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Weakly compactly generated Banach lattices

by

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Abstract. We study the different ways in which a weakly compact set can generate a Banach lattice. Among other things, it is shown that in an order continuous Banach lattice X, the existence of a weakly compact set $K \subset X$ such that X coincides with the band generated by K implies that X is weakly compactly generated.

1. The general problem. The purpose of this note is to study Banach lattices which are generated in one way or another by a weakly compact set. Namely, we will explore the connection between the existence of a weakly compact set which generates a Banach lattice as a linear space, a lattice, an ideal or a band. Our motivation starts with the question of J. Diestel of whether every Banach lattice which is generated, as a lattice, by a weakly compact set must be weakly compactly generated (i.e., as a linear space). Although this question remains open in full generality, we provide here several partial results in the affirmative sense.

Recall that a *Banach lattice* is a Banach space endowed with additional order and lattice structures which behave well with respect to the norm and linear structure. This is in particular highlighted by the fact that $\|x\| \leq \|y\|$ whenever $|x| \leq |y|$, or by the norm continuity of the lattice operations $\wedge$ and $\vee$. However, for the weak topology, the relation to the order and lattice structures is more subtle, in particular it is not always true that the lattice operations are weakly continuous. In fact, on infinite-dimensional Banach lattices the weak topology fails to be locally solid (see e.g. [1, Theorem 6.9]).

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A Banach space X is called *weakly compactly generated* (WCG) whenever there exists a weakly compact subset of X whose closed linear span coincides with X. This class of Banach spaces was first studied by Corson [12] and this study was pushed further by the fundamental work of Amir and Lindenstrauss [5]. Nowadays, WCG spaces play a relevant role in non-separable Banach space theory. For more complete information on WCG spaces, see [17, 22, 35].

Weakly compact sets and weakly compact operators in Banach lattices have been the object of research by several authors (cf. [2, 3, 11, 28], see also the monographs [4, Chapter 4.2] and [27, Chapter 2.5]). In particular, WCG Banach lattices have been considered in [8, 32].

Before introducing the main notions of the paper let us recall that a *sublattice* of a Banach lattice X is a subspace which is also closed under the lattice operations $\vee$ and $\wedge$. Also, an *ideal* Y of X is a subspace with the property that $|x| \leq |y|$ with $y \in Y$ implies that $x \in Y$. Finally, a *band* Z of X is an ideal for which $\sup(A) \in Z$ whenever $A \subset Z$ and $\sup(A)$ exists in X. Unless otherwise mentioned, all subspaces, sublattices, ideals and bands in this paper are assumed to be closed. Given a subset A of a Banach lattice X, we will denote by $\overline{\text{span}}(A)$, $L(A)$, $I(A)$ and $B(A)$ the smallest subspace (respectively, sublattice, ideal and band) of X containing A.

DEFINITION 1.1. Let X be a Banach lattice. We will say that

(i) X is *weakly compactly generated as a lattice* (LWCG) if there is a weakly compact set $K \subset X$ such that $X = L(K)$;
(ii) X is *weakly compactly generated as an ideal* (IWCG) if there is a weakly compact set $K \subset X$ such that $X = I(K)$;
(iii) X is *weakly compactly generated as a band* (BWCG) if there is a weakly compact set $K \subset X$ such that $X = B(K)$.

Since for a set $A \subset X$ the inclusions $\overline{\text{span}}(A) \subset L(A) \subset I(A) \subset B(A)$ always hold, we clearly have

$$\text{WCG} \Rightarrow \text{LWCG} \Rightarrow \text{IWCG} \Rightarrow \text{BWCG}.$$

Our interest is whether the converse implications hold. The equivalence between LWCG and WCG for general Banach lattices seems to be an open question; it was raised by J. Diestel during the conference "Integration, Vector Measures and Related Topics IV" held in La Manga del Mar Menor, Spain, 2011.

The paper is organized as follows:

In Section 2 we provide a first approach to the comparison between the notion of WCG Banach lattice and the weaker versions introduced above. For instance, we prove that LWCG = WCG for Banach lattices having weakly sequentially continuous lattice operations (Theorem 2.2). We also show that, in general, IWCG $\neq$ LWCG and BWCG $\neq$ IWCG (Example 2.4 and Proposition 2.9).

In Section 3 we prove that BWCG = WCG for order continuous Banach lattices (Theorem 3.1). Some related results on Dedekind complete Banach lattices are also given. As a by-product of our methods we provide applications to weakly precompactly generated Banach lattices.

In Section 4 we apply the factorization method of Davis–Figiel–Johnson–Pełczyński in our framework. For instance, it is shown that an IWCG Banach lattice not containing $C[0,1]$ is Asplund generated (Theorem 4.5).

In Section 5 we collect some results about the stability of weakly compact generation properties in Banach lattices. In general, the property of being LWCG is not inherited by sublattices. We discuss the three-space problem for LWCG Banach lattices (Example 5.2 and Theorem 5.4) and the connection of these properties with weakly Lindelöf determined Banach spaces.

We use standard Banach space/lattice terminology as can be found in [4, 25, 27]. By an *operator* between Banach spaces we mean a continuous linear map. The closed unit ball of a Banach space X is denoted by B_X and the dual of X is denoted by X^*. The weak* topology of X^* is denoted by w^*. The symbol X_+ stands for the positive cone of a Banach lattice X and we write $C_+ = C \cap X_+$ for any $C \subset X$.

