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Some Applications of the Katětov Order on Borel Ideals

by

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Summary. We construct an embedding of the algebra $\mathcal{P}(\omega)/\mathrm{Fin}$ into the family of summable ideals with the Katětov order. This construction will be used to solve two problems: about the relation between the Katětov order and the ideal Baire classes of functions, and about long chains of ideals alternately with and without the property of being a P-ideal.

1. Introduction. An *ideal* on the set ω of natural numbers is a family $\mathcal{I} \subset \mathcal{P}(\omega)$ (where $\mathcal{P}(\omega)$ denotes the power set of ω) which is closed under taking subsets and finite unions. We denote by Fin the ideal of all finite subsets of ω. We assume that all the ideals under consideration are proper ($\neq \mathcal{P}(\omega)$) and contain all finite sets.

Given two ideals $\mathcal{I}$ and $\mathcal{J}$ we write $\mathcal{I} \leq_K \mathcal{J}$ if there exists a function $f\colon \omega \to \omega$ such that $f^{-1}[A] \in \mathcal{J}$ whenever $A \in \mathcal{I}$. This preorder is called the *Katětov order* and was introduced by Katětov [4, 5].

Many topological and combinatorial properties could be described by finding a locally minimal (in the Katětov order) ideal among ideals having a given property (see [10], [12], [1] or [7]). In particular, Katětov investigated ideal convergence of sequences of continuous functions using this order. In [5] he proved that if $\mathcal{I} \leq_K \mathcal{J}$ then $\mathcal{B}_1^{\mathcal{I}}(T) \subset \mathcal{B}_1^{\mathcal{J}}(T)$ (where $\mathcal{B}_1^{\mathcal{I}}(T)$ is the family of $\mathcal{I}$-Baire class one functions over a topological space T, see Section 2 for a formal definition). In the same article he asked about the converse implication:

PROBLEM 1.1. If $\mathcal{B}_1^{\mathcal{I}}(T) \subset \mathcal{B}_1^{\mathcal{J}}(T)$ for any topological space T, does it follow that $\mathcal{I} \leq_K \mathcal{J}$?

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The answer can be deduced from [6] where the authors proved that $\mathcal{B}_1^{\mathcal{I}_d}(X) = \mathcal{B}_1^{\mathrm{Fin}}(X)$ where $\mathcal{I}_d$ is the ideal of sets of asymptotic density zero and X is a complete metric space. It is easy to prove that $\mathcal{I}_d \not\leq_K \mathrm{Fin}$, hence we have a negative answer to Katětov's problem. In Corollary 3.8 we give a stronger counterexample by showing that below any analytic P-ideal there is a family of size continuum of pairwise incomparable (in the Katětov order) ideals such that the Baire classes generated by them are equal.

In Section 4 we use the above construction to answer a question of Wilczyński. During the problem session at the 23th International Summer Conference on Real Functions Theory in Niedzica the following problem was formulated:

PROBLEM 1.2. Does there exist, for any $n \in \omega$, a sequence of ideals

$$\mathcal{I}_0 \subset \mathcal{I}_1 \subset \cdots \subset \mathcal{I}_n$$

such that $\mathcal{I}_i$ is a P-ideal iff i is even?

In Theorem 4.4 we give a positive answer to this question by producing even a transfinite sequence of ideals with this property.

2. Preliminaries. An ideal $\mathcal{I}$ is called *dense* if for any infinite set $A \subset \omega$ there exists an infinite set $B \subset A$ which belongs to $\mathcal{I}$.

An ideal $\mathcal{I}$ is a *P-ideal* if for every sequence $(A_n)_{n\in\omega}$ of sets from $\mathcal{I}$ there is $A \in \mathcal{I}$ such that $A_n \subset^\star A$, i.e. $A_n \setminus A \in \mathrm{Fin}$ for all n.

By identifying sets of naturals with their characteristic functions, we can treat $\mathcal{P}(\omega)$ as the Cantor cube with the natural product topology and therefore we can assign the topological complexity to ideals of sets of integers. In particular, an ideal $\mathcal{I}$ is *analytic* if it is a continuous image of a G_δ subset of the Cantor space.

