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A normal generating set for the Torelli group of a non-orientable closed surface

by

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Abstract. For a closed surface S, its Torelli group $\mathcal{I}(S)$ is the subgroup of the mapping class group of S consisting of elements acting trivially on $H_1(S;\mathbb{Z})$. When S is orientable, a generating set for $\mathcal{I}(S)$ is known (see Powell (1978)). We give a normal generating set of $\mathcal{I}(N_g)$ for $g \geq 4$, where N_g is a genus-g non-orientable closed surface.

1. Introduction. For a closed connected *non-orientable* surface S, the *mapping class group* $\mathcal{M}(S)$ of S is defined to be the group of isotopy classes of all diffeomorphisms of S. For a closed connected *orientable* surface S, the *mapping class group* $\mathcal{M}(S)$ of S is defined to be the group of isotopy classes of all *orientation-preserving* diffeomorphisms of S. In this paper, for $x, y \in \mathcal{M}(S)$ the composition yx means that we first apply x and then y. The *Torelli group* $\mathcal{I}(S)$ of S is the subgroup of $\mathcal{M}(S)$ consisting of elements acting trivially on $H_1(S;\mathbb{Z})$. Let Σ_g be a genus-g orientable closed surface. Powell [13] showed that $\mathcal{I}(\Sigma_g)$ is generated by mapping classes of two types. In [14], Putman proved Powell's result more conceptually. In addition, Johnson [7] showed that $\mathcal{I}(\Sigma_g)$ is finitely generated for $g \geq 3$. In this paper, we consider the case where S is a non-orientable closed surface.

Let N_g denote a genus-g non-orientable closed surface, that is, N_g is a connected sum of g real projective planes. According to another classification, N_g is a connected sum of a genus-h orientable closed surface with $g-2h$ real projective planes for $0 \leq h < g/2$. In this paper, we regard N_g as a surface which is obtained by attaching $g-2h$ Möbius bands to a genus-h compact orientable surface with $g-2h$ boundary components for $0 \leq h < g/2$ (see Figure 1). For $R = \mathbb{Z}$ and $\mathbb{Z}/2\mathbb{Z}$, let $\cdot : H_1(N_g;R) \times H_1(N_g;R) \to \mathbb{Z}/2\mathbb{Z}$ be the mod 2 intersection form, and let $\mathrm{Aut}(H_1(N_g;R),\cdot)$ be the group

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of automorphisms of $H_1(N_g;R)$ preserving the mod 2 intersection form. McCarthy–Pinkall [11] and Gadgil–Pancholi [5] proved that the natural homomorphism $\rho : \mathcal{M}(N_g) \to \mathrm{Aut}(H_1(N_g;\mathbb{Z}),\cdot)$ is surjective.

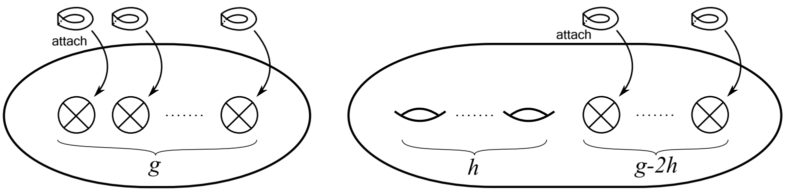
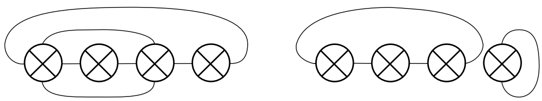


Fig. 1. A genus-g non-orientable closed surface N_g

Lickorish [9] showed that $\mathcal{M}(N_g)$ is generated by *Dehn twists* and *crosscap slides*. In addition, Lickorish [10] showed that the subgroup of $\mathcal{M}(N_g)$ generated by Dehn twists is an index 2 subgroup of $\mathcal{M}(N_g)$. Hence $\mathcal{M}(N_g)$ is not generated by Dehn twists. On the other hand, since a crosscap slide acts on $H_1(N_g;\mathbb{Z}/2\mathbb{Z})$ trivially, $\mathcal{M}(N_g)$ is not generated by crosscap slides. Chillingworth [2] found a finite generating set for $\mathcal{M}(N_g)$. Presentations for $\mathcal{M}(N_1)$ and $\mathcal{M}(N_2)$ are known classically. A finite presentation for $\mathcal{M}(N_3)$ was obtained by Birman–Chillingworth [1]. A finite presentation for $\mathcal{M}(N_4)$ was obtained by Szepietowski [16]. Finally, a finite presentation for $\mathcal{M}(N_g)$ was obtained by Paris–Szepietowski [12] and Stukow [15] for $g \geq 4$.



(a) A-circles (b) M-circles

Fig. 2

For a simple closed curve c on N_g, c is called an *A-circle* (resp. an *M-circle*) if its regular neighborhood is an annulus (resp. a Möbius band) (see Figure 2). Let a and m be an A-circle and an M-circle on N_g respectively. Suppose that a and m intersect transversely at only one point. We define a crosscap slide $Y_{m,a}$ as follows. Let K be a regular neighborhood of $a \cup m$ in N_g, and let M be a regular neighborhood of m in the interior of K. Note

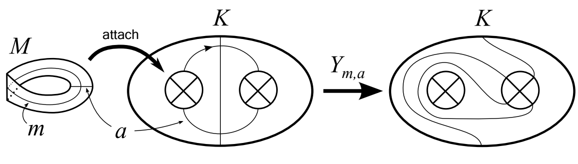


Fig. 3. A crosscap slide $Y_{m,a}$

that K is homeomorphic to the Klein bottle with a boundary. $Y_{m,a}$ is a diffeomorphism of N_g which pushes M once along a and back onto itself, keeping the boundary of K fixed (see Figure 3). For an A-circle c on N_g, we denote by t_c a Dehn twist about c, and the direction of the twist is indicated by a small arrow beside c as in Figure 4.

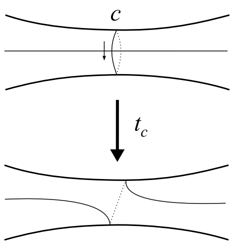
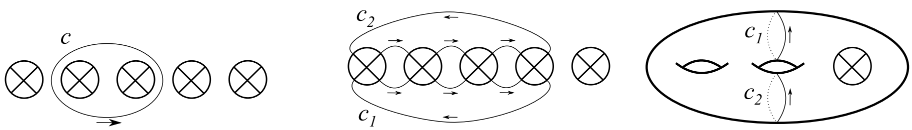


Fig. 4. A Dehn twist t_c about c

Let c be an A-circle on N_g such that $N_g \setminus c$ is not connected. We call t_c a *bounding simple closed curve map*, for short a *BSCC map* (see Figure 5(a)). Let c_1 and c_2 be A-circles on N_g such that $N_g \setminus c_i$ is connected, $N_g \setminus (c_1 \cup c_2)$ is disconnected and one of its connected components is orientable, which ensures that c_1 and c_2 are homologous. We call $t_{c_1} t_{c_2}^{-1}$ a *bounding pair map*, for short a *BP map* (see Figure 5(b)). In Section 2, we will see that BSCC maps and BP maps are in $\mathcal{I}(N_g)$.



(a) A BSCC map t_c (b) A BP map $t_{c_1} t_{c_2}^{-1}$

Fig. 5

For $h, b \geq 1$, let N_h^b be a non-orientable surface of genus h with b boundary components, and let Σ_h^b be an orientable surface of genus h with b boundary components. Our main result is the following.

THEOREM 1.1. *For $g \geq 5$, $\mathcal{I}(N_g)$ is generated by the following elements:*

- *BSCC maps t_c such that one of the connected components of $N_g \setminus c$ is homeomorphic to N_2^1, and the other is non-orientable.*
- *BP maps $t_{c_1} t_{c_2}^{-1}$ such that one of the connected components of $N_g \setminus (c_1 \cup c_2)$ is homeomorphic to Σ_1^2, and the other is non-orientable.*

$\mathcal{I}(N_4)$ is generated by the following elements:

- *BSCC maps t_c such that one of the connected components of $N_4 \setminus c$ is homeomorphic to N_2^1, and the other is non-orientable.*

- *BSCC maps t_c such that one of the connected components of $N_4 \setminus c$ is homeomorphic to N_2^1, and the other is orientable.*
- *BP maps $t_{c_1} t_{c_2}^{-1}$ such that one of the connected components of $N_4 \setminus (c_1 \cup c_2)$ is homeomorphic to Σ_1^2, and the other is an annulus as shown in Figure* 6*.*

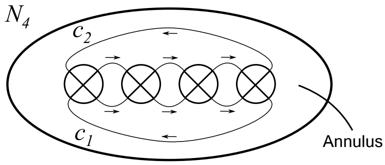


Fig. 6

In this theorem, the generating sets are infinite. We do not know whether or not $\mathcal{I}(N_g)$ can be finitely generated, and generated by only BSCC maps or BP maps.

Here is an outline of the proof of Theorem 1.1. Let $\Gamma_2(N_g)$ be the subgroup of $\mathcal{M}(N_g)$ consisting of the elements acting trivially on $H_1(N_g; \mathbb{Z}/2\mathbb{Z})$. We call $\Gamma_2(N_g)$ the *level* 2 *mapping class group* of N_g. Note that $\mathcal{I}(N_g) \subset \Gamma_2(N_g)$. Let $\Phi_g : \mathrm{Aut}(H_1(N_g; \mathbb{Z}), \cdot) \to \mathrm{Aut}(H_1(N_g; \mathbb{Z}/2\mathbb{Z}), \cdot)$ be the natural epimorphism. Consider the natural homomorphism $\rho' : \Gamma_2(N_g) \to \ker \Phi_g$. Then $\mathcal{I}(N_g) = \ker \rho'$. Let $\Gamma_2(n) = \ker(\mathrm{GL}(n; \mathbb{Z}) \to \mathrm{GL}(n; \mathbb{Z}/2\mathbb{Z}))$. We call $\Gamma_2(n)$ the *level* 2 *principal congruence subgroup* of $\mathrm{GL}(n; \mathbb{Z})$. McCarthy–Pinkall [11] showed that $\ker \Phi_g$ is isomorphic to $\Gamma_2(g-1)$. On the other hand, Szepietowski [18] gave a finite generating set for $\Gamma_2(N_g)$, and then the first author and Sato [6] gave a minimal generating set for $\Gamma_2(N_g)$. Fullarton [4] and the second author [8] gave a finite presentation for $\Gamma_2(n)$ independently. A presentation for $\Gamma_2(n)$ was also known to Margalit and Putman. Thus, we have a normal generating set for $\mathcal{I}(N_g)$ in $\Gamma_2(N_g)$.