2. Basic approach. Given a Banach lattice X, for a set $A \subset X$ we define

$$A^\wedge := \Big\{ \bigwedge_{i=1}^n a_i : n \in \mathbb{N},\ (a_i)_{i=1}^n \subset A \Big\},$$

$$A^\vee := \Big\{ \bigvee_{i=1}^n a_i : n \in \mathbb{N},\ (a_i)_{i=1}^n \subset A \Big\}.$$

We will denote $A^{\wedge\vee} := (A^\wedge)^\vee$ and $A^{\vee\wedge} := (A^\vee)^\wedge$. Using the distributive law of the lattice operations, it is easy to see that $A^{\vee\wedge} = A^{\wedge\vee}$ and that

$$L(A) = \overline{\operatorname{span}(A)^{\vee\wedge}} \tag{2.1}$$

(see e.g. [4, p. 204]). Recall that a subset $B \subset X$ of a Banach lattice is *solid* when $|x| \le |y|$ and $y \in B$ implies that $x \in B$. The *solid hull* $\operatorname{sol}(A)$ of A is the smallest solid subset of X containing A, which can be written as

$$\operatorname{sol}(A) = \bigcup_{x \in A} [-|x|, |x|].$$

It is not difficult to check that

$$I(A) = \overline{\operatorname{span}}(\operatorname{sol}(A)). \tag{2.2}$$

The disjoint complement of A is defined as

$$A^{\perp} = \{x \in X : |x| \wedge |y| = 0 \text{ for every } y \in A\}.$$

It is well known that

$$B(A) = A^{\perp\perp} \tag{2.3}$$

(see e.g. [27, Proposition 1.2.7]).

Recall that an operator between Banach lattices $T : X \to Y$ is:

- a *lattice homomorphism* if $T(x_1 \vee x_2) = (Tx_1) \vee (Tx_2)$ for $x_1, x_2 \in X$;
- *interval preserving* if it is positive and $T[0,x] = [0,Tx]$ for every $x \in X_+$.

Proposition 2.1. *Let X and Y be Banach lattices and $T : X \to Y$ an operator with dense range.*

(i) *If X is LWCG and T is a lattice homomorphism, then Y is LWCG.*
(ii) *If X is IWCG and T is an interval preserving lattice homomorphism, then Y is IWCG.*

Proof. (i) Since T is a lattice homomorphism, we have $L(T(A)) = \overline{T(L(A))}$ for any $A \subset X$. In particular, if $K \subset X$ is a weakly compact set such that $X = L(K)$, then $T(K)$ is a weakly compact set in Y such that $Y = \overline{T(X)} = L(T(K))$.

(ii) Since T is an interval preserving lattice homomorphism, $I(T(A)) = \overline{T(I(A))}$ for any $A \subset X$. Therefore, if $K \subset X$ is a weakly compact set such that $X = I(K)$, then $T(K)$ is a weakly compact set in Y satisfying $Y = \overline{T(X)} = I(T(K))$. ∎

Recall that a Banach lattice is said to have *weakly sequentially continuous lattice operations* if $(x_n \vee y_n)$ converges weakly to $x \vee y$ whenever (x_n) and (y_n) converge weakly to x and y, respectively. The basic examples of Banach lattices having weakly sequentially continuous lattice operations are AM-spaces (e.g. $C(K)$ spaces where K is a compact Hausdorff topological space), see e.g. [4, Theorem 4.31], and atomic order continuous Banach lattices (e.g. Banach spaces with unconditional basis), see e.g. [27, Proposition 2.5.23].

Theorem 2.2. *Let X be a Banach lattice having weakly sequentially continuous lattice operations. Then X is LWCG if and only if it is WCG.*

Proof. Let $K \subset X$ be a weakly compact set such that $L(K) = X$. By the Krein–Shmul'yan theorem (see e.g. [4, Theorem 3.42]), we can assume that K is absolutely convex. Hence $\operatorname{span}(K) = \bigcup_{n\in\mathbb{N}} nK$ is weakly σ-compact (that is, a countable union of weakly compact sets). Since X has weakly sequentially continuous lattice operations, for any weakly σ-compact set $A \subset X$ we see that both $A^{\vee}$ and $A^{\wedge}$ are weakly σ-compact. In particular, $\operatorname{span}(K)^{\vee\wedge}$

is weakly σ-compact, and since

$$X = L(K) \overset{(2.1)}{=} \overline{\mathrm{span}(K)^{\vee\wedge}},$$

we conclude that X is WCG. ∎

Corollary 2.3. *Let K be a compact Hausdorff topological space. Then:*

(i) *$C(K)$ is IWCG.*
(ii) *$C(K)$ is LWCG if and only if it is WCG.*

Proof. (i) follows from the fact that for the constant function 1_K we clearly have

$$C(K) = I(\{1_K\}).$$

(ii) is a direct consequence of Theorem 2.2 and the comments preceding it. ∎

Example 2.4. It is well known that $C(K)$ is WCG if and only if K is Eberlein compact [5] (cf. [17, Theorem 14.9]). If ω_1 denotes the first uncountable ordinal, then the ordinal segment $[0, \omega_1]$ with its usual topology is a compact space which is not Eberlein. Thus, $C[0, \omega_1]$ provides an example of an IWCG Banach lattice which is not LWCG. Another example of this situation is given by the space ℓ_∞ (see also Corollary 2.8 below).

In general, it is not true that the solid hull of a weakly relatively compact set is also weakly relatively compact (see e.g. [27, p. 108]). Banach lattices with this stability property are order continuous and were characterized in [10, Theorem 2.4]: these include atomic order continuous Banach lattices, as well as Banach lattices not containing c_0.

Theorem 2.5. *Let X be a Banach lattice with the property that the solid hull of any weakly relatively compact set is weakly relatively compact. Then X is BWCG if and only if it is WCG.*

Proof. Since X is order continuous, every ideal of X is a band (see e.g. [27, Corollary 2.4.4]) and so X is BWCG if and only if it is IWCG. Let $K \subset X$ be a weakly compact set such that $X = I(K)$. Then $\mathrm{sol}(K)$ is weakly relatively compact and

$$X = I(K) \overset{(2.2)}{=} \overline{\mathrm{span}}(\mathrm{sol}(K)),$$

hence X is WCG. ∎

It is clear that the discussion of this paper is only meaningful for non-separable Banach lattices. However, for Banach lattices with a separable predual we have some reformulations of the lattice versions of WCG: see Corollary 2.8 below. Recall first that a positive element u of a Banach lattice X is said to be:

- a *quasi-interior point* of X if for every $x \in X_+$ we have $\|x - x \wedge nu\| \to 0$ as $n \to \infty$, or equivalently $I(\{u\}) = X$ (cf. [4, Theorem 4.85]);
- a *weak order unit* of X if $\{u\}^\perp = \{0\}$, or equivalently $B(\{u\}) = X$.