A map $\phi\colon \mathcal{P}(\omega) \to [0,\infty]$ is a *submeasure on* ω if

$$\phi(\emptyset) = 0, \qquad \phi(A) \leq \phi(A \cup B) \leq \phi(A) + \phi(B),$$

for all $A, B \subset \omega$. It is *lower semicontinuous* (lsc for short) if for all $A \subset \omega$ we have

$$\phi(A) = \lim_{n\to\infty} \phi(A \cap \{0,1,\ldots,n-1\}).$$

For any lsc submeasure on ω, let $\|\cdot\|_\phi\colon \mathcal{P}(\omega) \to [0,\infty]$ be the submeasure defined by

$$\|A\|_\phi = \lim_{n\to\infty} \phi(A \setminus \{0,1,\ldots,n-1\}).$$

Let

$$\mathrm{Exh}(\phi) = \{A \subset \omega : \|A\|_\phi = 0\}.$$

All analytic P-ideals were characterized by Solecki [11].

Theorem 2.1. *The following conditions are equivalent for an ideal $\mathcal{I}$ on ω.*

(1) *$\mathcal{I}$ is an analytic P-ideal;*
(2) *$\mathcal{I} = \mathrm{Exh}(\phi)$ for some lsc submeasure ϕ on ω.*

It is easy to observe that:

Fact 2.2. *For any lsc submeasure ϕ, $\mathrm{Exh}(\phi)$ is dense iff $\lim_{n\to\infty} \phi(\{n\}) = 0$.*

For a function $g\colon \omega \to \mathbb{R}$ such that $\sum_{n\in\omega} g(n) = \infty$ the family

$$\mathcal{I}_g = \Big\{ A \subset \omega : \sum_{n\in A} g(n) < \infty \Big\}$$

is an analytic P-ideal called a *summable* ideal generated by g.

Let T be a topological Hausdorff space and $\mathcal{I}$ be an ideal on ω. We say that a sequence $(x_n)_{n\in\omega}$ in T is *$\mathcal{I}$-convergent* to $x \in T$ if

$$\{n \in \omega : x_n \notin U\} \in \mathcal{I}$$

for every open neighborhood U of x.

We say that a sequence $(f_n : T \to \mathbb{R})_{n\in\omega}$ of functions is *pointwise $\mathcal{I}$-convergent* if $(f_n(x))_{n\in\omega}$ is $\mathcal{I}$-convergent for every $x \in T$.

Using this definition we can introduce ideal Baire classes of functions. We say that a function f is of *$\mathcal{I}$-Baire class one* if it is an $\mathcal{I}$-pointwise limit of continuous functions. The family of all $\mathcal{I}$-Baire class one functions over a Hausdorff space T is denoted by $\mathcal{B}_1^{\mathcal{I}}(T)$.

Laczkovich and Recław [8] proved the following theorem.

Theorem 2.3. *If $\mathcal{I}$ is a non-pathological analytic P-ideal and T is a Hausdorff space, then $\mathcal{B}_1^{\mathcal{I}}(T) = \mathcal{B}_1(T)$.*

The definition of a non-pathological analytic P-ideal is found in [8]; in particular, all summable ideals are non-pathological.

3. Katětov's problem

Theorem 3.1. *Let $\mathcal{I}$ be a dense analytic P-ideal. There exists an embedding of the algebra $\mathcal{P}(\omega)/\mathrm{Fin}$ into the family of summable ideals included in $\mathcal{I}$.*

A weaker version of this theorem was proved independently by Meza-Alcántara [10] and published in [2].

To prove this theorem we start with the construction of a family of ideals. Fix a dense analytic P-ideal $\mathcal{I} = \mathrm{Exh}(\phi)$ for some lsc submeasure ϕ.

Let $(p_n)_{n\in\omega}$ be a sequence of natural numbers such that

(1) $p_0 = 0$,
(2) $\frac{p_n}{2(p_0+p_1+\cdots+p_{n-1})+1} > 2^{2^n}$,

(3) $\phi(\{m\}) < 1/2^{2^{n+1}}$ for all $m > p_0 + p_1 + \cdots + p_{n-1}$.

The fulfillment of the third condition is possible by Fact 2.2.