Here is an outline of this paper. In Section 2, we give the basics on the Torelli group of a non-orientable surface. In Section 3, we describe a finite generating set for $\Gamma_2(N_g)$, a finite presentation for $\Gamma_2(n)$ and an isomorphism from $\ker \Phi_g$ to $\Gamma_2(g-1)$. In Section 4, we obtain a normal generating set for $\mathcal{I}(N_g)$. In Section 5, we show that each normal generator of $\mathcal{I}(N_g)$ obtained in Section 4 is a product of BSCC maps and BP maps.

2. Basics on the Torelli group of a non-orientable surface. There are BSCC maps of two types. A BSCC map t_c is called a *BSCC map of type* $(1, h)$ if each connected component of $N_g \setminus c$ is non-orientable and one of these components is homeomorphic to N_h^1 for $1 \leq h \leq g/2$ (see Figure 5(a)). A BSCC map t_c is called a *BSCC map of type* $(2, h)$ if one component of $N_g \setminus c$ is homeomorphic to Σ_h^1 for $1 \leq h < g/2$, and the other is non-orientable (see

Figure 7). Note that a BSCC map t_c is trivial if c bounds a Möbius band (see [3, Theorem 3.4]).

There are BP maps of two types. A BP map $t_{c_1}t_{c_2}^{-1}$ is called a *BP map of type* $(1,h)$ if one component of $N_g \setminus (c_1 \cup c_2)$ is homeomorphic to Σ_h^2 for $1 \leq h < g/2-1$, and the other is non-orientable (see Figure 5(b)). A BP map $t_{c_1}t_{c_2}^{-1}$ is called a *BP map of type* $(2,h)$ if each component of $N_g \setminus (c_1 \cup c_2)$ is orientable and one component of $N_g \setminus (c_1 \cup c_2)$ is homeomorphic to Σ_h^2 for $1 \leq h \leq g/2-1$ (see Figure 7). Note that a BP map of type $(2,h)$ appears only if g is even.

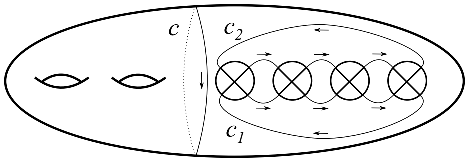
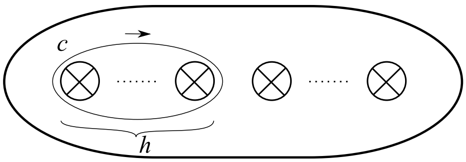


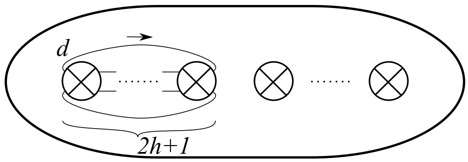
Fig. 7. A BSCC map t_c of type $(2,2)$ and a BP map $t_{c_1}t_{c_2}^{-1}$ of type $(2,1)$

First, we show the following.

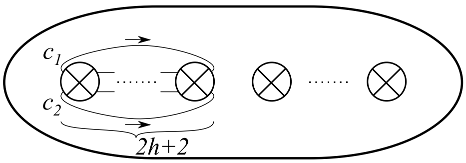
Remark 2.1. *All BSCC maps and BP maps are in $\mathcal{I}(N_g)$.*



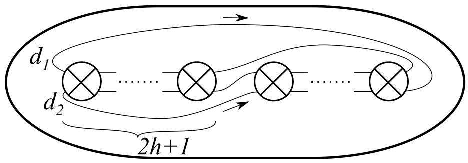
(a) A BSCC map t_c of type $(1,h)$



(b) A BSCC map t_d of type $(2,h)$



(c) A BP map $t_{c_1}t_{c_2}^{-1}$ of type $(1,h)$



(d) A BP map $t_{d_1}t_{d_2}^{-1}$ of type $(2,h)$

Fig. 8

Proof. Let c, d, c_1, c_2, d_1 and d_2 be simple closed curves on N_g as shown in Figure 8. Note that t_c is a BSCC map of type $(1,h)$, t_d is a BSCC map of type $(2,h)$, $t_{c_1}t_{c_2}^{-1}$ is a BP map of type $(1,h)$ and $t_{d_1}t_{d_2}^{-1}$ is a BP map of type $(2,h)$. In $\mathcal{M}(N_g)$, any BSCC map of type $(1,h)$ (resp. type $(2,h)$) is conjugate to $t_c^{\pm1}$ (resp. $t_d^{\pm1}$), and any BP map of type $(1,h)$ (resp. type $(2,h)$) is conjugate to $(t_{c_1}t_{c_2}^{-1})^{\pm1}$ (resp. $(t_{d_1}t_{d_2}^{-1})^{\pm1}$). Hence it suffices to show that t_c, t_d, $t_{c_1}t_{c_2}^{-1}$ and $t_{d_1}t_{d_2}^{-1}$ are in $\mathcal{I}(N_g)$.

For $1 \leq i \leq g$, let α_i be a simple closed curve on N_g as shown in Figure 9, and let $c_i = [\alpha_i] \in H_1(N_g;\mathbb{Z})$. By a natural handle decomposition whose cores of 1-handles are α_i, $H_1(N_g;\mathbb{Z})$ is generated by the c_i as a $\mathbb{Z}$-module (see Figure 13). We can see that t_c, t_d, $t_{c_1}t_{c_2}^{-1}$ and $t_{d_1}t_{d_2}^{-1}$ act trivially on each c_i. ∎

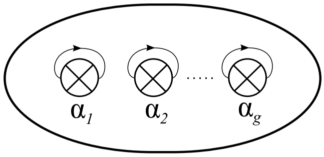


Fig. 9. The loops $\alpha_1, \alpha_2, \ldots, \alpha_g$

Next we prove the following.

LEMMA 2.2. *Let* $g \geq 5$.

(1) (a) *Any BSCC map of type* $(2, g/2-1)$ *with* g *even is a product of BP maps of type* $(1,1)$.
(b) *Any other BSCC map is a product of BSCC maps of type* $(1,2)$.

(2) *Any BP map of type* $(1,h)$ *is a product of BP maps of type* $(1,1)$. *Any BP map of type* $(2,h)$ *is a product of BP maps of type* $(2,1)$.

Proof. In the proof, we use the ideas of Johnson [7].

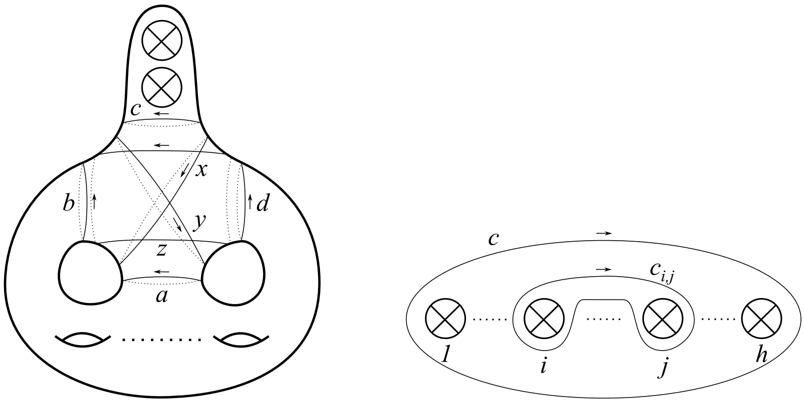
(1)(a) We first show that a BSCC map of type $(2, g/2-1)$ is a product of BP maps. Let t_c be a BSCC map of type $(2, g/2-1)$. Then the curve c is as shown in Figure 10(a). Let x, y, z, a, b and d be simple closed curves as shown in Figure 10(a). By the lantern relation, we have the relation $t_dt_ct_bt_a = t_zt_yt_x$. Since a, b, c and d do not intersect other loops, we have $t_c = (t_zt_a^{-1})(t_yt_d^{-1})(t_xt_b^{-1})$. Note that $t_xt_b^{-1}$, $t_yt_d^{-1}$ and $t_zt_a^{-1}$ are BP maps. Hence a BSCC map of type $(2, g/2-1)$ is a product of BP maps. As we will show in (2), these BP maps are products of BP maps of type $(1,1)$. Hence we obtain the claim.

(1)(b) Let t_c be a BSCC map of type $(1,h)$ or $(2,(g-h)/2)$ for $h \geq 3$. Then c is as shown in Figure 10(b). Let $c_{i,j}$ be a simple closed curve for $1 \leq i < j \leq h$ as shown in Figure 10(b). We have

$$t_c = \prod_{1 \leq i \leq h-1} (t_{c_{i,i+1}}t_{c_{i,i+2}}\cdots t_{c_{i,h-1}}t_{c_{i,h}}). \tag{I}$$

Equality (I) will be proved in Appendix A. Since each $t_{c_{i,j}}$ is a BSCC map of type $(1,1)$, we obtain the claim.

(2) For $h \geq 1$, let $d_0, d_1, \ldots, d_h$ be simple closed curves as shown in Figure 11. Suppose that d_0 and d_h are not separating curves. Note that $t_{d_0}t_{d_h}^{-1}$ is a BP map of type $(1,h)$ or $(2,h)$. Then we have the equality



(a) Loops a, b, c, d and x, y, z (b) Loops c and $c_{i,j}$

Fig. 10

$t_{d_0}t_{d_h}^{-1} = (t_{d_0}t_{d_1}^{-1})(t_{d_1}t_{d_2}^{-1}) \cdots (t_{d_{h-1}}t_{d_h}^{-1})$. Since each $t_{d_i}t_{d_{i+1}}^{-1}$ is a BP map of type $(1,1)$ or $(2,1)$, we obtain the claim. ∎

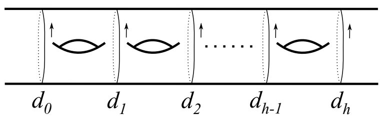


Fig. 11. Loops $d_0, d_1, \ldots, d_h$

3. Preliminaries

3.1. Generators for $\Gamma_2(N_g)$. For $I = \{i_1, \ldots, i_k\} \subset \{1, \ldots, g\}$, we define an oriented simple closed curve α_I as in Figure 12. For short, we denote $\alpha_{\{i\}}$ by α_i. We define $Y_{i_1;i_2,\ldots,i_k} = Y_{\alpha_{i_1},\alpha_I}$, $T_{i_1,\ldots,i_k} = t_{\alpha_I}$ if k is even.

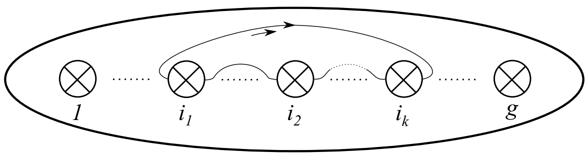


Fig. 12. The curve α_I for $I = \{i_1, \ldots, i_k\}$

Szepietowski [18] gave a generating set for $\Gamma_2(N_g)$:

Theorem 3.1 ([18]). *For $g \geq 4$, $\Gamma_2(N_g)$ is generated by the following elements:*

(1) $Y_{i;j}$ *for* $1 \leq i \leq g-1$, $1 \leq j \leq g$ *and* $i \neq j$,
(2) $T^2_{i,j,k,l}$ *for* $1 \leq i < j < k < l \leq g$.