In particular, every Banach lattice having a quasi-interior point (resp. weak order unit) is IWCG (resp. BWCG).

PROPOSITION 2.6. *Let X be a Banach lattice. Then X has a quasi-interior point* (*resp. weak order unit*) *if and only if $X = I(C)$* (*resp. $X = B(C)$*) *for some separable set $C \subset X$.*

Proof. It suffices to prove the "if" parts. We can assume that C is norm bounded. Let $(x_n)_{n\in\mathbb{N}}$ be a dense sequence in C and define

$$u := \sum_{n\in\mathbb{N}} \frac{|x_n|}{2^n} \in X_+.$$

Since $x_n \in I(\{u\}) \subset B(\{u\})$ for all $n \in \mathbb{N}$, we have

$$I(C) \subset I(\{u\}) \quad \text{and} \quad B(C) \subset B(\{u\}).$$

So, u is a quasi-interior point (resp. a weak order unit) of X whenever $X = I(C)$ (resp. $X = B(C)$). ■

The *density character* of a topological space T, denoted by $\operatorname{dens}(T)$, is the minimal cardinality of a dense subset of T. For an arbitrary Banach space X we have

$$\operatorname{dens}(X) \geq \operatorname{dens}(X^*, w^*)$$

(see e.g. [17, p. 576]), and equality holds whenever X is WCG (see e.g. [17, Theorem 13.3]). We next show that equality holds for any LWCG Banach lattice.

THEOREM 2.7. *Let X be an LWCG Banach lattice. Then*

$$\operatorname{dens}(X) = \operatorname{dens}(X^*, w^*).$$

Proof. It suffices to prove that $\operatorname{dens}(X) \leq \operatorname{dens}(X^*, w^*)$. Let $K \subset X$ be a weakly compact set such that

$$X = L(K) \overset{(2.1)}{=} \overline{\operatorname{span}(K)^{\vee\wedge}}.$$

Consider the WCG subspace $Y := \overline{\operatorname{span}}(K) \subset X$. According to the comments preceding the theorem, $\operatorname{dens}(Y) = \operatorname{dens}(Y^*, w^*)$. Since $Y^{\vee\wedge}$ is dense in X, we have $\operatorname{dens}(Y) = \operatorname{dens}(X)$. Moreover, since the restriction operator $X^* \to Y^*$ is w^*-w^*-continuous and onto, we have $\operatorname{dens}(Y^*, w^*) \leq \operatorname{dens}(X^*, w^*)$. It follows that $\operatorname{dens}(X) \leq \operatorname{dens}(X^*, w^*)$, as required. ■

COROLLARY 2.8. *Let X be a Banach lattice such that X^* is w^*-separable* (*e.g. $X = Y^*$ for a separable Banach lattice Y*).

(i) *X is LWCG if and only if X is separable.*
(ii) *X is IWCG if and only if X has a quasi-interior point.*
(iii) *X is BWCG if and only if X has a weak order unit.*

Proof. (i) is an immediate consequence of Theorem 2.7. Since any weakly compact subset of X is separable, (ii) and (iii) follow from Proposition 2.6. ∎

The following illustrates the difference between BWCG and IWCG.

PROPOSITION 2.9. *For $1 < p < \infty$ the Lorentz space $L_{p,\infty}[0,1]$ is BWCG but not IWCG.*

Proof. Recall that, for $1 < p < \infty$, the Lorentz space $L_{p,\infty}[0,1]$ consists of those (equivalence classes of) measurable functions $f : [0,1] \to \mathbb{R}$ for which

$$\|f\|_{p,\infty} := \sup_{t>0} t\lambda(\{x \in [0,1] : |f(x)| > t\})^{1/p} < \infty,$$

where λ denotes the Lebesgue measure on $[0,1]$. Although the expression $\|f\|_{p,\infty}$ just defines a lattice quasi-norm, it is actually equivalent to a lattice norm (cf. [7, p. 219, Lemma 4.5 and Theorem 4.6]).

It is clear that $L_{p,\infty}[0,1]$ is BWCG since $\chi_{[0,1]}$ is a weak order unit of it. On the other hand, it is well known that $L_{p,\infty}[0,1]$ is the dual of a separable Banach lattice, namely, the Lorentz space $L_{p',1}[0,1]$ with $1/p + 1/p' = 1$ (cf. [7, p. 220, Theorem 4.7]). Therefore, to prove that $L_{p,\infty}[0,1]$ is not IWCG it suffices to check that it has no quasi-interior point (Corollary 2.8). Although this is probably known to any expert in the field, we include a proof since we did not find a suitable reference.

