac{1}{k^{2(1+\alpha)}}\int\limits_{|X|_\alpha\le 2k} \vartheta'\Big(\frac{|X|_\alpha^{2(1+\alpha)}}{k^{2(1+\alpha)}}\Big)\overline{u}\nabla_\alpha|X|_\alpha^{2(1+\alpha)}\cdot\nabla_\alpha u\,dX\\
&\qquad \le \frac{C}{k^{2(1+\alpha)}}\Big(\int\limits_{|X|_\alpha\le 2k}|\overline{u}|^2\,dX\Big)^{1/2}\Big(\int\limits_{|X|_\alpha\le 2k}|\nabla_\alpha|X|_\alpha^{2(1+\alpha)}|^2|\nabla_\alpha u|^2\,dX\Big)^{1/2}\\
&\qquad \le \frac{C}{k}\Big(\int\limits_{|X|_\alpha\le 2k}|\overline{u}|^2\,dX\Big)^{1/2}\Big(\int\limits_{|X|_\alpha\le 2k}|\nabla_\alpha u|^2\,dX\Big)^{1/2}\le\frac{\epsilon}{4},
\end{aligned}
$$

for all $k\ge K_2$.

Applying Gronwall's inequality, we obtain

$$\begin{aligned}\int_{\mathbb{R}^N} \vartheta\Big(\frac{|X|_\alpha^{2(1+\alpha)}}{k^{2(1+\alpha)}}\Big)\big(\mathcal{C}(\overline{u},u) - 2F(X,u)\big)\,dX \\ \le e^{-2\mu t}\int_{\mathbb{R}^N} \vartheta\Big(\frac{|X|_\alpha^{2(1+\alpha)}}{k^{2(1+\alpha)}}\Big)\big(\mathcal{C}(\overline{u}_0,u_0) - 2F(X,u_0)\big)\,dX \\ + \epsilon(1-e^{-2\mu t}), \quad \forall k \ge K(\epsilon).\end{aligned}$$

Now since $U_0 \in \mathcal{B}$, there exists a constant $M > 0$ such that

$$\int_{\mathbb{R}^N} \vartheta\Big(\frac{|X|_\alpha^{2(1+\alpha)}}{k^{2(1+\alpha)}}\Big)(\mathcal{C}(\overline{u}_0,u_0) - 2F(X,u_0))\,dX \le M.$$

By (2.5) and the definition of ϑ, we get the conclusion of the lemma. ■

From Lemma 3.1, for any solution $U = (u(t), u_t(t))$ with the initial data $U_0 = (u_0, u_1) \in \mathcal{B}$, we have

$$\lim_{T,k\to\infty} \frac{1}{T}\int_0^T \int_{|X|_\alpha \ge k} (|u|^2 + |u_t|^2 + |\nabla_\alpha u|^2)\,dX\,dt = 0. \tag{3.5}$$

LEMMA 3.2. *Assume that $f(X,u)$ satisfies the conditions* (1.3)–(1.6) *and $U_n \rightharpoonup U$ in H. Then for every $t \ge 0$,*

$$S(t)U_n \rightharpoonup S(t)U \quad \text{in } H. \tag{3.6}$$

Proof. The proof is similar to the one of [10, Lemma 1]. We omit the details. ■

LEMMA 3.3. *Assume that $f(X,u)$ satisfies the condition* (1.3), *Ω is a bounded domain in $\mathbb{R}^N$, $u_n \rightharpoonup u$ in $S_1^2(\Omega)$, and $v_n \to v$ in $L^2(\Omega)$. Then*

$$\int_\Omega f(X,u_n)v_n\,dX \to \int_\Omega f(X,u)v\,dX,$$
$$\int_\Omega F(X,u_n)\,dX \to \int_\Omega F(X,u)\,dX.$$

Proof. The proof is a simple modification of [14, proof of Theorem 3]. ■

LEMMA 3.4. *Assume that $f(X,u)$ satisfies conditions* (1.3)–(1.6), *B is a bounded subset of H, and $\epsilon > 0$. Then there exists a $T_0 = T_0(\epsilon, B)$ such that for any sequence $\{U_n\}$ in B, weakly converging to U in H, we have*

$$\limsup_{n\to\infty} \|S(T)U_n\|_H \le \|S(T)U_0\|_H + \epsilon \quad \textit{for all } T \ge T_0. \tag{3.7}$$