In addition, the first author and Sato [6] gave a minimal generating set for $\Gamma_2(N_g)$:

THEOREM 3.2 ([6]). *For $g \geq 4$, $\Gamma_2(N_g)$ is generated by the following elements:*

(1) $Y_{i;j}$ *for* $1 \leq i \leq g-1$, $1 \leq j \leq g$ *and* $i \neq j$,
(2) $T^2_{1,j,k,l}$ *for* $1 < j < k < l \leq g$.

3.2. $\ker \Phi_g$ **and** $\Gamma_2(g-1)$**.** McCarthy–Pinkall claimed that $\ker \Phi_g$ is isomorphic to $\Gamma_2(g-1)$ in their unpublished preprint [11]. In this subsection, we recall their result and proof.

Let $c_i = [\alpha_i] \in H_1(N_g;\mathbb{Z})$ for $1 \leq i \leq g$, and let $c = c_1 + \cdots + c_g$ ($= [\alpha_{\{1,\dots,g\}}]$). Then, by a natural handle decomposition as in Figure 13, as a $\mathbb{Z}$-module, $H_1(N_g;\mathbb{Z})$ has a presentation

$$H_1(N_g;\mathbb{Z}) = \langle c_1, \dots, c_g \mid 2c = 0 \rangle.$$

As a $\mathbb{Z}$-module we have

$$H_1(N_g;\mathbb{Z})/\langle c \rangle = \langle c_1, \dots, c_g \mid c = 0 \rangle = \langle c_1, \dots, c_{g-1} \rangle \cong \mathbb{Z}^{g-1},$$

where the last isomorphism sends c_i to the ith canonical normal vector e_i for $1 \leq i \leq g-1$. For $x \in H_1(N_g;\mathbb{Z})$, we denote by $\overline{x}$ the image of x under the projection $H_1(N_g;\mathbb{Z}) \to \mathbb{Z}^{g-1}$. Explicitly, for $x = \sum_{j=1}^{g} x_j c_j \in H_1(N_g;\mathbb{Z})$, we have $\overline{x} = \sum_{j=1}^{g-1}(x_j - x_g)e_j \in \mathbb{Z}^{g-1}$. We regard $\mathrm{Aut}(H_1(N_g;\mathbb{Z})/\langle c \rangle)$ as $\mathrm{GL}(g-1;\mathbb{Z})$.

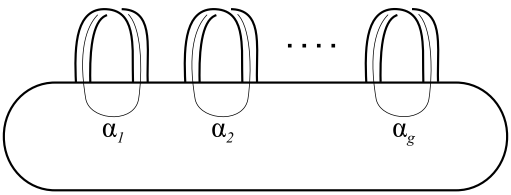


Fig. 13. A handle decomposition of N_g whose cores of the 1-handles are α_i.

For $L \in \mathrm{Aut}(H_1(N_g;\mathbb{Z}))$, since $2L(c) = L(2c) = 0$ and c is the only non-trivial element of $H_1(N_g;\mathbb{Z})$ satisfying $2c = 0$, we have $L(c) = c$. Hence $L \in \mathrm{Aut}(H_1(N_g;\mathbb{Z}))$ induces $\overline{L} \in \mathrm{GL}(g-1;\mathbb{Z})$. More precisely, $\overline{L}$ is defined as $\overline{L}(e_i) = \overline{L(c_i)}$. By this correspondence, we obtain the following.

PROPOSITION 3.3. *The correspondence $f : \ker \Phi_g \to \Gamma_2(g-1)$ defined by $f(L) = \overline{L}$ is an isomorphism.*

Proof. We first show that $f(\ker \Phi_g)$ is in $\Gamma_2(g-1)$ and f is a homomorphism. By the definition of Φ_g, we have $L(c_i) \equiv c_i \bmod 2$ for $1 \leq i \leq g$. Hence $\overline{L}(e_i) = \overline{L(c_i)} \equiv \overline{c_i} = e_i \bmod 2$ for $1 \leq i \leq g-1$. Therefore $f(L) \in \Gamma_2(g-1)$.

In addition, for $L, L' \in \ker \varPhi_g$,

$$\overline{LL'}(e_i) = \overline{LL'(c_i)} = \overline{L(L'(c_i))} = \overline{L}(\overline{L'(c_i)}) = \overline{L}(\overline{L'}(e_i)).$$

Thus, f is a homomorphism.

We next show the injectivity of f. For $L \in \ker \varPhi_g$, suppose that $\overline{L}$ is the identity. Then either $L(c_i) = c_i$ or $L(c_i) = c_i + c$. By the definition of $\varPhi_g$, we have $L(c_i) \equiv c_i \bmod 2$ for $1 \leq i \leq g$. Hence L is the identity. Therefore f is injective.

Finally, we show the surjectivity of f. For any $A = (a_{ij}) \in \varGamma_2(g-1)$, we define $\widetilde{A} \in \ker \varPhi_g$ to be

$$\widetilde{A}(c_i) = \begin{cases} \displaystyle\sum_{j=1}^{g-1} a_{ji} c_j & (i \neq g), \\ \displaystyle\sum_{j=1}^{g-1} \Big(1 - \sum_{k=1}^{g-1} a_{jk}\Big) c_j + c_g & (i = g). \end{cases}$$

Then $\widetilde{A}(c) = c$, and since a_{ii} is odd and a_{ij} is even for $i \neq j$, we obtain $\widetilde{A}(c_i) \equiv c_i \bmod 2$. Hence $\widetilde{A} \in \ker \varPhi_g$. Moreover $f(\widetilde{A}) = A$. Therefore f is surjective.

Thus f is an isomorphism. ∎

3.3. A presentation for $\varGamma_2(g-1)$. For $1 \leq i, j \leq n$ with $i \neq j$, let E_{ij} denote the matrix whose (i,j) entry is 2, the diagonal entries are 1 and the others are 0, and let F_i denote the matrix whose (i,i) entry is -1, the other diagonal entries are 1 and all others are 0. It is known that $\varGamma_2(n)$ is generated by E_{ij} and F_i (see [11]). A finite presentation for $\varGamma_2(n)$ was independently given by Fullarton [4] and the second author [8], and was also known to Margalit and Putman.

Theorem 3.4 (cf. [4], [8]). *For $n \geq 1$, $\varGamma_2(n)$ has a finite presentation with generators E_{ij} and F_i, for $1 \leq i, j \leq n$, and with relators:*

(1) F_i^2,
(2) $(E_{ij}F_i)^2$, $(E_{ij}F_j)^2$, $(F_iF_j)^2$ *(when $n \geq 2$)*,
(3) **(a)** $[E_{ij}, E_{ik}]$, $[E_{ij}, E_{kj}]$, $[E_{ij}, F_k]$, $[E_{ij}, E_{ki}]E_{kj}^2$ *(when $n \geq 3$)*,
(b) $(E_{ji}E_{ij}^{-1}E_{kj}^{-1}E_{jk}E_{ik}E_{ki}^{-1})^2$ *for $i < j < k$ (when $n \geq 3$)*,
(4) $[E_{ij}, E_{kl}]$ *(when $n \geq 4$)*,

where $[X, Y] = X^{-1}Y^{-1}XY$ and $1 \leq i, j, k, l \leq n$ are distinct.

For $1 \leq i \leq g-1$ and $1 \leq j \leq g$ with $i \neq j$, let $Y_{ij} = f((Y_{i;j})_*)$, where $\varphi_* \in \operatorname{Aut}(H_1(N_g; \mathbb{Z}), \cdot)$ is induced by $\varphi \in \mathcal{M}(N_g)$. Then $Y_{ij} = E_{ij}F_i$ if $j < g$ and $Y_{ig} = F_i$. We now prove

Proposition 3.5. *For $g-1 \geq 1$, $\Gamma_2(g-1)$ has a finite presentation with generators Y_{ij} for $1 \leq i \leq g-1$ and $1 \leq j \leq g$ with $i \neq j$, and with relators:*

(1) Y_{ij}^2 *for* $1 \leq i \leq g-1$ *and* $1 \leq j \leq g$,
(2) $[Y_{ik}, Y_{jk}]$ *for* $1 \leq i, j \leq g-1$ *and* $1 \leq k \leq g$,
(3) $[Y_{ij}, Y_{ik}Y_{jk}]$ *for* $1 \leq i, j \leq g-1$ *and* $1 \leq k \leq g$,
(4) $[Y_{ij}, Y_{kl}]$ *for* $1 \leq i, k \leq g-1$ *and* $1 \leq j, l \leq g$,
(5) $(Y_{ij}Y_{ik}Y_{il})^2$ *for* $1 \leq i \leq g-1$ *and* $1 \leq j, k, l \leq g$,
(6) $(Y_{ji}Y_{ij}Y_{kj}Y_{jk}Y_{ik}Y_{ki})^2$ *for* $1 \leq i, j, k \leq g-1$,

where $[X, Y] = X^{-1}Y^{-1}XY$ *and* i, j, k, l *are distinct.*

Proof. First, we show that the relators of $\Gamma_2(g-1)$ in Proposition 3.5 are obtained from those in Theorem 3.4. Since $F_i^2 = 1$, we may identify F_i^{-1} with F_i in $\Gamma_2(g-1)$.

(1) We have

$$Y_{ij}^2 = \begin{cases} (E_{ij}F_i)^2 & (j < g), \\ F_i^2 & (j = g). \end{cases}$$

Hence we obtain the relator Y_{ij}^2 in $\Gamma_2(g-1)$.

(2) For $k < g$, we see that

$$Y_{ik}Y_{jk} = \underset{\mathbf{(3)(a)}}{\underline{E_{ik}F_iE_{jk}}}F_j = E_{jk}\,\underset{\mathbf{(2),(3)(a)}}{\underline{E_{ik}F_iF_j}} = E_{jk}F_jE_{ik}F_i = Y_{jk}Y_{ik}.$$

In addition,

$$Y_{ig}Y_{jg} = F_iF_j \underset{\mathbf{(2)}}{=} F_jF_i = Y_{jg}Y_{ig}.$$

Hence we obtain the relator $[Y_{ik}, Y_{jk}]$ in $\Gamma_2(g-1)$.