Suppose $L_{p,\infty}[0,1]$ has a quasi-interior point, say v. Consider $f_0 \in L_{p,\infty}[0,1]$ defined by $f_0(x) := 1/x^{1/p}$ for $x \in [0,1]$. Observe that $\lambda(\{x \in [0,1] : f_0(x) > t\}) = 1/t^p$ for every $t > 0$ and so $\|f_0\|_{p,\infty} = 1$. Set

$$u := \frac{v + f_0}{\|v + f_0\|_{p,\infty}} \in L_{p,\infty}[0,1].$$

Clearly, u is also a quasi-interior point of $L_{p,\infty}[0,1]$. Note that for any $t > 0$,

$$\{x \in [0,1] : f_0(x) > t\|v + f_0\|_{p,\infty}\} \subset \{x \in [0,1] : u(x) > t\},$$

and so, bearing in mind that $\|u\|_{p,\infty} = 1$, we get

$$\frac{1}{(t\|v + f_0\|_{p,\infty})^p} \leq \lambda(\{x \in [0,1] : u(x) > t\}) \leq \frac{1}{t^p}.$$

Hence we can choose $t_0 > 0$ large enough such that

$$0 < \lambda(\{x \in [0,1] : u(x) > t_0\}) < 1.$$

Let $A_0 := \{x \in [0,1] : u(x) \leq t_0\}$, $A_1 := [0,1] \setminus A_0$ and $r_0 := \lambda(A_0) \in (0,1)$. There exists a measure-preserving transformation $\sigma : [0,1] \to [0,1]$

such that $\sigma(A_0) = [0, r_0]$ and $\sigma(A_1) = [r_0, 1]$ (see e.g. [7, p. 81, Proposition 7.4]). Define $f_\sigma := f \circ \sigma \in L_{p,\infty}[0,1]$. We claim that

$$\|f_\sigma - f_\sigma \wedge Nu\|_{p,\infty} = 1 \tag{2.4}$$

for every $N \in \mathbb{N}$. This would imply that u cannot be a quasi-interior point, a contradiction.

In order to prove (2.4), note first that since $\|f_\sigma\|_{p,\infty} = 1$ and $u > 0$, we have $\|f_\sigma - f_\sigma \wedge Nu\|_{p,\infty} \le 1$. For the converse inequality, fix $\varepsilon > 0$ and choose $t > 0$ large enough such that

$$\frac{1}{(t+Nt_0)^p} \le r_0 \quad \text{and} \quad \frac{t}{t+Nt_0} \ge 1-\varepsilon.$$

Define

$$B := \sigma^{-1}\left(\left[0, \frac{1}{(t+Nt_0)^p}\right)\right) \subset A_0$$

and note that for every $x \in B$ we have

$$f_\sigma(x) > t + Nt_0 \ge t + Nu(x),$$

hence $(f_\sigma \wedge Nu)(x) = Nu(x)$ and so $f_\sigma(x) - (f_\sigma \wedge Nu)(x) > t$. Thus

$$\begin{aligned}\|f_\sigma - f_\sigma \wedge Nu\|_{p,\infty} &\ge t\lambda(\{x \in [0,1] : f_\sigma(x) - (f_\sigma \wedge Nu)(x) > t\})^{1/p} \\ &\ge t\lambda(B)^{1/p} = \frac{t}{t+Nt_0} \ge 1-\varepsilon.\end{aligned}$$

As $\varepsilon > 0$ is arbitrary, (2.4) holds and the proof is complete. ∎

3. Order continuous Banach lattices. The next result provides an improvement of Theorem 2.5.

THEOREM 3.1. *Let X be an order continuous Banach lattice. Then X is BWCG if and only if it is WCG.*

For the proof we need two lemmas. Recall that a subset K of a Banach space is called *weakly precompact* (or conditionally weakly compact) if every sequence in K has a weakly Cauchy subsequence. Thanks to Rosenthal's ℓ_1-theorem (see e.g. [17, Theorem 5.37]), this is equivalent to saying that K is bounded and contains no sequence equivalent to the usual basis of ℓ_1.

LEMMA 3.2. *Let X be an order continuous Banach lattice, $K \subset X$ a weakly precompact set and $A \subset \mathrm{sol}(K)$ a set of pairwise disjoint vectors. Then* $\mathrm{sol}(A)$ *is weakly compact.*

Proof. Let $(y_n)_{n\in\mathbb{N}}$ be a sequence in $\mathrm{sol}(A) \subset \mathrm{sol}(K)$. By passing to a further subsequence, not relabeled, we can assume that one of the following cases holds.

CASE 1: *There is $x \in A$ such that $y_n \in [-|x|, |x|]$ for all $n \in \mathbb{N}$.* Since every order interval of an order continuous Banach lattice is weakly compact (see e.g. [27, Theorem 2.4.2]), $(y_n)_{n\in\mathbb{N}}$ admits a subsequence which weakly converges to some vector in $[-|x|,|x|] \subset \mathrm{sol}(A)$.

CASE 2: *There is a sequence $(x_n)_{n\in\mathbb{N}}$ of distinct elements of A such that $y_n \in [-|x_n|,|x_n|]$ for all $n \in \mathbb{N}$.* In particular, $(y_n)_{n\in\mathbb{N}}$ is a disjoint sequence. Since K is weakly precompact and $y_n \in \mathrm{sol}(K)$ for all $n \in \mathbb{N}$, the sequence $(y_n)_{n\in\mathbb{N}}$ weakly converges to $0 \in \mathrm{sol}(A)$ (see e.g. [27, Proposition 2.5.12(iii)]).

This proves that $\mathrm{sol}(A)$ is a weakly compact set. ∎

LEMMA 3.3. *Let X be an order continuous Banach lattice, $C \subset X$ a solid set and $A \subset C_+$ a maximal set of pairwise disjoint vectors. Then $C \subset I(A)$.*

Proof. We follow the ideas of [25, Proposition 1.a.9]. For each $x \in A$, let $P_x : X \to X$ be the band projection onto $B(\{x\})$, so that

$$P_x(z) = \bigvee_{n\in\mathbb{N}} (z \wedge nx) = \lim_{n\to\infty} \bigvee_{k=1}^{n} (z \wedge kx)$$

for all $z \in X_+$ (see e.g. [25, pp. 8–10 and Proposition 1.a.8]).