Proof. Define

$$E(u(t),u_t(t)) := \tfrac{1}{2}\big(\|\nabla_\alpha u(t)\|^2_{L^2(\mathbb{R}^N)} + \|u(t)\|^2_{L^2(\mathbb{R}^N)} + \|u_(t)\|^2_{L^2(\mathbb{R}^N)}\big),$$
$$[B(0,R)] := \{X \in \mathbb{R}^N : |X|_\alpha \le R\} \quad \text{for } R>0,$$

and let $S(t)U_n = (u_n(t), u_{nt}(t))$ be the solution of problem (2.1)–(2.2) with the initial data $U_n = (u_n(0), u_{nt}(0))$, and $S(t)U_0 = (u(t), u_t(t))$ be the solution with the initial data $U_0 = (u(0), u_t(0))$. Lemma 2.5 implies

$$\sup_{t,n\ge 0} \|S(t)U_n\|_H \le C. \tag{3.8}$$

Multiplying (1.1) by $u_t + \frac{\gamma}{2}u$ and integrating over $\mathbb{R}^N$, similarly to Lemma 2.5, we get

$$\left\|u_t+\frac{\gamma}{2}u\right\|^2_{L^2(\mathbb{R}^N)}+\left(1-\frac{\gamma^2}{4}\right)\|u\|^2_{L^2(\mathbb{R}^N)}+\|\nabla_\alpha u\|^2_{L^2(\mathbb{R}^N)}-2\int_{\mathbb{R}^N} F(X,u)\,dX \le C_1.$$

From (2.6),

$$\left\|u_t+\frac{\gamma}{2}u\right\|^2_{L^2(\mathbb{R}^N)}+\left(1+\frac{\gamma^2}{4}\right)\|u\|^2_{L^2(\mathbb{R}^N)}+\|\nabla_\alpha u\|^2_{L^2(\mathbb{R}^N)}-2\int_{\mathbb{R}^N} F(X,u)\,dX \le C_2.$$

Multiplying (1.1) by $u_t + \frac{\gamma}{2}u$ and integrating over $[0,T]\times\mathbb{R}^N$, we obtain

$$\begin{aligned}
&\frac{\gamma}{2}\int_0^T\int_{\mathbb{R}^N} \big(|u_t|^2+|u|^2+|\nabla_\alpha u|^2-f(X,u)u\big)\,dX\,dt\\
&= \frac{1}{2}\left\{\left\|u_t(0)+\frac{\gamma}{2}u(0)\right\|^2_{L^2(\mathbb{R}^N)}+\left(1+\frac{\gamma^2}{4}\right)\|u(0)\|^2_{L^2(\mathbb{R}^N)}+\|\nabla_\alpha u(0)\|^2_{L^2(\mathbb{R}^N)}\right\}\\
&\quad -\int_{\mathbb{R}^N} F(X,u(0))\,dX+\int_{\mathbb{R}^N} F(X,u(T))\,dX\\
&\quad -\frac{1}{2}\left\{\left\|u_t(T)+\frac{\gamma}{2}u(T)\right\|^2_{L^2(\mathbb{R}^N)}+\left(1+\frac{\gamma^2}{4}\right)\|u(T)\|^2_{L^2(\mathbb{R}^N)}+\|\nabla_\alpha u(T)\|^2_{L^2(\mathbb{R}^N)}\right\}.
\end{aligned}$$

Hence

$$\left|\int_0^T \Big[E(u(t),u_t(t)) - \int_{\mathbb{R}^N} f(X,u)u\,dX\Big]\,dt\right| \le C. \tag{3.9}$$

Similarly to the case of (3.9), since B is bounded in H and $U_n \in B$, for every $T \ge 0$ we have

$$\left|\int_0^T \Big[E(u_n(t),u_{nt}(t)) - \int_{\mathbb{R}^N} f(X,u_n)u_n\,dX\Big]\,dt\right| \le C. \tag{3.10}$$

Multiplying (1.1) by u_t and integrating over $[t,T]\times\mathbb{R}^N$ we obtain

$$E(u(T),u_t(T)) - \int_{\mathbb{R}^N} F(X,u(X,T))\,dX + \gamma\int_t^T \|u_\tau\|^2_{L^2(\mathbb{R}^N)}\,d\tau = E(u(t),u_t(t)) - \int_{\mathbb{R}^N} F(X,u(X,t))\,dX. \tag{3.11}$$