(3) For $k < g$, we see that

$$\begin{aligned} Y_{ij}Y_{ik}Y_{jk} &= E_{ij}\underset{\mathbf{(2)}}{\underline{F_iE_{ik}}}\,\underset{\mathbf{(3)(a),(2)}}{\underline{F_iE_{jk}F_j}} = \underset{\mathbf{(3)(a),(2)}}{\underline{E_{ij}E_{ik}^{-1}F_i}}E_{jk}F_jF_i = E_{ik}^{-1}F_i\underset{\mathbf{(3)(a)}}{\underline{E_{ij}^{-1}E_{jk}F_jF_i}} \\ &= E_{ik}^{-1}\underset{\mathbf{(2)}}{\underline{F_iE_{ik}^{-2}}}E_{jk}\underset{\mathbf{(2)}}{\underline{E_{ij}^{-1}F_j}}F_i = E_{ik}F_iE_{jk}F_jE_{ij}F_i = Y_{ik}Y_{jk}Y_{ij}. \end{aligned}$$

In addition,

$$Y_{ij}Y_{ig}Y_{jg} = \underset{\mathbf{(2)}}{\underline{E_{ij}F_i}}\,\underset{\mathbf{(2)}}{\underline{F_iF_j}} = F_i\underset{\mathbf{(2)}}{\underline{E_{ij}^{-1}F_jF_i}} = F_iF_jE_{ij}F_i = Y_{ig}Y_{jg}Y_{ij}.$$

Hence we obtain the relator $[Y_{ij}, Y_{ik}Y_{jk}]$ in $\Gamma_2(g-1)$.

(4) For $j, l < g$, we see that

$$Y_{ij}Y_{kl} = E_{ij}\,\underset{\mathbf{(3)(a),(2)}}{\underline{F_iE_{kl}F_k}} = \underset{\mathbf{(4),(3)(a)}}{\underline{E_{ij}E_{kl}F_k}}F_i = E_{kl}F_kE_{ij}F_i = Y_{kl}Y_{ij}.$$

In addition,

$$Y_{ij}Y_{kg} = E_{ij}F_iF_k \underset{\mathbf{(2),(3)(a)}}{=} F_kE_{ij}F_i = Y_{kg}Y_{ij}.$$

Hence we obtain the relator $[Y_{ij}, Y_{kl}]$ in $\Gamma_2(g-1)$.

(5) For the relator $(Y_{ij}Y_{ik}Y_{il})^2$, since $Y_{im} = Y_{im}^{-1}$, applying conjugations and taking their inverses, it suffices to consider the case $j < k < l$. For $l < g$, we find that

$$\begin{aligned} Y_{ij}Y_{ik}Y_{il} &= E_{ij}\underset{\mathbf{(2)}}{\underline{F_iE_{ik}}}\,\underset{\mathbf{(2)}}{\underline{F_iE_{il}}}F_i = \underset{\mathbf{(3)(a),(2)}}{\underline{E_{ij}E_{ik}^{-1}F_iE_{il}^{-1}}}F_iF_i = E_{ik}^{-1}F_i\underset{\mathbf{(3)(a),(2)}}{\underline{E_{ij}^{-1}E_{il}^{-1}F_i}}F_i \\ &= E_{ik}^{-1}\underset{\mathbf{(2)}}{\underline{F_iE_{il}^{-1}}}F_iE_{ij}F_i = \underset{\mathbf{(3)(a),(2)}}{\underline{E_{ik}^{-1}E_{il}F_i}}F_iE_{ij}F_i = E_{il}F_iE_{ik}F_iE_{ij}F_i \\ &= Y_{il}Y_{ik}Y_{ij} = Y_{il}^{-1}Y_{ik}^{-1}Y_{ij}^{-1}. \end{aligned}$$

In addition,

$$Y_{ij}Y_{ik}Y_{ig} = \underset{\mathbf{(2),(3)(a)}}{\underline{E_{ij}F_iE_{ik}F_i}}F_i = F_iE_{ik}F_iE_{ij}F_i = Y_{ig}Y_{ik}Y_{ij} = Y_{ig}^{-1}Y_{ik}^{-1}Y_{ij}^{-1}.$$

Hence we obtain the relator $(Y_{ij}Y_{ik}Y_{il})^2$ in $\Gamma_2(g-1)$.

(6) We see that

$$\begin{aligned} Y_{ji}Y_{ij}\cdot Y_{kj}Y_{jk}\cdot Y_{ik}Y_{ki} &= (E_{ji}F_jE_{ij}F_i)(E_{kj}F_kE_{jk}F_j)(E_{ik}\underline{F_i}E_{ki}F_k) \\ &\underset{\mathbf{(2)}}{=} (E_{ji}F_jE_{ij}F_i)(E_{kj}F_kE_{jk}\underline{F_j})(E_{ik}E_{ki}^{-1})F_iF_k \\ &\underset{\mathbf{(3)(a)}}{=} (E_{ji}F_jE_{ij}F_i)(E_{kj}\underline{F_k}E_{jk})(E_{ik}E_{ki}^{-1})F_jF_iF_k \\ &\underset{\mathbf{(2)}}{=} (E_{ji}F_jE_{ij}\underline{F_i})(E_{kj}E_{jk}^{-1})(E_{ik}^{-1}E_{ki})F_kF_jF_iF_k \\ &\underset{\mathbf{(3)(a),(2)}}{=} (E_{ji}\underline{F_j}E_{ij})(E_{kj}E_{jk}^{-1})(E_{ik}E_{ki}^{-1})F_iF_kF_jF_iF_k \\ &\underset{\mathbf{(2),(3)(a)}}{=} (E_{ji}E_{ij}^{-1})(E_{kj}^{-1}E_{jk})(E_{ik}E_{ki}^{-1})F_jF_iF_kF_jF_iF_k \\ &\underset{\mathbf{(1),(2)}}{=} (E_{ji}E_{ij}^{-1})(E_{kj}^{-1}E_{jk})(E_{ik}E_{ki}^{-1}). \end{aligned}$$

By **(3)(b)** of Theorem 3.4, we obtain the relator $(Y_{ji}Y_{ij}\cdot Y_{kj}Y_{jk}\cdot Y_{ik}Y_{ki})^2$ in $\Gamma_2(g-1)$.

Next, we show that the relators of $\Gamma_2(g-1)$ in Theorem 3.4 are obtained from those in Proposition 3.5. Since $Y_{ij}^2 = 1$, we may identify Y_{ij}^{-1} with Y_{ij} in $\Gamma_2(g-1)$. Note that $E_{ij} = Y_{ij}Y_{ig}$ and $F_i = Y_{ig}$.

(1) Since $F_i^2 = Y_{ig}^2$, we have the relator F_i^2 in $\Gamma_2(g-1)$.

(2) Since $E_{ij}F_i = Y_{ij}$, we have the relator $(E_{ij}E_i)^2$ in $\Gamma_2(g-1)$. We see that (in the underlinings, we refer to items of Proposition 3.5)

$$(E_{ij}F_j)^2 = \underset{(1)}{\underline{Y_{ij}}}\ \underset{(1),(2)}{\underline{Y_{ig}Y_{jg}}} \cdot Y_{ij}Y_{ig}Y_{jg} = Y_{ij}^{-1}(Y_{ig}Y_{jg})^{-1} \cdot Y_{ij}Y_{ig}Y_{jg} = [Y_{ij}, Y_{ig}Y_{jg}].$$

Hence we obtain the relator $(E_{ij}F_j)^2$ in $\Gamma_2(g-1)$. We find that

$$(F_iF_j)^2 = Y_{ig}\underset{(2)}{\underline{Y_{jg}Y_{ig}}}Y_{jg} = Y_{ig}^2Y_{jg}^2.$$

Hence we obtain the relator $(F_iF_j)^2$ in $\Gamma_2(g-1)$.

(3)(a) We see that

$$E_{ij}E_{ik} = \underset{(5)}{\underline{Y_{ij}Y_{ig}Y_{ik}}}Y_{ig} = Y_{ik}Y_{ig}Y_{ij}Y_{ig} = E_{ik}E_{ij}.$$

Hence we obtain the relator $[E_{ij}, E_{ik}]$ in $\Gamma_2(g-1)$. We find that

$$E_{ij}E_{kj} = Y_{ij}\underset{(4),(2)}{\underline{Y_{ig}Y_{kj}Y_{kg}}} = \underset{(2),(4)}{\underline{Y_{ij}Y_{kj}Y_{kg}}}Y_{ig} = Y_{kj}Y_{kg}Y_{ij}Y_{ig} = E_{kj}E_{ij}.$$

Hence we obtain the relator $[E_{ij}, E_{kj}]$ in $\Gamma_2(g-1)$. We see that

$$E_{ij}F_k = Y_{ij}Y_{ig}Y_{kg} \underset{(2),(4)}{=} Y_{kg}Y_{ij}Y_{ig} = F_kE_{ij}.$$

Hence we obtain the relator $[E_{ij}, F_k]$ in $\Gamma_2(g-1)$. We see that

$$\begin{aligned} E_{ij}E_{ki}E_{kj}^2 &= Y_{ij}Y_{ig}\underset{(5)}{\underline{Y_{ki}Y_{kg}Y_{kj}}}Y_{kg}Y_{kj}Y_{kg} = Y_{ij}\underset{(4),(2)}{\underline{Y_{ig}Y_{kj}Y_{kg}}}Y_{ki}Y_{kg}Y_{kj}Y_{kg} \\ &= \underset{(2)}{\underline{Y_{ij}Y_{kj}}}\ \underset{(3)}{\underline{Y_{kg}Y_{ig}Y_{ki}}}Y_{kg}Y_{kj}Y_{kg} = \underset{(3)}{\underline{Y_{kj}Y_{ij}Y_{ki}}}Y_{kg}\underset{(2),(4)}{\underline{Y_{ig}Y_{kg}Y_{kj}Y_{kg}}} \\ &= Y_{ki}Y_{kj}Y_{ij}\underset{(1)}{\underline{Y_{kg}Y_{kg}}}Y_{kj}Y_{kg}Y_{ig} = Y_{ki}Y_{kj}\underset{(2),(4)}{\underline{Y_{ij}Y_{kj}Y_{kg}}}Y_{ig} \\ &= Y_{ki}\underset{(1)}{\underline{Y_{kj}Y_{kj}}}Y_{kg}Y_{ij}Y_{ig} = Y_{ki}Y_{kg}Y_{ij}Y_{ig} = E_{ki}E_{ij}. \end{aligned}$$

Hence we obtain the relator $[E_{ij}, E_{ki}]E_{kj}^2$ in $\Gamma_2(g-1)$.