In order to see that $C \subset Y := I(A)$ it is enough to prove that $C_+ \subset Y$ (because C is solid). To this end, pick $z \in C_+$. For every $x \in A$ we have $P_x(z) \in Y$ (bear in mind that $\bigvee_{k=1}^{n}(z\wedge kx) \in n\,\mathrm{sol}(A) \subset Y$ for all $n \in \mathbb{N}$) and $0 \le P_x(z) \le z$. Moreover, the sum $\sum_{x\in A} P_x(z)$ unconditionally converges to some $y \in [0, z]$ (see the proof of [25, Proposition 1.a.9]). We claim that $z = y$. Indeed, if this were not the case, then $z - y > 0$ and, since $z - y \in C_+$ (remember that C is solid), by the maximality of A there would be at least one $x \in A$ such that $x \wedge (z - y) \neq 0$. However, this is impossible since

$$0 \le x \wedge (z - y) \le x \wedge (z - P_x(z)) = 0.$$

Here, the last equality follows from the fact that P_x is the band projection onto the band generated by x. Hence, $z = y \in Y$. ∎

Proof of Theorem 3.1. Suppose X is BWCG. Since X is order continuous, every ideal of X is a band (see e.g. [27, Corollary 2.4.4]) and so X is IWCG. Hence there is a weakly compact set $K \subset X$ such that

$$X = I(K) \overset{(2.2)}{=} \overline{\mathrm{span}}(\mathrm{sol}(K)).$$

Fix a maximal set $A \subset \mathrm{sol}(K)_+$ of pairwise disjoint vectors. By Lemma 3.3 (applied to $C := \mathrm{sol}(K)$), we have $\mathrm{sol}(K) \subset I(A) = \overline{\mathrm{span}}(\mathrm{sol}(A))$ and so $X = \overline{\mathrm{span}}(\mathrm{sol}(A))$. Since $\mathrm{sol}(A)$ is weakly compact (by Lemma 3.2), it follows that X is WCG. ∎

REMARK 3.4. The proof of Theorem 3.1 makes it clear that an order continuous Banach lattice X is WCG if and only if there is a *weakly precompact* set $K \subset X$ such that $X = B(K)$.

Following [23, p. 28], a Banach space X is called *weakly precompactly generated* (WPG) if there is a weakly precompact set $K \subset X$ such that $X = \overline{\operatorname{span}}(K)$.

Corollary 3.5. *Let X be an order continuous Banach lattice. Then X is WCG if and only if it is WPG.*

It is known that order continuous Banach lattices with order continuous dual are WCG (see [8, p. 194]). We next provide another proof of this fact. For geometrical properties of this class of Banach lattices, see [18].

Corollary 3.6. *Let X be a Banach lattice. If X and X^* are order continuous, then X is WCG.*

Proof. The assumption implies that B_X is weakly precompact (see e.g. [4, Theorem 4.25]). Hence X is WPG and Corollary 3.5 applies. ∎

Let us now turn to the larger class of Dedekind complete (and σ-complete) Banach lattices. Recall that a Banach lattice X is called *Dedekind complete* (respectively, *σ-Dedekind complete*) if every order bounded set (respectively, sequence) has a supremum in X. It is well known that every Banach lattice which is the dual of another Banach lattice is Dedekind complete.

Theorem 3.7. *Let X be a Banach lattice and $Z \subset X$ a Dedekind complete sublattice. If $I(Z)$ is LWCG, then Z is LWCG.*

Proof. Note that

$$Y := \{x \in X : \exists z \in Z \text{ with } |x| \leq z\}$$

is the smallest (not necessarily closed) ideal of X containing Z, so that $I(Z) = \overline{Y}$. By the Lipecki–Luxemburg–Schep theorem (see e.g. [4, Theorem 2.29]), the identity on Z can be extended to a lattice homomorphism $T_0 : Y \to Z$ (we use the Dedekind completeness of Z and the fact that Z is a majorizing sublattice of Y). By the density, T_0 admits a further extension to a lattice homomorphism $T : I(Z) \to Z$. Since T is surjective and $I(Z)$ is LWCG, Proposition 2.1 ensures that Z is LWCG. ∎

Corollary 3.8. *Let X be a Dedekind σ-complete Banach lattice. If every ideal of X is LWCG, then X is WCG.*

Proof. According to Theorem 3.1, it suffices to prove that X is order continuous. Suppose it is not. Since X is Dedekind σ-complete, X contains a sublattice Z which is lattice isomorphic to ℓ_∞ (see e.g. [4, Theorem 4.51]). In particular, Z is Dedekind complete and not LWCG. From Theorem 3.7 it follows that $I(Z)$ cannot be LWCG, a contradiction. ∎

These results motivate the question: *Can an LWCG Banach lattice contain a sublattice isomorphic to ℓ_∞?* If the answer were negative, then every Dedekind σ-complete LWCG Banach lattice would be WCG.

4. Applications of the Davis–Figiel–Johnson–Pełczyński factorization. The Davis–Figiel–Johnson–Pełczyński (DFJP) [13] factorization method is a keystone of Banach space theory. Given an absolutely convex bounded subset W of a Banach space X, the DFJP interpolation Banach space obtained from W is denoted by $\Delta(W,X)$ (cf. [4, Theorem 5.37]). As a set, $\Delta(W,X)$ is a linear subspace of X. The identity map $J:\Delta(W,X)\to X$ is an operator and $J(B_{\Delta(W,X)})\supset W$. The space $\Delta(W,X)$ is reflexive (resp. contains no isomorphic copy of ℓ_1) if and only if W is weakly relatively compact (resp. weakly precompact)—see e.g. [4, Theorem 5.37] (resp. [21, Theorem 5.3.6]).

Bearing in mind that the absolutely convex hull of any weakly precompact set in a Banach space is also weakly precompact (see e.g. [31, p. 377]), we deduce from the DFJP factorization method that *a Banach space X is WPG if and only if there exist a Banach space Y not containing ℓ_1 and an operator $T:Y\to X$ with dense range.* As an application we get the following result (cf. [33, Corollary 2.3.1]).