For all $n \geq 1$ let

$$S_n = \{p_0 + p_1 + \cdots + p_{n-1}, p_0 + p_1 + \cdots + p_{n-1} + 1, \ldots, p_0 + p_1 + \cdots + p_n\}.$$

Obviously $\{S_n\}_{n\in\omega}$ is a partition of the naturals. For each n define two measures ϕ_n^0 and ϕ_n^1 on S_n by

$$\phi_n^0(A) = \frac{|A|}{2^{2^{n+1}}}, \qquad \phi_n^1(A) = \frac{|A|}{2^{2^n}}.$$

For each infinite set $M \subset \omega$ define the ideal

$$\mathcal{I}_M = \Big\{A \subset \omega : \sum_{n\in M} \phi_n^1(A \cap S_n) < \infty, \sum_{n\in\omega\setminus M} \phi_n^0(A \cap S_n) < \infty\Big\}.$$

Lemma 3.2. *For each infinite set M the ideal $\mathcal{I}_M$ is a summable ideal contained in $\mathcal{I}$.*

Proof. To prove the summability it is enough to observe that $\mathcal{I}_M$ is generated by the function

$$f_M(i) = \begin{cases} 1/2^{2^n} & \text{if } i \in S_n \text{ and } n \in M, \\ 1/2^{2^{n+1}} & \text{if } i \in S_n \text{ and } n \notin M. \end{cases}$$

To justify the inclusion in $\mathcal{I}$ notice that $\phi \leq f_M$. ∎

Lemma 3.3. *Let A, B be infinite subsets of ω. If $B \subset^\star A$, then $\mathcal{I}_A \subset \mathcal{I}_B$.*

Proof. This follows from the easy observation that $B \subset^\star A$ implies that $f_B(n) \leq f_A(n)$ for sufficiently large n. ∎

Lemma 3.4. *$\mathcal{I}_A \leq_K \mathcal{I}_B$ iff $B \subset^\star A$.*

Proof. The implication $\Leftarrow$ follows from Lemma 3.3 and from the implication

$$\mathcal{I} \subset \mathcal{J} \;\Rightarrow\; \mathcal{I} \leq_K \mathcal{J}.$$

To prove the converse, suppose that $B \setminus A$ is infinite and $\mathcal{I}_A \leq_K \mathcal{I}_B$. Hence there exists a function $f\colon \omega \to \omega$ such that $f^{-1}(I) \in \mathcal{I}_B$ for any $I \in \mathcal{I}_A$.

For each $n \in \omega$ we have two possibilities:

(a) $\{i \in S^n : f(i) \in \bigcup_{i\geq n} S_i\}$ has at least 2^{2^n} elements,
(b) $\{i \in S^n : f(i) \in \bigcup_{i<n} S_i\}$ has at least $p_n/2$ elements.

One of them holds for infinitely many n in $B \setminus A$. Let N be the set of all such n.

Suppose that N consists of those n for which (a) holds. For each $n \in N$ choose $D_n \subset S_n$ such that $|D_n| = 2^{2^n}$ and $f(D_n) \subset \bigcup_{i \geq n} S_i$. Define

$$E = \bigcup_{n \in \omega} f(D_n).$$

Notice that for each n,

$$\sum_{m \in \omega} \phi_m^0(f(D_n) \cap S_m) \leq \frac{1}{2^{2^n}}.$$

Hence $E \in \mathcal{I}_A$. On the other hand, for each $n \in N$ we have $|f^{-1}(E) \cap S_n| \geq 2^{2^n}$, hence $\phi_n^1(f^{-1}(E) \cap S_n) = 1$. Finally $f^{-1}(E) \notin \mathcal{I}_B$, a contradiction.

Suppose now that N consists of those n for which (b) holds. For each $n \in N$ choose e_n such that $|f^{-1}(\{e_n\}) \cap S_n| > 2^{2^n}$ (this is possible by the condition (1) on $(p_n)_n$ and the pigeonhole principle). Let $E \subset \{e_n\}_{n \in N}$ be such that

- $|E \cap S_i| \leq 1$ for all $i \in \omega$,
- $|f^{-1}(E) \cap S_n| > 2^{2^n}$ for infinitely many $n \in N$.