From (3.9) and (3.11), we have

$$\begin{aligned} E(u(T),u_t(T)) &- \int_{\mathbb{R}^N} F(X,u(X,T))\,dX + \frac{\gamma}{T}\int_0^T\int_t^T \|u_\tau\|^2_{L^2(\mathbb{R}^N)}\,d\tau\,dt \\ &= \frac{1}{T}\int_0^T\Big[E(u(t),u_t(t)) - \int_{\mathbb{R}^N} F(X,u(X,t))\,dX\Big]\,dt \\ &\geq \frac{1}{T}\int_0^T\int_{\mathbb{R}^N}\big(-F(X,u(X,t)) + f(X,u)u\big)\,dX\,dt - \frac{C}{T}. \end{aligned} \tag{3.12}$$

From (3.10) and (3.11), we get

$$\begin{aligned} E(u_n(T),u_{nt}(T)) &- \int_{\mathbb{R}^N} F(X,u_n(X,T))\,dX \\ &+ \frac{\gamma}{T}\int_0^T\int_t^T \|u_{n\tau}\|^2_{L^2(\mathbb{R}^N)}\,d\tau\,dt \\ &\leq \frac{1}{T}\int_0^T\int_{\mathbb{R}^N}\big(-F(X,u_n(X,t)) + f(X,u_n)u_n\big)\,dX\,dt + \frac{C}{T}. \end{aligned} \tag{3.13}$$

From (3.12) and (3.13), it follows that

$$\begin{aligned} E(u_n(T),u_{nt}(T)) &- \int_{\mathbb{R}^N} F(X,u_n(X,T))\,dX + \frac{\gamma}{T}\int_0^T\int_t^T \|u_{n\tau}\|^2_{L^2(\mathbb{R}^N)}\,d\tau\,dt \\ &\leq \frac{1}{T}\int_0^T\int_{\mathbb{R}^N}\big(F(X,u(X,t)) - F(X,u_n(X,t))\big)\,dX\,dt \\ &\quad + \frac{1}{T}\int_0^T\int_{\mathbb{R}^N}\big(f(X,u_n)u_n - f(X,u)u\big)\,dX\,dt + E(u(T),u_t(T)) \\ &\quad - \int_{\mathbb{R}^N} F(X,u(X,T))\,dX + \frac{\gamma}{T}\int_0^T\int_t^T \|u_\tau\|^2_{L^2(\mathbb{R}^N)}\,d\tau\,dt + \frac{2C}{T}, \end{aligned}$$

hence

$$
\begin{aligned}
(3.14)\qquad & E(u_n(T),u_{nt}(T)) \\
&\le \frac{1}{T}\int_0^T\int_{\mathbb{R}^N}\big(F(X,u(X,t))-F(X,u_n(X,t))\big)\,dX\,dt \\
&\quad+\frac{1}{T}\int_0^T\int_{\mathbb{R}^N}\big(f(X,u_n)u_n-f(X,u)u\big)\,dX\,dt+E(u(T),u_t(T)) \\
&\quad+\Big(\int_{\mathbb{R}^N}F(X,u_n(X,T))\,dX-\int_{\mathbb{R}^N}F(X,u(X,T))\,dX\Big)+\frac{2C}{T} \\
&\quad+\frac{\gamma}{T}\Big(\int_0^T\int_t^T\|u_\tau\|^2_{L^2(\mathbb{R}^N)}\,d\tau\,dt-\int_0^T\int_t^T\|u_{n\tau}\|^2_{L^2(\mathbb{R}^N)}\,d\tau\,dt\Big).
\end{aligned}
$$

From $U_n \rightharpoonup U$ it follows that $u_{nt} \rightharpoonup u_t$, and by the weak lower semicontinuity of norms, we have

$$\liminf_{n\to\infty}\|u_{nt}\|^2_{L^2(\mathbb{R}^N)}\ge\|u_t\|^2_{L^2(\mathbb{R}^N)}.$$