Proposition 4.1. *If X is a WPG Banach space, then X contains no subspace isomorphic to ℓ_∞.*

Proof. The property of being WPG is clearly inherited by *complemented* subspaces, and therefore it suffices to prove that ℓ_∞ is not WPG. Suppose that ℓ_∞ is WPG. Let Y be a Banach space not containing ℓ_1 and $T:Y\to\ell_\infty$ an operator with dense range. Then the adjoint $T^*:\ell_\infty^*\to Y^*$ is injective. In particular, (B_{Y^*},w^*) contains a homeomorphic copy of $\beta\mathbb{N}$. Now, a result by Talagrand [34] ensures that Y contains a subspace isomorphic to $\ell_1(\mathfrak{c})$, a contradiction. ∎

In order to apply Proposition 4.1 to Banach lattices, recall that the following statements are equivalent for a Banach lattice X (see e.g. [4, Theorem 4.69]):

(i) X^* is order continuous;
(ii) X^* contains no subspace isomorphic to c_0;
(iii) X^* contains no subspace isomorphic to ℓ_∞;
(iv) X contains no sublattice which is lattice isomorphic to ℓ_1.

Corollary 4.2. *Let X be a Banach lattice. Then X^* is WCG if and only if it is WPG.*

Proof. In view of the comments above and Proposition 4.1, if X^* is WPG, then X^* is order continuous and so the result follows from Corollary 3.5. ∎

The question of whether LWCG = WCG for arbitrary Banach lattices can be reduced to Banach lattices with order continuous dual, thanks to the following result.

THEOREM 4.3. *Let X be a Banach lattice. If X is LWCG* (*resp. IWCG*), *then there exist an LWCG* (*resp. IWCG*) *Banach lattice Y and a lattice homomorphism* (*resp. an interval preserving lattice homomorphism*) $J : Y \to X$ *such that:*

(i) Y^* *is order continuous;*
(ii) $X = \overline{J(Y)}$.

Proof. Let $K \subset X$ be a weakly compact set such that $X = L(K)$ (resp. $X = I(K)$) and let $W := \mathrm{co}(\mathrm{sol}(K))$ be its convex solid hull (which is absolutely convex and bounded). Then $\Psi := \Delta(W, X)$ is a Banach lattice, the identity operator $J : \Psi \to X$ is an interval preserving lattice homomorphism and $J(\Psi)$ is a (not necessarily closed) ideal of X (see e.g. [4, Theorem 5.41]).

Moreover, from the weak compactness of K it follows that Ψ^* is order continuous (see e.g. [4, Theorem 5.43]). Since J is a weak-to-weak homeomorphism when restricted to B_Ψ (see e.g. [4, p. 313, Exercise 11]), the set $K_0 =: J^{-1}(K)$ is weakly compact in Ψ. Then $Y := L(K_0)$ (resp. $Y := I(K_0)$) is an LWCG sublattice (resp. IWCG ideal) of Ψ. Since the property of having order continuous dual in inherited by sublattices (see the comments preceding Corollary 4.2), Y^* is order continuous. Finally, from the fact that J is an interval preserving lattice homomorphism it follows that $X = \overline{J(Y)}$ (see the proof of Proposition 2.1). ■

REMARK 4.4. The DFJP factorization and the result from [8] isolated in Corollary 3.6 provide an alternative proof of Theorem 3.1. Indeed, let Ψ and J be as in the proof of Theorem 4.3. If we assume further that X is order continuous, then so is Ψ (see e.g. [4, Theorem 5.41]). From the order continuity of Ψ and Ψ^* we infer that Ψ is WCG (see [8, p. 194]). Finally, the equality $X = \overline{J(\Psi)}$ ensures that X is WCG.

An order continuous Banach lattice cannot contain a subspace isomorphic to $C[0,1]$ (see e.g. [27, Corollary 5.1.12]). In Theorem 4.5 below we give an improvement of Theorem 4.3 within the class of Banach lattices not containing $C[0,1]$.

Recall first that a Banach space X is said to be *Asplund* if every separable subspace of X has separable dual or, equivalently, X^* has the Radon–Nikodým property [14, p. 198]. A Banach space X is said to be *Asplund generated* if there exist an Asplund Banach space Y and an operator $T : Y \to X$ with dense range. By the DFJP factorization, every WCG Banach space is Asplund generated.

THEOREM 4.5. *Let X be a Banach lattice not containing subspaces isomorphic to $C[0,1]$. If X is LWCG (resp. IWCG), then there exist an LWCG (resp. IWCG) Banach lattice Y and a lattice homomorphism (resp. an interval preserving lattice homomorphism) $J : Y \to X$ such that:*

(i) *Y is Asplund;*
(ii) *$X = \overline{J(Y)}$.*

In particular, X is Asplund generated.

Proof. Fix a weakly compact set $K \subset X$ such that $X = L(K)$ (resp. $X = I(K)$) and consider the set $W := \mathrm{co}(\mathrm{sol}(K))$. Since X contains no isomorphic copy of $C[0,1]$, the convex solid hull of any weakly precompact subset of X is weakly precompact (see [19, Corollary II.4]), and so is W. Let Ψ, J and Y be as in the proof of Theorem 4.3. Since W is weakly precompact, Ψ contains no isomorphic copy of ℓ_1. Hence the Banach lattice Ψ is Asplund (see [14, p. 95] and [20, Theorem 7]) and the same holds for its subspace Y. ∎

In view of the previous theorem, if the equality LWCG = WCG were true for Asplund Banach lattices, then it would also be true for all Banach lattices not containing isomorphic copies of $C[0,1]$.