The first condition guarantees that $E \in \mathcal{I}_A$. By the second condition $\phi_n^1(f^{-1}(E) \cap S_n) \geq 1$ for infinitely many $n \in N \subset B$. This implies that $f^{-1}(E) \notin \mathcal{I}_B$, a contradiction. ∎

Proof of Theorem 3.1. Consider the mapping

$$\mathcal{P}(\omega)/\mathrm{Fin} \ni M \mapsto \mathcal{I}_M.$$

It is well defined since if $A \bigtriangleup B \in \mathrm{Fin}$, then $f_A = f_B$ almost everywhere, hence $\mathcal{I}_A = \mathcal{I}_B$. The fact that it is an embedding follows from Lemma 3.4. ∎

Definition 3.5. We call a family $\{\mathcal{I}_\alpha\}_\alpha$ of ideals an $\leq_K$-*antichain* if

$$\mathcal{I}_\alpha \leq_K \mathcal{I}_{\alpha'} \ \Leftrightarrow\ \alpha = \alpha'.$$

Notice that this definition is different from the classical definition of the antichain in a Boolean algebra, where there is no element of the algebra smaller than two distinct elements of the antichain. Such a definition would be too strong since for any two ideals $\mathcal{I}, \mathcal{J}$ we have $\mathcal{I} \oplus \mathcal{J} \leq_K \mathcal{I}, \mathcal{J}$ where $\mathcal{I} \oplus \mathcal{J}$ is the disjoint union of $\mathcal{I}$ and $\mathcal{J}$.

Hrušák and García Ferreira [3] showed that below any dense ideal there is a $\leq_K$-antichain of size continuum. The ideals constructed by them are generated by maximal almost disjoint families, so by the result of Mathias [9] they are not analytic. Since there exists an almost disjoint family in $\mathcal{P}(\omega)$ of cardinality continuum, we get the following corollary:

Corollary 3.6. *Let $\mathcal{I}$ be a dense analytic P-ideal. Below $\mathcal{I}$, there exists a $\leq_K$-antichain of cardinality continuum consisting of summable ideals.*

Here we cannot replace a dense analytic P-ideal by any dense ideal. Recall that if $\mathcal{A}$ is an almost disjoint family in $\mathcal{P}(\omega)$ then

$$\{B \subset \omega : A \cap B \text{ is infinite only for finitely many } A \in \mathcal{A}\}$$

is an ideal.

Theorem 3.7. *If $\mathcal{I}$ is an ideal generated by an almost disjoint family $\mathcal{A}$, and $\mathcal{J}$ is any dense P-ideal, then $\mathcal{J} \not\leq_K \mathcal{I}$.*

Proof. Suppose that $\mathcal{J} \leq_K \mathcal{I}$. Let $f\colon \omega \to \omega$ be a function from the definition of the Katětov order.

Choose a countable family $\{A_n\}_{n\in\omega} \subset \mathcal{A}$. Let N be the set of all n such that $f(A_n)$ is infinite.

Suppose that N is infinite. Since $\mathcal{J}$ is a dense ideal, for each n we can choose an infinite set $E_n \subset f(A_n)$ and $E_n \in \mathcal{J}$. Let $E \in \mathcal{J}$ be such that $E_n \subset^\star E$ for each $n \in N$. Notice that $f^{-1}(E) \cap A_n$ is infinite for each $n \in N$, hence $f^{-1}(E) \notin \mathcal{J}$—a contradiction.

Suppose that N is finite. For each $n \in \omega \setminus N$ choose $e_n \in f(A_n)$ such that $f^{-1}(e_n)$ is infinite. Let $E \subset \{e_n\}_{n\in\omega}$ be an infinite set such that $E \in \mathcal{J}$. Observe that $f^{-1}(E) \cap A_n$ is infinite for each $n \in N$. Hence $f^{-1}(E) \notin \mathcal{I}$—a contradiction. ∎

Finally, as a corollary from Theorem 2.3 and Corollary 3.6 we get the following answer to Katětov's problem:

Corollary 3.8. *Let $\mathcal{I}$ be a dense analytic P-ideal. There exists a family $\{\mathcal{I}_\alpha\}_{\alpha<\mathfrak{c}}$ of ideals pairwise incomparable in the Katětov order such that $\mathcal{I}_\alpha \leq_K \mathcal{I}$ and $\mathcal{B}_1^{\mathcal{I}_\alpha}(T) = \mathcal{B}_1(T)$ for each $\alpha < \mathfrak{c}$ and every Hausdorff space T.*

4. Wilczyński's problem.

Recall the definition of the *bounded number* $\mathfrak{b}$:

$$\mathfrak{b} = \min\{|F| : F \subset \omega^\omega \text{ and } \forall_{g\in\omega^\omega}\, \exists_{f\in F}\, \forall_{n\in\omega}\, \exists_{m>n}\; g(m) < f(m)\}.$$

Recall that $\aleph_0 < \mathfrak{b} \leq \mathfrak{c}$. Obviously CH implies $\mathfrak{b} = \mathfrak{c}$, but in the Cohen model we have $\aleph_1 = \mathfrak{b} < \mathfrak{c}$.