Thus for any $\epsilon>0$ there exists $N_0>0$ such that for all $N>N_0$,

$$\int_0^T\int_t^T\|u_\tau\|^2_{L^2(\mathbb{R}^N)}\,d\tau\,dt-\int_0^T\int_t^T\|u_{n\tau}\|^2_{L^2(\mathbb{R}^N)}\,d\tau\,dt\le\epsilon. \tag{3.15}$$

From Lemma 3.3 we have

$$\lim_{n\to\infty}\frac{1}{T}\int_0^T\int_{[B(0,R)]}\big(F(X,u(X,t))-F(X,u_n(X,t))\big)\,dX\,dt=0, \tag{3.16}$$

$$\lim_{n\to\infty}\frac{1}{T}\int_0^T\int_{[B(0,R)]}\big(f(X,u_n)u_n-f(X,u)u\big)\,dX\,dt=0, \tag{3.17}$$

$$\lim_{n\to\infty}\int_{[B(0,R)]}\big(F(X,u_n(X,T))-F(X,u(X,T))\big)\,dX=0. \tag{3.18}$$

On the other hand, as in [14] we have

$$
\begin{aligned}
&\Big|\frac{1}{T}\int_0^T\int_{\mathbb{R}^N\setminus[B(0,R)]}\big(F(X,u(X,t))-F(X,u_n(X,t))\big)\,dX\,dt\Big| \\
&\le\frac{C}{T}\int_0^T\Big\{\|u_n-u\|_{L^2(\mathbb{R}^N\setminus[B(0,R)])} \\
&\quad\times\Big[\Big(\|u\|^{\rho+1}_{L^{2(\rho+1)}(\mathbb{R}^N\setminus[B(0,R)])}+\|u_n\|^{\rho+1}_{L^{2(\rho+1)}(\mathbb{R}^N\setminus[B(0,R)])}\Big) \\
&\quad+\|f(X,0)\|_{L^2(\mathbb{R}^N\setminus[B(0,R)])} \\
&\quad+\|g\|_{L^{N_\alpha}(\mathbb{R}^N\setminus[B(0,R)])}\Big(\|u\|_{L^{\frac{2N_\alpha}{N_\alpha-2}}(\mathbb{R}^N\setminus[B(0,R)])}+\|u_n\|_{L^{\frac{2N_\alpha}{N_\alpha-2}}(\mathbb{R}^N\setminus[B(0,R)])}\Big)\Big]\Big\}\,dt,
\end{aligned}
$$

and

$$\begin{aligned}&\Bigg|\frac{1}{T}\int_0^T\int_{\mathbb{R}^N\setminus[B(0,R)]}\big(f(X,u_n)u_n-f(X,u)u\big)\,dX\,dt\Bigg|\\&\quad\le\frac{C}{T}\int_0^T\Big\{\|u_n-u\|_{L^2(\mathbb{R}^N\setminus[B(0,R)])}\big(\|g\|_{L^{N_\alpha}(\mathbb{R}^N\setminus[B(0,R)])}\|u_n\|_{L^{\frac{2N_\alpha}{N_\alpha-2}}(\mathbb{R}^N\setminus[B(0,R)])}\\&\qquad+\|u_n\|^{\rho}_{L^{r_1\rho}(\mathbb{R}^N\setminus[B(0,R)])}\|u_n\|_{L^{r_2}(\mathbb{R}^N\setminus[B(0,R)])}+\|f(X,u)\|^2_{L^2(\mathbb{R}^N\setminus[B(0,R)])}\\&\qquad+\|u\|^{\rho}_{L^{r_1\rho}(\mathbb{R}^N\setminus[B(0,R)])}\|u_n\|_{L^{r_2}(\mathbb{R}^N\setminus[B(0,R)])}\big)\Big\}\,dt\end{aligned}$$

where $r_1=2N_\alpha/((N_\alpha-2)\rho)$ if $\rho\neq 0$, $r_1=N_\alpha$ if $\rho=0$, and $1/r_1+1/r_2=1/2$.