(3)(b) Since we have already obtained relators **(1)**, **(2)** and **(3)(a)** of Theorem 3.4, using these relators we deduce that

$$\begin{aligned} (E_{ji}E_{ij}^{-1})(E_{kj}^{-1}E_{jk})(E_{ik}E_{ki}^{-1}) &= (E_{ji}E_{ij}^{-1})(E_{kj}^{-1}E_{jk})(E_{ik}E_{ki}^{-1})F_jF_iF_kF_jF_iF_k \\ &= (E_{ji}F_jE_{ij})(E_{kj}E_{jk}^{-1})(E_{ik}E_{ki}^{-1})F_iF_kF_jF_iF_k \\ &= (E_{ji}F_jE_{ij}F_i)(E_{kj}E_{jk}^{-1})(E_{ik}^{-1}E_{ki})F_kF_jF_iF_k \\ &= (E_{ji}F_jE_{ij}F_i)(E_{kj}F_kE_{jk})(E_{ik}E_{ki}^{-1})F_jF_iF_k \\ &= (E_{ji}F_jE_{ij}F_i)(E_{kj}F_kE_{jk}F_j)(E_{ik}E_{ki}^{-1})F_iF_k \\ &= (E_{ji}F_jE_{ij}F_i)(E_{kj}F_kE_{jk}F_j)(E_{ik}F_iE_{ki}F_k) \\ &= Y_{ji}Y_{ij} \cdot Y_{kj}Y_{jk} \cdot Y_{ik}Y_{ki}. \end{aligned}$$

By Proposition 3.5(6), we obtain the relator $(E_{ji}E_{ij}^{-1}E_{kj}^{-1}E_{jk}E_{ik}E_{ki}^{-1})^2$ in $\Gamma_2(g-1)$.

(4) We see that

$$E_{ij}E_{kl} = Y_{ij}\underbrace{Y_{ig}Y_{kl}Y_{kg}}_{(2),(4)} = \underbrace{Y_{ij}Y_{kl}Y_{kg}}_{(4)}Y_{ig} = Y_{kl}Y_{kg}Y_{ij}Y_{ig} = E_{kl}E_{ij}.$$

Hence we obtain the relator $[E_{ij}, E_{kl}]$ in $\Gamma_2(g-1)$.

This completes the proof. ■

4. A normal generating set for $\mathcal{I}(N_g)$. Let $f : \ker \Phi_g \to \Gamma_2(g-1)$ be the isomorphism introduced in Subsection 3.2. In order to obtain a presentation for $\Gamma_2(g-1)$ whose generators are $Y_{ij} := f((Y_{i;j})_*)$, $T_{1jkl}^2 := f((T_{1,j,k,l}^2)_*)$, we need to express T_{1jkl}^2 as a product of Y_{ij}'s.

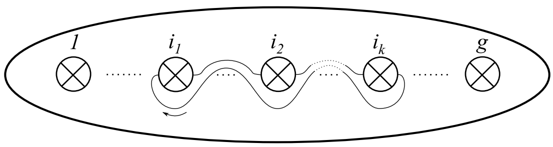


Fig. 14. The curve α'_I for $I = \{i_1, \dots, i_k\}$

For $I = \{i_1, \dots, i_k\} \subset \{1, \dots, g\}$, we define a simple closed curve α'_I as shown in Figure 14 and $T'_{i,j,k,l} = t_{\alpha'_{\{i,j,k,l\}}}$. Note that $T_{i,j,k,l}T'^{-1}_{i,j,k,l}$ is a BP map. In addition, for $1 \le m \le g$ with $m \ne i, j, k, l$, there exist A-circles $\beta_{m,i}$, $\beta_{m,j}$, $\beta_{m,k}$ and $\beta_{m,l}$ intersecting α_m in only one point such that

$$T_{i,j,k,l}^{-1}T'^{-1}_{i,j,k,l} = \prod_{m \ne i,j,k,l} Y_{\alpha_m,\beta_{m,l}}Y_{\alpha_m,\beta_{m,k}}Y_{\alpha_m,\beta_{m,j}}Y_{\alpha_m,\beta_{m,i}}.$$

For example, when (i,j,k,l) is $(1,2,3,4)$, for $m \ge 5$ and $t = 1,2,3,4$ we have

$$Y_{\alpha_m,\beta_{m,t}} = \begin{cases} Y_{m;t}^{-1} & (t = 1,2,3), \\ Y_{m;m-1}^{-2} \cdots Y_{m;6}^{-2}Y_{m;5}^{-2}Y_{m;4}^{-1} & (t = 4). \end{cases}$$

Therefore,

$$\begin{aligned} T_{1jkl}^{-2} &= f((T_{1,j,k,l}^{-1}T'^{-1}_{1,j,k,l})_*) \\ &= f\Big(\Big(\prod_{m \ne 1,j,k,l} Y_{\alpha_m,\beta_{m,l}}Y_{\alpha_m,\beta_{m,k}}Y_{\alpha_m,\beta_{m,j}}Y_{\alpha_m,\beta_{m,1}}\Big)_*\Big). \end{aligned}$$

Note that any crosscap slide $Y_{\alpha_m,\beta}$ is a product of some $Y_{i;j}$ for $1 \le i \le g-1$

and $1 \le j \le g$ with $i \neq j$. For example, we have

$$\text{(II)} \quad Y_{g;i} = (Y_{1;2}^2 \cdots Y_{1;g-1}^2 Y_{1;i}^{-1} Y_{1;g}) \cdots (Y_{i-1;i}^2 \cdots Y_{i-1;g-1}^2 Y_{i-1;i}^{-1} Y_{i-1;g}) \cdot (Y_{i+1;i+2}^2 \cdots Y_{i+1;g-1}^2 Y_{i+1;i}^{-1} Y_{i+1;g} Y_{i+1;i}^2) \cdots (Y_{g-2;g-1}^2 Y_{g-2;i}^{-1} Y_{g-2;g} Y_{g-2;i}^2) \cdot (Y_{g-1;i}^{-1} Y_{g-1;g} Y_{g-1;i}^2) Y_{i;g}.$$

Equality (II) will be proved in Appendix A. Thus T_{1jkl}^2 can be expressed as a product of Y_{ij}'s.

4.1. A normal generating set for $\mathcal{I}(N_g)$ in $\Gamma_2(N_g)$. Let G be a group. For $x_1, \ldots, x_n \in G$, we say that N is *normally generated* by $x_1, \ldots, x_n$ in G if N is a minimal normal subgroup of G which contains $x_1, \ldots, x_n$.

In this subsection, we prove the following.

PROPOSITION 4.1. *For $g \ge 4$, $\mathcal{I}(N_g)$ is normally generated by the following elements in $\Gamma_2(N_g)$:*

(1) $Y_{i;j}^2$ *for* $1 \le i \le g-1$ *and* $1 \le j \le g$,
(2) $[Y_{i;k}, Y_{j;k}]$ *for* $1 \le i, j \le g-1$ *and* $1 \le k \le g$,
(3) $[Y_{i;j}, Y_{i;k} Y_{j;k}]$ *for* $1 \le i \le g-1$ *and* $1 \le j, k \le g$,
(4) $[Y_{i;j}, Y_{k;l}]$ *for* $1 \le i, k \le g-1$ *and* $1 \le j, l \le g$,
(5) $(Y_{i;j} Y_{i;k} Y_{i;l})^2$ *for* $1 \le i \le g-1$ *and* $1 \le j, k, l \le g$,
(6) $(Y_{j;i} Y_{i;j} Y_{k;j} Y_{j;k} Y_{i;k} Y_{k;i})^2$ *for* $1 \le i, j, k \le g-1$,
(7) $T_{1,j,k,l}^2 (\prod_{m \neq 1,j,k,l} Y_{\alpha_m, \beta_{m,l}} Y_{\alpha_m, \beta_{m,k}} Y_{\alpha_m, \beta_{m,j}} Y_{\alpha_m, \beta_{m,1}})$ *for* $1 < j < k < l \le g$,

where i, j, k, l are distinct.

Let $\Gamma = \langle g_1, \ldots, g_n \mid r_1, \ldots, r_k \rangle$ be a finitely presented group, and let G be a group generated by $\tilde{g}_1, \ldots, \tilde{g}_n$. For $r_i = g_{i(1)}^{\varepsilon_1} \cdots g_{i(m)}^{\varepsilon_m}$, define $\tilde{r}_i = \tilde{g}_{i(1)}^{\varepsilon_1} \cdots \tilde{g}_{i(m)}^{\varepsilon_m}$, where $\varepsilon_j = \pm 1$. Let $\widetilde{N}$ be a normal subgroup of G normally generated by $\tilde{r}_1, \ldots, \tilde{r}_k$. We first prove the following.

LEMMA 4.2. *Let $\mu : G \to \Gamma$ be a homomorphism sending $\tilde{g}_i$ to g_i. Then $\widetilde{N} = \ker \mu$.*

Proof. Let $F = \langle g_1, \ldots, g_n \rangle$, and let N be a normal subgroup of F normally generated by $r_1, \ldots, r_k$. Let $\pi : F \to \Gamma$ be a natural epimorphism, and let $\nu : F \to G$ be a homomorphism sending g_i to $\tilde{g}_i$. Since μ is surjective, we have $\pi = \mu\nu$. Then we obtain the following short exact sequences and commutative diagram:

$$\begin{array}{ccccccccc}
1 & \longrightarrow & N & \longrightarrow & F & \xrightarrow{\pi} & \Gamma & \longrightarrow & 1 \\
 & & \downarrow{\scriptstyle \nu|_N} & & \downarrow{\scriptstyle \nu} & & \| & & \\
1 & \longrightarrow & \ker \mu & \longrightarrow & G & \xrightarrow{\mu} & \Gamma & \longrightarrow & 1
\end{array}$$

For any $\widetilde{R} \in \ker \mu$, there exists $R \in F$ such that $\nu(R) = \widetilde{R}$. Then $\pi(R) = \mu\nu(R) = \mu(\widetilde{R}) = 1$. Hence $R \in \ker \pi = N$. Therefore $\nu|_N : N \to \ker \mu$ is surjective. Since $\nu(N) = \widetilde{N}$, we conclude that $\widetilde{N} = \ker \mu$. ∎

Proof of Proposition 4.1. Let F be a free group of rank $\binom{g}{3}+\binom{g}{2}$ generated by Y_{ij} for $1 \leq i \leq g-1$ and $1 \leq j \leq g$ with $i \neq j$, and by T_{1jkl}^2 for $1 < j < k < l \leq g$, where $T_{1jkl} = f((T_{1,j,k,l})_*)$, and let N be the normal subgroup of F normally generated by:

(1) Y_{ij}^2 for $1 \leq i \leq g-1$ and $1 \leq j \leq g$,
(2) $[Y_{ik}, Y_{jk}]$ for $1 \leq i, j \leq g-1$ and $1 \leq k \leq g$,
(3) $[Y_{ij}, Y_{ik}Y_{jk}]$ for $1 \leq i \leq g-1$ and $1 \leq j, k \leq g$,
(4) $[Y_{ij}, Y_{kl}]$ for $1 \leq i, k \leq g-1$ and $1 \leq j, l \leq g$,
(5) $(Y_{ij}Y_{ik}Y_{il})^2$ for $1 \leq i \leq g-1$ and $1 \leq j, k, l \leq g$,
(6) $(Y_{ji}Y_{ij}Y_{kj}Y_{jk}Y_{ik}Y_{ki})^2$ for $1 \leq i, j, k \leq g-1$,
(7) $T_{1jkl}^2 f((\prod_{m\neq 1,j,k,l} Y_{\alpha_m,\beta_{m,l}} Y_{\alpha_m,\beta_{m,k}} Y_{\alpha_m,\beta_{m,j}} Y_{\alpha_m,\beta_{m,1}})_*)$ for $1 < j < k < l \leq g$,

where i, j, k, l are distinct. By Proposition 3.5 and the fact that

$$f\Big(\Big(\prod_{m\neq 1,j,k,l} Y_{\alpha_m,\beta_{m,l}} Y_{\alpha_m,\beta_{m,k}} Y_{\alpha_m,\beta_{m,j}} Y_{\alpha_m,\beta_{m,1}}\Big)_*\Big) = T_{1jkl}^{-2},$$

$\Gamma_2(g-1)$ is the quotient of F by N. Let $\widetilde{N}$ be the normal subgroup of $\Gamma_2(N_g)$ normally generated by:

(1) $Y_{i;j}^2$ for $1 \leq i \leq g-1$ and $1 \leq j \leq g$,
(2) $[Y_{i;k}, Y_{j;k}]$ for $1 \leq i, j \leq g-1$ and $1 \leq k \leq g$,
(3) $[Y_{i;j}, Y_{i;k}Y_{j;k}]$ for $1 \leq i \leq g-1$ and $1 \leq j, k \leq g$,
(4) $[Y_{i;j}, Y_{k;l}]$ for $1 \leq i, k \leq g-1$ and $1 \leq j, l \leq g$,
(5) $(Y_{i;j}Y_{i;k}Y_{i;l})^2$ for $1 \leq i \leq g-1$ and $1 \leq j, k, l \leq g$,
(6) $(Y_{j;i}Y_{i;j}Y_{k;j}Y_{j;k}Y_{i;k}Y_{k;i})^2$ for $1 \leq i, j, k \leq g-1$,
(7) $T_{1,j,k,l}^2(\prod_{m\neq 1,j,k,l} Y_{\alpha_m,\beta_{m,l}} Y_{\alpha_m,\beta_{m,k}} Y_{\alpha_m,\beta_{m,j}} Y_{\alpha_m,\beta_{m,1}})$ for $1 < j < k < l \leq g$,

where i, j, k, l are distinct. Let $\pi : F \to \Gamma_2(g-1)$ be a natural epimorphism, and let $\nu : F \to \Gamma_2(N_g)$ be a homomorphism sending Y_{ij} to $Y_{i;j}$, and T_{1jkl}^2 to $T_{1,j,k,l}^2$. Then we have the following short exact sequences and commutative diagram:

$$\begin{array}{ccccccccc}
1 & \longrightarrow & N & \longrightarrow & F & \xrightarrow{\pi} & \Gamma_2(g-1) & \longrightarrow & 1 \\
 & & \downarrow{\scriptstyle \nu|_N} & & \downarrow{\scriptstyle \nu} & & \downarrow{\scriptstyle f^{-1}} & & \\
1 & \longrightarrow & \ker \rho' & \longrightarrow & \Gamma_2(N_g) & \xrightarrow{\rho'} & \ker \Phi_g & \longrightarrow & 1
\end{array}$$

By Lemma 4.2, we have $\widetilde{N} = \ker \rho'$. Since on the other hand $\mathcal{I}(N_g) = \ker \rho'$, we obtain the claim. ∎

4.2. A normal generating set for $\mathcal{I}(N_g)$ in $\mathcal{M}(N_g)$. In this subsection, we prove the following.

PROPOSITION 4.3. *For $g \geq 4$, $\mathcal{I}(N_g)$ is normally generated by the following elements in $\mathcal{M}(N_g)$:*

(1) $Y^2_{1;2}$,
(2) $[Y_{1;3}, Y_{2;3}]$,
(3) $[Y_{1;2}, Y_{1;3}Y_{2;3}]$, $[Y_{1;3}, Y_{1;2}Y_{3;2}]$,
(4) $[Y_{1;2}, Y_{3;4}]$, $[Y_{1;3}, Y_{2;4}]$,
(5) $(Y_{1;2}Y_{1;3}Y_{1;4})^2$,
(6) $(Y_{2;1}Y_{1;2}Y_{3;2}Y_{2;3}Y_{1;3}Y_{3;1})^2$,
(7) $T^2_{1,2,3,4}(\prod_{5\leq m\leq g} Y^{-2}_{m;m-1}\cdots Y^{-2}_{m;6}Y^{-2}_{m;5}Y^{-1}_{m;4}Y^{-1}_{m;3}Y^{-1}_{m;2}Y^{-1}_{m;1})$.

For $1 \leq i < j \leq g$, let c_{ij} be a simple closed curve on N_g as shown in Figure 15, and let $U_{i,j}$ be a diffeomorphism of N_g as shown in Figure 15. Note that $Y_{i;j} = U_{i,j}T_{i,j}$ and $U^2_{i,j} = t_{c_{ij}} = Y^2_{i;j}$.

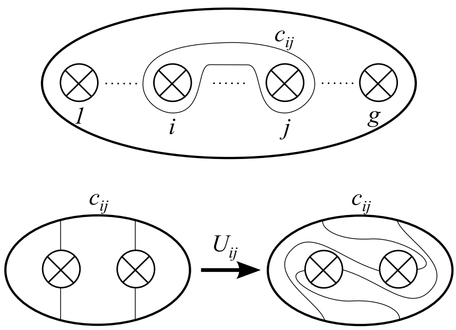


Fig. 15. The loop c_{ij} and the diffeomorphism U_{ij} of N_g

Since $\mathcal{I}(N_g)$ is a normal subgroup of $\Gamma_2(N_g)$, the normal generating set for $\mathcal{I}(N_g)$ in $\Gamma_2(N_g)$ is a normal generating set for $\mathcal{I}(N_g)$ in $\mathcal{M}(N_g)$. In addition, each normal generator for $\mathcal{I}(N_g)$ in Proposition 4.1 is conjugate to some normal generator for $\mathcal{I}(N_g)$ in Proposition 4.3 by a product of some $U_{i,j}$. For example,

$$Y^2_{i;j} = (U_{i-i,i}\cdots U_{1,2})(U_{j-i,j}\cdots U_{2,3})Y^2_{1;2}(U^{-1}_{2,3}\cdots U^{-1}_{j-1,j})(U^{-1}_{1,2}\cdots U^{-1}_{i-1,i})$$

for $i < j$. This finishes the proof of Proposition 4.3.

5. Proof of Theorem 1.1. Fix a basepoint $* \in N_{g-1}$. Let $\gamma_1, \ldots, \gamma_{g-1}$ be oriented loops on N_{g-1} starting at $*$ as shown in Figure 16. Note that $\pi_1(N_{g-1}, *)$ is generated by $[\gamma_1], \ldots, [\gamma_{g-1}]$. For $1 \leq i \leq g$, let $s_i : \pi_1(N_{g-1}, *) \to \mathcal{M}(N_g)$ be the crosscap pushing map defined in [18] such that $s_i([\gamma_j]) = Y_{i;j}$ if $j < i$, and $s_i([\gamma_j]) = Y_{i;j+1}$ if $j \geq i$. We note that the crosscap pushing map is an anti-homomorphism: $s_i([\alpha][\beta]) = s_i([\beta])s_i([\alpha])$ for

$[\alpha], [\beta] \in \pi_1(N_{g-1}, *)$. For $[\alpha] \in \pi_1(N_{g-1}, *)$, $s_i([\alpha])$ is a crosscap slide if α is an M-circle, and $s_i([\alpha])$ is a product of two Dehn twists if α is an A-circle.

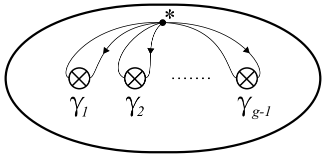


Fig. 16. The loops $\gamma_1, \dots, \gamma_{g-1}$

The following is a corollary of Proposition 4.3.

COROLLARY 5.1. *For $g \geq 4$, $\mathcal{I}(N_g)$ is normally generated by the following elements in $\mathcal{M}(N_g)$:*

(1) $Y_{1;2}^2$,
(2) $[Y_{2;3}, Y_{1;3}^{-1}]$,
(3) $[Y_{1;2}, Y_{1;3}Y_{2;3}]$, $[Y_{1;3}, Y_{3;2}Y_{1;2}]$,
(4) $[Y_{1;2}, Y_{3;4}]$, $[Y_{2;3}^{-2}Y_{1;3}Y_{2;3}^2, Y_{2;4}]$,
(5) $(Y_{1;2}Y_{1;3}Y_{1;4})^2$,
(6) $(Y_{2;1}^{-1}Y_{1;2}Y_{3;2}Y_{2;3}^{-1}Y_{1;3}^{-1}Y_{3;1})^2$,
(7) $T_{1,2,3,4}^2(\prod_{5\leq m\leq g} Y_{m;m-1}^{-2} \cdots Y_{m;6}^{-2}Y_{m;5}^{-2}Y_{m;4}^{-1}Y_{m;3}^{-1}Y_{m;2}^{-1}Y_{m;1}^{-1})$.

We now prove Theorem 1.1.

Proof of Theorem 1.1. We show that each normal generator for $\mathcal{I}(N_g)$ in Corollary 5.1 is a product of BSCC maps and BP maps.

(1) We have $Y_{1;2}^2 = s_1([\gamma_1^2]) = t_{c_1}t_{c_2}$, where c_1 and c_2 are simple closed curves as shown in Figure 17. Note that t_{c_1} is a BSCC map of type $(1, 2)$. Since c_2 bounds a Möbius band, t_{c_2} is trivial. Hence $Y_{1;2}^2$ is a BSCC map of type $(1, 2)$.

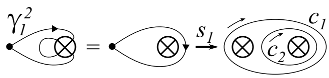


Fig. 17

(2) We have $Y_{1;3}Y_{2;3}Y_{1;3}^{-1} = Y_{1;3}s_2([\gamma_2])Y_{1;3}^{-1} = s_2([\gamma_1^2\gamma_2])$ (see Figure 18). Hence

$$\begin{aligned}[Y_{2;3}, Y_{1;3}^{-1}] &= Y_{2;3}^{-1}Y_{1;3}Y_{2;3}Y_{1;3}^{-1} = s_2([\gamma_2^{-1}])s_2([\gamma_1^2\gamma_2]) \\ &= s_2([\gamma_1^2\gamma_2][\gamma_2^{-1}]) = s_2([\gamma_1^2]) = t_{c_1}t_{c_2},\end{aligned}$$

where c_1 and c_2 are simple closed curves as shown in Figure 18. Similar to (1), we find that $[Y_{2;3}, Y_{1;3}^{-1}]$ is a BSCC map of type $(1, 2)$.