5. Miscellaneous properties. Rosenthal [30] gave the first instance of a WCG Banach space with a non-WCG subspace. Likewise, LWCG/IWCG/BWCG are not hereditary properties:

EXAMPLE 5.1. Let X be the Banach space constructed in [6, Section 2], which is WCG and has an uncountable unconditional basis $\mathcal{E} = \{e_{(\sigma,m)} : \sigma \in \mathbb{N}^{\mathbb{N}}, m \in \mathbb{N}\}$. In particular, X is an LWCG Banach lattice. Define $x_\sigma := \sum_{m\in\mathbb{N}} 2^{-m/2} \cdot e_{(\sigma,m)}$ for any $\sigma \in \mathbb{N}^{\mathbb{N}}$. In [6, Theorem 2.6] it was proved that $\mathcal{B} = \{x_\sigma : \sigma \in \mathbb{N}^{\mathbb{N}}\}$ is a block basis of $\mathcal{E}$ such that $Y := \overline{\mathrm{span}}(\mathcal{B})$ is not WCG. Note that Y is a sublattice of X (because the coordinates of the x_σ's with respect to $\mathcal{E}$ are positive) which is not LWCG (by Theorem 2.2). In fact, Y cannot be BWCG (by Theorem 3.1).

It is well known that being WCG is not a three-space property, that is, there exist non-WCG Banach spaces X having a WCG subspace $Y \subset X$ such that X/Y is WCG. For complete information on the three-space problem for WCG Banach spaces, see [9, Section 4.10] and the references therein. If X is a Banach lattice and $Y \subset X$ is an *ideal*, then X/Y is a Banach lattice and the quotient operator $X \to X/Y$ is a lattice homomorphism (see e.g. [25, p. 3]). Some counterexamples to the three-space problem for WCG spaces fit into the Banach lattice setting, like the following construction which goes back to [24] (cf. [9, Section 4.10]).

Example 5.2. Let $2^{<\omega}$ be the dyadic tree (finite sequences of 0s and 1s), 2^ω the set of its branches (countable infinite sequences of 0s and 1s) and K the one-point compactification of $2^{<\omega} \cup 2^\omega$ equipped with the topology defined by: (i) all points from $2^{<\omega}$ are isolated; (ii) any $x = (x_k)_{k<\omega} \in 2^\omega$ has a neighborhood basis made of the sets $\{x\} \cup \{(x_k)_{k<m} : m > n\}$ for $n < \omega$. Then $L := 2^\omega \cup \{\infty\}$ is a closed subset of K and so $Y := \{f \in C(K) : f|_L \equiv 0\}$ is an ideal of $C(K)$. It is not difficult to check that Y is isomorphic to c_0, and that the quotient space $C(K)/Y$ is isomorphic to $C(L)$, which in turn is isomorphic to $c_0(\mathfrak{c})$. Hence Y and $C(K)/Y$ are WCG. On the other hand, $C(K)$ is not WCG, because it is not separable and every weakly compact subset of $C(K)$ is separable (since K is separable). For the same reason, $C(K)$ is not LWCG (cf. Corollary 2.3).

However, a Banach space X is WCG if there exists a *reflexive* subspace $Y \subset X$ such that X/Y is WCG (see [24]; cf. [9, Proposition 4.10.d]). Theorem 5.4 below collects some positive results on the three-space problem for WCG and LWCG Banach lattices. We first need a result on WPG Banach spaces which might be of independent interest.

Proposition 5.3. *Let X be a Banach space and $Y \subset X$ a subspace containing no isomorphic copy of ℓ_1. If X/Y is WPG, then X is WPG.*

Proof. Let $q : X \to X/Y$ be the quotient operator and $K \subset X/Y$ a weakly precompact set such that $X/Y = \overline{\operatorname{span}}(K)$. Since q is open and K is bounded, there is a bounded set $G \subset X$ such that $q(G) = K$. Since Y contains no subspace isomorphic to ℓ_1 and K is weakly precompact, G is weakly precompact as well (see e.g. [9, 2.4.a]). Then $G_1 := G \cup B_Y \subset X$ is weakly precompact. We claim that $Z := \overline{\operatorname{span}}(G_1)$ equals X. Suppose that $X \neq Z$. By the Hahn–Banach separation theorem, there is $x^* \in X^* \setminus \{0\}$ such that $x^*(x) = 0$ for all $x \in Z$. In particular, x^* vanishes on Y and so it factorizes as $x^* = \phi \circ q$ for some $\phi \in (X/Y)^*$. Note that ϕ vanishes on $q(Z)$. But $X/Y = \overline{q(Z)}$ (because $q(Z)$ is a linear subspace of X/Y containing $q(G) = K$), hence $\phi = 0$ and so $x^* = 0$, a contradiction. This shows that $X = Z$, as claimed. Therefore X is WPG. ∎

Theorem 5.4. *Let X be a Banach lattice and $Y \subset X$ an ideal.*

(i) *If X is LWCG, then X/Y is LWCG.*
(ii) *If Y is reflexive and X/Y is LWCG, then X is LWCG.*
(iii) *If X is order continuous, Y contains no isomorphic copy of ℓ_1 and X/Y is WCG, then X is WCG.*

Proof. (i) This follows at once from Proposition 2.1 because the quotient operator $q : X \to X/Y$ is a surjective lattice homomorphism.

(ii) Let $K \subset X/Y$ be a weakly compact set such that $X/Y = L(K)$. Bearing in mind that q is open and that K is bounded and weakly closed, we can find a bounded and weakly closed set $K_0 \subset X$ such that $q(K_0) = K$. Since Y is reflexive and K is weakly compact, K_0 is weakly compact as well (see e.g. [9, 2.4.b]). Then the set $K_1 := K_0 \cup B_Y \subset X$ is weakly compact. We claim that $X = L(K_1)$. Indeed, define $Z := L(K_1)$. Since q is a lattice homomorphism and Z is a sublattice, $q(Z)$ is a (not necessarily closed) sublattice of X/Y. Recalling $q(Z) \supset q(K_0) = K$, we conclude that $q(Z)$ is dense in X/Y. As in the proof of Proposition 5.3, it follows that $X = Z = L(K_1)$ and so X is LWCG.

(iii) This follows from Corollary 3.5 and Proposition 5.3. ∎

In connection with part (iii) of the previous theorem, note that if X is an order continuous Banach lattice and $Y \subset X$ is an ideal, then the quotient space X/Y is order continuous (see e.g. [4, p. 205, Exercise 13]).