So it follows from Lemma 3.1 and (3.5) that for each $\epsilon>0$, there exist T_0 and R_0 such that when $T\ge T_0$ and $R\ge R_0$, we have

$$\begin{aligned}\text{(3.19)}\qquad&\frac{1}{T}\int_0^T\int_{\mathbb{R}^N\setminus[B(0,R)]}\big(F(X,u(X,t))-F(X,u_n(X,t))\big)\,dX\,dt\\&+\frac{1}{T}\int_0^T\int_{\mathbb{R}^N\setminus[B(0,R)]}\big(f(X,u_n)u_n-f(X,u)u\big)\,dX\,dt\\&+\int_{\mathbb{R}^N\setminus[B(0,R)]}\big(F(X,u_n(X,T))-F(X,u(X,T))\big)\,dX+\frac{C}{T}\le\epsilon.\end{aligned}$$

Taking into account (3.15)–(3.19) in (3.14) and passing to the limit we get

$$\limsup_{n\to\infty}E(u_n(T),u_{nt}(T))\le E(u(T),u_t(T))+\epsilon,$$

which yields (3.7). ∎

We now formulate the main result of the section.

Theorem 3.5. *Assume that $f(X,u)$ satisfies conditions* (1.3)–(1.6). *Then the semigroup $\{S(t)\}_{t\ge0}$ associated with problem* (2.1)–(2.2) *is asymptotically compact in the phase space H, i.e., if $\{U_n\}_{n=1}^\infty$ is a bounded sequence in H and $\{t_n\}_{n=1}^\infty$ is a time sequence such that $t_n\to\infty$ as $n\to\infty$, then $\{S(t_n)U_n\}_{n=1}^\infty$ is precompact in H.*

Proof. From (3.1), we know that there exists a bounded subset $\mathcal{B}$ in H such that $S(t_n)U_n\subset\mathcal{B}$ for all $n\in\mathbb{N}$. It follows that there exist $U\in H$ and a subsequence $\{n_k\}_{k=1}^\infty$ such that

$$S(t_{n_k})U_{n_k}\rightharpoonup U\quad\text{in } H,$$

which implies that

$$\liminf_{k\to\infty} \|S(t_{n_k})U_{n_k}\|_H \ge \|U\|_H. \tag{3.20}$$

Now, to prove that there exists a subsequence of $S(t_{n_k})U_{n_k}$ converging strongly to U in H, we will construct a subsequence $S(t_{n_{k_j}})U_{n_{k_j}}$ such that

$$\limsup_{j\to\infty} \|S(t_{n_{k_j}})U_{n_{k_j}}\|_H \le \|U\|_H.$$

Indeed, from Lemma 3.4 for any $l > 0$ there exists a $T_0 = T_0(l, \mathcal{B})$ such that for any $\{\varphi_i\} \subset \mathcal{B}$ with $\varphi_i \rightharpoonup \varphi$ in H we have

$$\limsup_{i\to\infty} \|S(T_0)\varphi_i\|_H \le \|S(T_0)\varphi\|_H + 1/l. \tag{3.21}$$

For $t_{n_k} \ge T_0$, we then also have

$$S(t_{n_k} - T_0)U_{n_k} \subset \mathcal{B}.$$

Therefore, there is a U_{T_0} and a subsequence $\{n_{k_{j(l)}}\}_{j(l)=1}^{\infty}$ of $\{n_k\}_{k=1}^{\infty}$ such that

$$S(t_{n_{k_{j(l)}}} - T_0)U_{n_{k_{j(l)}}} \rightharpoonup U_{T_0} \quad \text{in } H, \tag{3.22}$$

which, by using Lemma 3.2, implies that

$$S(t_{n_{k_{j(l)}}})U_{n_{k_{j(l)}}} = S(T_0)S(t_{n_{k_{j(l)}}} - T_0)U_{n_{k_{j(l)}}} \rightharpoonup S(T_0)U_{T_0} \quad \text{in } H.$$

The uniqueness of the weak limit yields $U = S(T_0)U_{T_0}$.

Taking $\varphi_{k_{j(l)}} = S(t_{n_{k_{j(l)}}} - T_0)U_{n_{k_{j(l)}}}$ in (3.21) we obtain

$$\limsup_{j(l)\to\infty} \|S(T_0)S(t_{n_{k_{j(l)}}} - T_0)U_{n_{k_{j(l)}}}\|_H \le \|S(T_0)U_{T_0}\|_H + 1/l.$$

Hence,

$$\limsup_{j(l)\to\infty} \|S(t_{n_{k_{j(l)}}})U_{n_{k_{j(l)}}}\|_H \le \|U\|_H + 1/l. \tag{3.23}$$

For $l = 1$ from (3.23) there exists a $j_1(1)$ such that

$$\|S(t_{n_{k_{j_1(1)}}})U_{n_{k_{j_1(1)}}}\|_H \le \|U\|_H + 2.$$

Denote $j_1(1)$ by j_1.