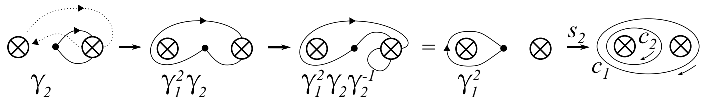


Fig. 18

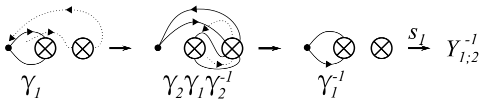
(3) We have

$$Y_{2;3}^{-1}Y_{1;3}^{-1}Y_{1;2}Y_{1;3}Y_{2;3} = Y_{2;3}^{-1}Y_{1;3}^{-1}s_1([\gamma_1])Y_{1;3}Y_{2;3} = Y_{2;3}^{-1}s_1([\gamma_2\gamma_1\gamma_2^{-1}])Y_{2;3}$$
$$= s_1([\gamma_1^{-1}]) = Y_{1;2}^{-1}$$

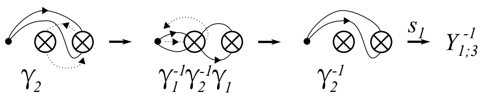
(see Figure 19(a)). Hence $[Y_{1;2}, Y_{1;3}Y_{2;3}] = Y_{1;2}^{-1}Y_{2;3}^{-1}Y_{1;3}^{-1}Y_{1;2}Y_{1;3}Y_{2;3} = Y_{1;2}^{-2}$. Similarly,

$$Y_{1;2}^{-1}Y_{3;2}^{-1}Y_{1;3}Y_{3;2}Y_{1;2} = Y_{1;2}^{-1}Y_{3;2}^{-1}s_1([\gamma_2])Y_{3;2}Y_{1;2} = Y_{1;2}^{-1}s_1([\gamma_1^{-1}\gamma_2^{-1}\gamma_1])Y_{1;2}$$
$$= s_1([\gamma_2^{-1}]) = Y_{1;3}^{-1}$$

(see Figure 19(b)). Hence $[Y_{1;3}, Y_{3;2}Y_{1;2}] = Y_{1;3}^{-2}$. Similar to (1), we conclude that $Y_{1;2}^{-2}$ and $Y_{1;3}^{-2}$ are BSCC maps of type $(1, 2)$.



(a)



(b)

Fig. 19

(4) In $\mathcal{I}(N_g)$, it is clear that $[Y_{1;2}, Y_{3;4}]$ and $[Y_{2;3}^{-2}Y_{1;3}Y_{2;3}^2, Y_{2;4}]$ are equal to 1. Note that $Y_{2;3}^{-2}Y_{1;3}Y_{2;3}^2 = s_1([\gamma_1^{-2}\gamma_2\gamma_1^2])$ (see Figure 20).

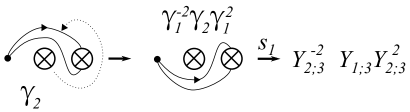


Fig. 20

(5) We have $(Y_{1;2}Y_{1;3}Y_{1;4})^2 = s_1([(\gamma_3\gamma_2\gamma_1)^2]) = t_{c_1}t_{c_2}$, where c_1 and c_2 are simple closed curves as shown in Figure 21. Note that t_{c_1} is a BSCC map of type $(1, 2)$ if $g \geq 5$ and a BSCC map of type $(2, 1)$ if $g = 4$. Since c_2 bounds a Möbius band, t_{c_2} is trivial. Hence $(Y_{1;2}Y_{1;3}Y_{1;4})^2$ is a BSCC map of type $(1, 2)$ if $g \geq 5$, and a BSCC map of type $(2, 1)$ if $g = 4$.

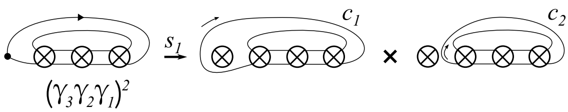


Fig. 21. The black cross "×" means the composition of t_{c_1} and t_{c_2}

(6) By [17, proof of Lemma 3.1], we have $Y_{j;i}^{-1}Y_{i;j} = Y_{j;i}Y_{i;j}^{-1} = T_{i,j}^2$ for $1 \le i < j \le g$. Hence $(Y_{2;1}^{-1}Y_{1;2}Y_{3;2}Y_{2;3}^{-1}Y_{1;3}^{-1}Y_{3;1})^2 = (T_{1,2}^2T_{2,3}^2T_{1,3}^{-2})^2$.

Lemma 5.2. *We have (see Figure 22):*

(a) $T_{1,2}^2 = L_1Y_{3;1}Y_{3;2}$, *where*

$$L_1 = (Y_{g;1}Y_{g;2}Y_{g;3}^2\cdots Y_{g;g-1}^2)\cdots(Y_{4;1}Y_{4;2}Y_{4;3}^2).$$

(b) $T_{2,3}^2 = L_2Y_{1;2}Y_{1;3}$, *where*

$$L_2 = (Y_{g;1}^2Y_{g;2}Y_{g;3}Y_{g;4}^2\cdots Y_{g;g-1}^2)\cdots(Y_{4;1}^2Y_{4;2}Y_{4;3}).$$

(c) $T_{1,3}^{-2} = L_3Y_{2;3}Y_{2;1}$, *where*

$$L_3 = (Y_{g;3}^{-1}Y_{g;1}Y_{g;2}^2\cdots Y_{g;g-1}^2)\cdots(Y_{4;3}^{-1}Y_{4;1}Y_{4;2}^2Y_{4;3}^2).$$

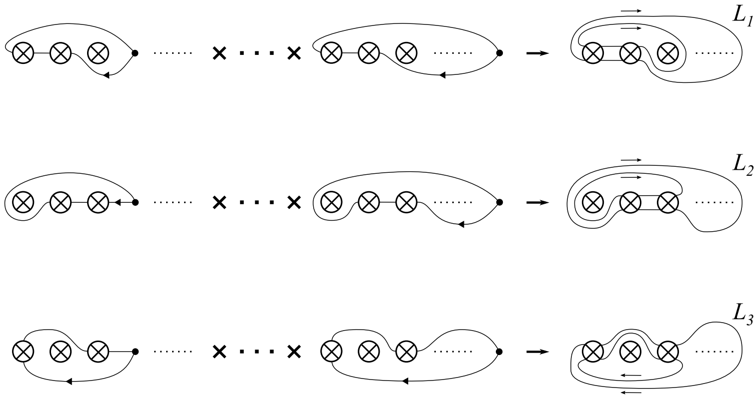


Fig. 22. In this figure, and also in Figures 24, 25 and 27, the black crosses "×" mean compositions of crosscap pushing maps.

We have

$$\begin{aligned} &Y_{3;2}^{-2}Y_{3;1}^{-1}(Y_{2;1}^{-1}Y_{1;2}Y_{3;2}Y_{2;3}^{-1}Y_{1;3}^{-1}Y_{3;1})^2Y_{3;1}Y_{3;2}^2 \\ &\qquad = Y_{3;2}^{-2}Y_{3;1}^{-1}\cdot L_1Y_{3;1}Y_{3;2}\cdot Y_{3;2}Y_{2;3}^{-1}\cdot L_3Y_{2;3}Y_{2;1} \\ &\qquad\quad \cdot Y_{2;1}^{-1}Y_{1;2}\cdot L_2Y_{1;2}Y_{1;3}\cdot Y_{1;3}^{-1}Y_{3;1}\cdot Y_{3;1}Y_{3;2}^2 \\ &\qquad = R_1R_3R_2Y_{1;2}^2Y_{3;1}^2Y_{3;2}^2, \end{aligned}$$

where $R_1 = Y_{3;2}^{-2}Y_{3;1}^{-1}L_1Y_{3;1}Y_{3;2}^2$, $R_2 = Y_{1;2}L_2Y_{1;2}^{-1}$ and $R_3 = Y_{2;3}^{-1}L_3Y_{2;3}$ (see Figure 23). Similar to (1), since $Y_{1;2}^2$, $Y_{3;1}^2$ and $Y_{3;2}^2$ are BSCC maps of type

$(1, 2)$, it suffices to show that $R_1R_3R_2$ is a product of BSCC maps.

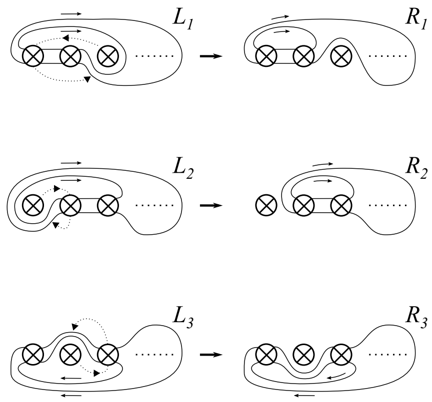


Fig. 23

Let d_0, d_1, d_2, d_3 and d_4 be simple closed curves as in Figure 24. Note that t_{d_0} is a BSCC map of type $(1, 3)$ or $(1, g-3)$, and t_{d_3} is a BSCC map of type $(2, 1)$. In addition, since d_4 bounds a Möbius band, we have $t_{d_4} = 1$. Let $R_{32} = t_{d_1}t_{d_2}$. By the lantern relation, R_{32} is a product of R_3R_2 and t_{d_0}. Let $R_{132} = t_{d_3}t_{d_4}$. By the lantern relation, R_{132} is a product of R_1R_{32} and t_{d_0}. Hence R_{132} is a product of $R_1R_3R_2$ and t_{d_0}. Therefore $R_1R_3R_2$ is a product of BSCC maps of type one and a BSCC map of type $(2, 1)$. In particular, if $g = 4$, since t_{d_0} is trivial, $R_1R_3R_2$ is a BSCC map of type $(2, 1)$, and if $g \geq 5$, by Lemma 2.2, $R_1R_3R_2$ is a product of BSCC maps of type $(1, 2)$.