Definition 4.1. Let γ be an ordinal number. We call a family $\{\mathcal{I}_\alpha\}_{\alpha<\gamma}$ of ideals an *increasing $\leq_K$-chain* if for any $\alpha, \alpha' < \gamma$,

$$\alpha \leq \alpha' \;\Leftrightarrow\; \mathcal{I}_\alpha \leq_K \mathcal{I}_{\alpha'}.$$

Since there are increasing chains of length $\mathfrak{b}$ in $\mathcal{P}(\omega)/\mathrm{Fin}$, we get the following corollary from Theorem 3.1.

Corollary 4.2. *Let $\mathcal{I}$ be an analytic P-ideal. Below $\mathcal{I}$, there exists an increasing $\leq_K$-chain of length $\mathfrak{b}$ of summable ideals.*

In Theorem 4.4 we will use this $\leq_K$-chain to answer Wilczyński's problem, but first we need the following proposition:

Proposition 4.3. *If $\mathcal{I}_0 \subset \mathcal{I}_1 \subset \cdots \subset \mathcal{I}_n \subset \cdots$ is a strictly increasing sequence of ideals, then $\bigcup_{n\in\omega} \mathcal{I}_n$ is an ideal which is not a P-ideal.*

Proof. It is easy to observe that $\mathcal{I} = \bigcup_{n\in\omega} \mathcal{I}_n$ is an ideal.

We now show that $\mathcal{I}$ is not a P-ideal. Since $(\mathcal{I}_n)_{n\in\omega}$ is a strictly increasing sequence of ideals, for each $n \in \omega$ we can choose $A_n \in \mathcal{I}_{n+1} \setminus \mathcal{I}_n$. Suppose that A is such that $A_n \setminus A$ is finite for each $n \in \omega$. Fix $n \in \omega$. Since $A_n \notin \mathcal{I}_n$, also $A \notin \mathcal{I}_n$. Hence $A \notin \mathcal{I}$. ∎

Let $\alpha = \lambda + n$ be an ordinal, where $n \in \omega$ and λ is a limit number or zero. We call α an *even* [*odd*] ordinal if n is even [odd].

Theorem 4.4. *Let $\mathcal{I}$ be a dense analytic P-ideal. There exists a sequence $(\mathcal{K}_\alpha)_{\alpha<\mathfrak{b}}$ of dense analytic ideals such that*

(1) *$\mathcal{K}_\alpha \subset \mathcal{K}_\beta \subset \mathcal{I}$ for $\alpha < \beta < \mathfrak{b}$,*
(2) *$\mathcal{K}_\alpha$ is a P-ideal iff α is even.*

Proof. Let $\{\mathcal{I}_\alpha\}_{\alpha<\mathfrak{b}}$ be an increasing $\leq_K$-chain of ideals from Corollary 4.2 (constructed as in the proof of Theorem 3.1). By Lemma 3.3, if $\alpha < \beta < \mathfrak{b}$, then $\mathcal{I}_\alpha \subset \mathcal{I}_\beta$.

For $\alpha < \mathfrak{b}$ define ideals

$$\mathcal{J}_\alpha = \begin{cases} \mathcal{I}_{\omega\cdot\alpha} & \text{if } \alpha \text{ is an even ordinal,} \\ \bigcup_{n\in\omega} \mathcal{I}_{\omega\cdot\alpha+n} & \text{if } \alpha \text{ is an odd ordinal.} \end{cases}$$

Since $(\mathcal{I}_\alpha)_{\alpha<\mathfrak{b}}$ is an increasing sequence of ideals, the sequence $(\mathcal{J}_\alpha)_{\alpha<\mathfrak{b}}$ is also increasing. Conclusion (2) holds by Proposition 4.3. ∎

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