For $l = 2$ from (3.23) there exists a $j_2(2)$ such that $n_{k_{j_1}} < n_{k_{j_2(2)}}$ and

$$\|S(t_{n_{k_{j_2(2)}}})U_{n_{k_{j_2(2)}}}\|_H \le \|U\|_H + 1.$$

Denote $j_2(2)$ by j_2.

Similarly, for $l = m$ from (3.23) there exists a number $j_m(m)$ such that $n_{k_{j_{m-1}}} < n_{k_{j_m(m)}}$ and

$$\|S(t_{n_{k_{j_m(m)}}})U_{n_{k_{j_m(m)}}}\|_H \le \|U\|_H + 2/m.$$

Denote $j_m(m)$ by j_m.

Obviously, the sequence $\{n_{k_j}\}_{j=1}^{\infty}$ is as desired since

$$\limsup_{j\to\infty}\|S(t_{n_{k_j}})U_{n_{k_j}}\|_H \le \|U\|_H. \ \blacksquare$$

From Theorem 3.5, we obtain

THEOREM 3.6. *Assume that $f(X,u)$ satisfies conditions* (1.3)–(1.6). *Then problem* (2.1)–(2.2) *has a global attractor in H which is a compact invariant subset that attracts every bounded set of H with respect to the norm topology.*

EXAMPLE 3.7. Consider the problem

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} + \gamma\dfrac{\partial u}{\partial t} + u = G_{1/2}u + f(X,u) & \text{for } X=(x,y)\in\mathbb{R}^2,\ t>0,\\ u(x,y,0)=u_0(x,y), \quad \dfrac{\partial u}{\partial t}(x,y,0)=u_1(x,y) & \text{for } (x,y)\in\mathbb{R}^2, \end{cases}$$

where $u_0\in S_1^2(\mathbb{R}^2)$, $u_1\in L^2(\mathbb{R}^2)$, γ is a positive constant and

$$G_{1/2}u = \frac{\partial^2 u}{\partial x^2} + |x|\frac{\partial^2 u}{\partial y^2},$$

and

$$f(X,u) = \begin{cases} \dfrac{-u(1-u^2)}{|X|^4+1} & \text{for } |u|\le 1,\ |X|^2 = x^2+y^2,\\ \dfrac{u(1-u^2)}{|X|^4+1} & \text{for } |u|\ge 1. \end{cases}$$

We obtain

$$F(X,u) = \begin{cases} \dfrac{-1}{|X|^4+1}\left(\dfrac{u^2}{2}-\dfrac{u^4}{4}\right) & \text{for } |u|\le 1,\\ \dfrac{1}{|X|^4+1}\left(\dfrac{u^2}{2}-\dfrac{u^4}{4}-\dfrac{1}{2}\right) & \text{for } |u|\ge 1. \end{cases}$$

Obviously $F(X,u(X))\le 0$ for all $X\in\mathbb{R}^2$ and $u\in S_1^2(\mathbb{R}^N)$.

It is easily checked that the function $f(X,u)$ satisfies conditions (1.3)–(1.6) in which we can take $C_1=2$, $\rho=4$, $N_{1/2}=5/2$, $h(x,y)=0$,

$$g(x,y) = \frac{1}{|X|^4+1} + \frac{1}{(|X|^4+1)^2}, \qquad g_1(x,y) = -\frac{1}{4(|X|^4+1)},$$

and $C_2 = 1/4$. Applying Theorem 3.6 we conclude that there exists a global compact attractor in $S_1^2(\mathbb{R}^2)\times L^2(\mathbb{R}^2)$ for the associated semigroup $S(t)$.

Note that we cannot take $g_1(x,y)$ satisfying $g_1(x_0,y_0)\ge 0$ at a particular point (x_0,y_0) for any constant C_2. Indeed, if $g_1(x_0,y_0)\ge 0$, from (1.5) we have $-\frac{1}{4((x_0^2+y_0^2)^2+1)}\ge 0$ for $u=1$, a contradiction.

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