A Banach space X is said to be *weakly Lindelöf determined* (WLD) if (B_{X^*}, w^*) is a *Corson compact*, i.e. it is homeomorphic to a set $S \subset [-1,1]^\Gamma$ for some non-empty set Γ such that $\{\gamma \in \Gamma : s(\gamma) \neq 0\}$ is countable for all $s \in S$. Every WCG space is WLD, but the converse does not hold in general. For a complete account on this class of Banach spaces, we refer the reader to [16, 17, 22].

Theorem 5.5. *Let X be a Banach lattice such that the order intervals of X and X^* are separable and w^*-separable, respectively. If there is a WLD subspace $Y \subset X$ such that $X = I(Y)$, then X is WLD.*

Before proving Theorem 5.5, let us mention that a Banach space is WCG if (and only if) it is Asplund generated and WLD (see e.g. [16, Theorem 8.3.4])). This fact together with Theorems 4.5 and 5.5 yields:

Corollary 5.6. *Let X be a Banach lattice such that the order intervals of X and X^* are separable and w^*-separable, respectively. If X is IWCG and contains no subspace isomorphic to $C[0,1]$, then X is WCG.*

In order to prove Theorem 5.5 we first need two lemmas. Given a Banach space X, we say that a set $C \subset X$ *countably supports* X^* if for every $x^* \in X^*$ the set $\{x \in C : x^*(x) \neq 0\}$ is countable.

Lemma 5.7. *Let X be a Banach lattice such that the order intervals of X^* are w^*-separable. If $C \subset X$ countably supports X^*, then for every $x^* \in X^*$ the set $\{x \in C : x^*(|x|) \neq 0\}$ is countable.*

Proof. Since every element of X^* is the difference of two positive functionals, it suffices to check that the set $\{x \in C : x^*(|x|) \neq 0\}$ is countable for every $x^* \in X^*_+$. Fix a w^*-dense sequence $(x^*_n)_{n\in\mathbb{N}}$ in $[-x^*, x^*]$. Then for every $x \in X$ we have

$$x^*(|x|) = \sup\big\{y^*(x) : y^* \in [-x^*, x^*]\big\} = \sup_{n\in\mathbb{N}} x^*_n(x)$$

(see e.g. [4, Theorem 1.23]). Therefore

$$\{x \in C : x^*(|x|) \neq 0\} \subset \bigcup_{n\in\mathbb{N}} \{x \in C : x_n^*(x) \neq 0\},$$

and so $\{x \in C : x^*(|x|) \neq 0\}$ is countable, as required. ∎

Lemma 5.8. *Let X be a Banach lattice such that the order intervals of X and X^* are separable and w^*-separable, respectively. If $C \subset X$ countably supports X^*, then there is a set $P \subset \mathrm{sol}(C)$ such that $\mathrm{sol}(C) \subset \overline{P}$ and P countably supports X^*.*

Proof. For every $x \in C$ we take a countable dense set $A_x \subset [-|x|, |x|]$. Therefore, $P := \bigcup_{x\in C} A_x$ is dense in $\mathrm{sol}(C) = \bigcup_{x\in C}[-|x|, |x|]$. Fix $x^* \in X_+^*$. By Lemma 5.7, the set $C_0 := \{x \in C : x^*(|x|) \neq 0\}$ is countable. Since $x^*(y) = 0$ for every $y \in [-|x|, |x|]$ whenever $x \in C \setminus C_0$, we have

$$\{y \in P : x^*(y) \neq 0\} \subset \bigcup_{x\in C_0} A_x,$$

and so $\{y \in P : x^*(y) \neq 0\}$ is countable. As $x^* \in X_+^*$ is arbitrary and every element of X^* is the difference of two positive functionals, P countably supports X^*. ∎

Proof of Theorem 5.5. Any WLD Banach space admits an M-basis (see e.g. [22, Corollary 5.42]). Let $\{(y_i, y_i^*) : i \in I\} \subset Y \times Y^*$ be an M-basis of Y, that is, a biorthogonal system such that $Y = \overline{\mathrm{span}}(\{y_i : i \in I\})$ and $\{y_i^* : i \in I\}$ separates the points of Y. We can assume without loss of generality that $\|y_i\| \leq 1$ for all $i \in I$. The fact that Y is WLD ensures that $C := \{y_i : i \in I\}$ countably supports X^* (see e.g. [22, Theorem 5.37]). Let $P \subset \mathrm{sol}(C)$ be such that $\mathrm{sol}(C) \subset \overline{P}$ and P countably supports X^* (Lemma 5.8). Since $X = I(Y)$, we have

$$X = I(C) \overset{(2.2)}{=} \overline{\mathrm{span}}(\mathrm{sol}(C)) = \overline{\mathrm{span}}(P).$$

It is now clear that the mapping

$$\phi : B_{X^*} \to [-1, 1]^P, \qquad \phi(x^*) := (x^*(x))_{x\in P},$$

is a w^*-pointwise homeomorphic embedding witnessing that (B_{X^*}, w^*) is a Corson compact. ∎

Besides the separable case, the following Banach lattices have the property that the order intervals of their dual are w^*-separable:

(i) WLD Banach spaces with unconditional basis, like $c_0(\Gamma)$ and $\ell_p(\Gamma)$ for any $1 < p < \infty$ and any non-empty set Γ. In this case, the order intervals of the dual have the stronger property of being w^*-metrizable.

(ii) $C(K)$, whenever K is a compact space with the property that $L^1(\mu)$ is separable for every regular Borel probability μ on K. This class of compact spaces includes all compacta which are Eberlein, Radon–Nikodým, Rosenthal or linearly ordered, among others (see [15, 26, 29]). In this case, the order intervals of the dual are norm separable.

On the other hand, it is not difficult to check that $L^1(\{0,1\}^{\omega_1})$ is a Banach lattice for which the conclusion of Lemma 5.7 fails.

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