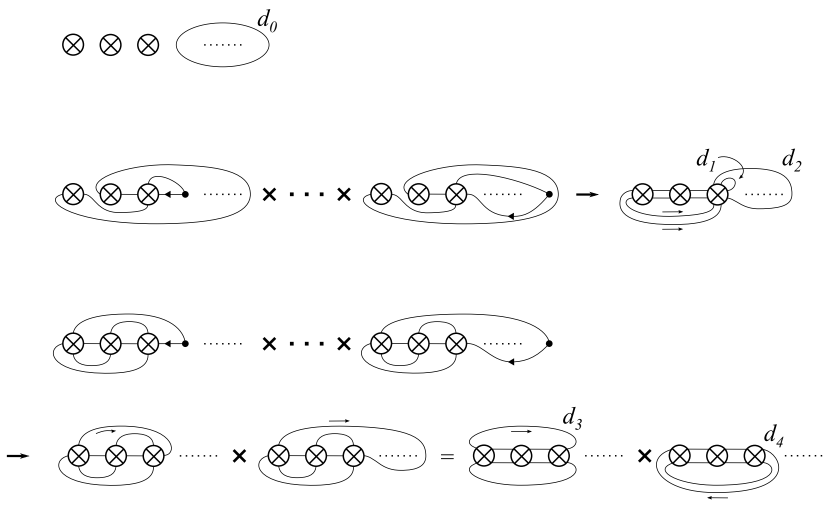


Fig. 24

(7) We have

$$T^{-1}_{1,2,3,4}T'^{-1}_{1,2,3,4} = \prod_{5\le m\le g} s_m([\gamma_1^{-1}\gamma_2^{-1}\gamma_3^{-1}\gamma_4^{-1}\gamma_5^{-2}\gamma_6^{-2}\cdots\gamma_{m-1}^{-2}])$$
$$= \prod_{5\le m\le g} Y^{-2}_{m;m-1}\cdots Y^{-2}_{m;6}Y^{-2}_{m;5}Y^{-1}_{m;4}Y^{-1}_{m;3}Y^{-1}_{m;2}Y^{-1}_{m;1}$$

(see Figure 25). Hence

$$T^{2}_{1,2,3,4}\Big(\prod_{5\le m\le g} Y^{-2}_{m;m-1}\cdots Y^{-2}_{m;6}Y^{-2}_{m;5}Y^{-1}_{m;4}Y^{-1}_{m;3}Y^{-1}_{m;2}Y^{-1}_{m;1}\Big) = T_{1,2,3,4}T'^{-1}_{1,2,3,4}.$$

Thus it is a BP map of type $(1,1)$.

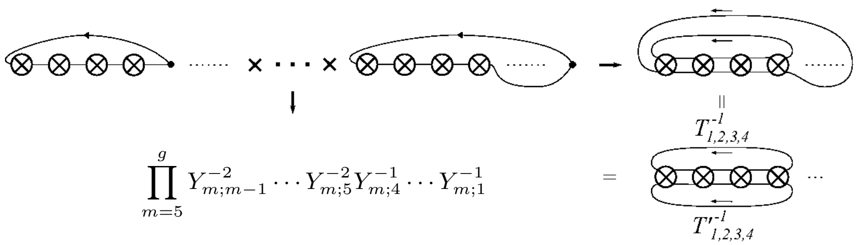


Fig. 25

This completes the proof. ∎

Appendix A. In this appendix, we prove equality (I) of Section 2 and equality (II) of Section 4.

For (I), it suffices to show the following lemma.

Lemma A.1. *For $1 \le i_1 < \cdots < i_k \le g$, let $c_{i_1,\dots,i_k}$ be a simple closed curve on N_g as shown in Figure 26. Then for $2 \le h \le g$ we have*

$$t_{c_{1,\dots,h}} = (t_{c_{1,2}}\cdots t_{c_{1,h-1}}t_{c_{1,h}})\cdots(t_{c_{h-2,h-1}}t_{c_{h-2,h}})(t_{c_{h-1,h}}).$$

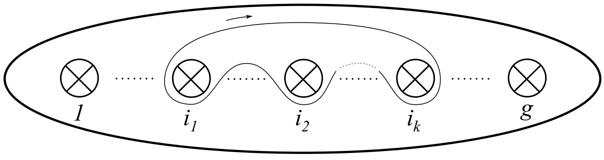


Fig. 26. The curve $c_{i_1,\dots,i_k}$ on N_g

Proof. We first note that $t_{c_{i,j}} = Y_{i;j}^2$. We see that

$$\begin{aligned} t_{c_{1,\dots,h}} &= (t_{c_{1,\dots,h}} t_{c_{2,\dots,h}}^{-1}) \cdots (t_{c_{h-2,h-1,h}} t_{c_{h-1,h}}^{-1})(t_{c_{h-1,h}} t_{c_h}^{-1}) \\ &= (s_1([\gamma_{h-1}^2 \cdots \gamma_1^2])) \cdots (s_{h-2}([\gamma_{h-1}^2 \gamma_{h-2}^2]))(s_{h-1}([\gamma_{h-1}^2])) \\ &= (Y_{1;2}^2 \cdots Y_{1;h-1}^2 Y_{1;h}^2) \cdots (Y_{h-2;h-1}^2 Y_{h-2;h}^2)(Y_{h-1;h}^2) \\ &= (t_{c_{1,2}} \cdots t_{c_{1,h-1}} t_{c_{1,h}}) \cdots (t_{c_{h-2,h-1}} t_{c_{h-2,h}})(t_{c_{h-1,h}}). \end{aligned}$$

Thus we obtain the claim. ■

EXAMPLE A.2. For $1 \le i \le g-1$ we have

$$\begin{aligned} Y_{g;i} = {} & (Y_{1;2}^2 \cdots Y_{1;g-1}^2 Y_{1;i}^{-1} Y_{1;g}) \cdots (Y_{i-1;i}^2 \cdots Y_{i-1;g-1}^2 Y_{i-1;i}^{-1} Y_{i-1;g}) \\ & \cdot (Y_{i+1;i+2}^2 \cdots Y_{i+1;g-1}^2 Y_{i+1;i}^{-1} Y_{i+1;g} Y_{i+1;i}^2) \cdots (Y_{g-2;g-1}^2 Y_{g-2;i}^{-1} Y_{g-2;g} Y_{g-2;i}^2) \\ & \cdot (Y_{g-1;i}^{-1} Y_{g-1;g} Y_{g-1;i}^2) Y_{i;g}. \end{aligned}$$

Proof. Note that $Y_{g;i} Y_{i;g}^{-1} = T_{i,g}^2$. We see that

$$\begin{aligned} T_{i,g}^2 = {} & \prod_{1 \le m \le i-1} s_m([\gamma_{g-1}\gamma_{i-1}^{-1}\gamma_{g-2}^2 \cdots \gamma_m^2]) \prod_{i+1 \le m \le g-2} s_m([\gamma_i^2 \gamma_{g-1} \gamma_i^{-1} \gamma_{g-2}^2 \cdots \gamma_m^2]) \\ & \cdot\ s_{g-1}([\gamma_i^2 \gamma_{g-1} \gamma_i^{-1}]) \\ = {} & \prod_{1 \le m \le i-1} (Y_{m;m+1}^2 \cdots Y_{m;g-1}^2 Y_{m;i}^{-1} Y_{m;g}) \\ & \times \prod_{i+1 \le m \le g-2} (Y_{m;m+1}^2 \cdots Y_{m;g-1}^2 Y_{m;i}^{-1} Y_{m;g} Y_{m;i}^2)(Y_{g-1;i}^{-1} Y_{g-1;g} Y_{g-1;i}^2) \end{aligned}$$

(see Figure 27). Thus we obtain the claim. ■

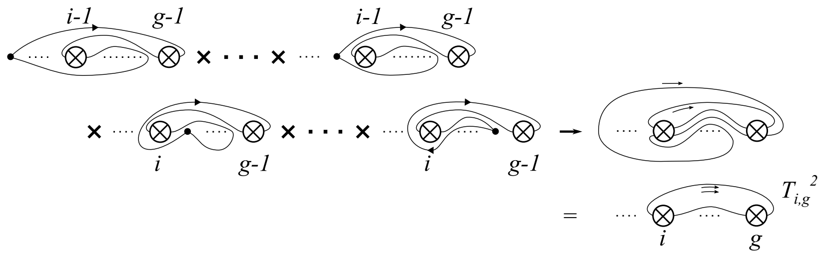


Fig. 27

Thus we obtain equality (Ⅱ).

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References

[1] J. S. Birman and D. R. J. Chillingworth, *On the homeotopy group of a non-orientable surface*, Math. Proc. Cambridge Philos. Soc. 71 (1972), 437–448. Erratum: Math. Proc. Cambridge Philos. Soc. 136 (2004), 441–441.

[2] D. R. J. Chillingworth, *A finite set of generators for the homeotopy group of a non-orientable surface*, Math. Proc. Cambridge Philos. Soc. 65 (1969), 409–430.

[3] D. B. A. Epstein, *Curves on 2-manifolds and isotopies*, Acta Math. 115 (1966), 83–107.

[4] N. J. Fullarton, *A generating set for the palindromic Torelli group*, Algebr. Geom. Topol. 15 (2015), 3535–3567.

[5] S. Gadgil and D. Pancholi, *Homeomorphisms and the homology of non-orientable surfaces*, Proc. Indian Acad. Sci. (Math. Sci.) 115 (2005), 251–257.

[6] S. Hirose and M. Sato, *A minimal generating set of the level 2 mapping class group of a non-orientable surface*, Math. Proc. Cambridge Philos. Soc. 157 (2014), 345–355.

[7] D. Johnson, *Homeomorphisms of a surface which act trivially on homology*, Proc. Amer. Math. Soc. 75 (1979), 119–125.

[8] R. Kobayashi, *A finite presentation of the level 2 principal congruence subgroup of* $\mathrm{GL}(n;\mathbb{Z})$, Kodai Math. J. 38 (2015), 534–559.

[9] W. B. R. Lickorish, *Homeomorphisms of non-orientable two-manifolds*, Math. Proc. Cambridge Philos. Soc. 59 (1963), 307–317.

[10] W. B. R. Lickorish, *On the homeomorphisms of a non-orientable surface*, Math. Proc. Cambridge Philos. Soc. 61 (1965), 61–64.

[11] J. D. McCarthy and U. Pinkall, *Representing homology automorphisms of nonorientable surfaces*, Max Planck Inst. preprint MPI/SFB 85-11, revised version written in 2004; http://www.math.msu.edu/~mccarthy.

[12] L. Paris and B. Szepietowski, *A presentation for the mapping class group of a nonorientable surface*, Bull. Soc. Math. France 143 (2015), 503–566.

[13] J. Powell, *Two theorems on the mapping class group of a surface*, Proc. Amer. Math. Soc. 68 (1978), 347–350.

[14] A. Putman, *Cutting and pasting in the Torelli group*, Geom. Topol. 11 (2007), 829–865.

[15] M. Stukow, *A finite presentation for the mapping class group of a nonorientable surface with Dehn twists and one crosscap slide as generators*, J. Pure Appl. Algebra 218 (2014), 2226–2239.

[16] B. Szepietowski, *A presentation for the mapping class group of the closed non-orientable surface of genus 4*, J. Pure Appl. Algebra 213 (2009), 2001–2016.

[17] B. Szepietowski, *Crosscap slides and the level 2 mapping class group of a nonorientable surface*, Geom. Dedicata 160 (2012), 169–183.

[18] B. Szepietowski, *A finite generating set for the level 2 mapping class group of a nonorientable surface*, Kodai Math. J. 36 (2013), 1–14.

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