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# Michael's theorem for Lipschitz cells in o-minimal structures

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**Abstract.** A version of Michael's theorem for multivalued mappings definable in o-minimal structures with $M$-Lipschitz cell values ($M$ a common constant) is proven. Uniform equi-$LC^n$ property for such families of cells is checked. An example is given showing that the assumption about the common Lipschitz constant cannot be omitted.

**1. Introduction.** Assume that $R$ is any real closed field and an expansion of $R$ to some o-minimal structure is given. Throughout the paper we will be talking about definable sets and mappings referring to this o-minimal structure. (For fundamental definitions and results on o-minimal structures the reader is referred to [vdD] or [C].) In this article we adopt the following definition of a closed cell.

A subset $S$ of $R^m$ ($m \in \mathbb{Z}$, $m > 0$) will be called a *closed* (respectively, *closed $M$-Lipschitz*) *cell* in $R^m$, where $M \in R$, $M > 0$, if

(i) when $m = 1$: $S$ is a closed interval $[\alpha, \beta]$ ($\alpha, \beta \in R$, $\alpha \leq \beta$), or $S = [\alpha, +\infty)$, or $S = (-\infty, \alpha]$ ($\alpha \in R$), or $S = R$, and

(ii) when $m > 1$: $S = [f_1, f_2] := \{(y', y_m) : y' \in S',\ f_1(y') \leq y_m \leq f_2(y')\}$, where $y' = (y_1, \ldots, y_{m-1})$, $S'$ is a closed (respectively, closed $M$-Lipschitz) cell in $R^{m-1}$, $f_i : S' \to R$ ($i = 1, 2$) are continuous (respectively, $M$-Lipschitz) definable functions such that $f_1(y') \leq f_2(y')$ for each $y' \in S'$, or $S = [f, +\infty) = \{(y', y_m) : y' \in S',\ y_m \geq f(y')\}$, or $S = (-\infty, f] = \{(y', y_m) : y' \in S', y_m \leq f(y')\}$, or $S = S' \times R$, where $S'$ is as before and $f : S' \to R$ is continuous (respectively, $M$-Lipschitz).

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Let $F : A \rightrightarrows R^m$ be a multivalued mapping defined on a subset $A$ of $R^n$; i.e. a mapping which assigns to each $x \in A$ a nonempty subset $F(x)$ of $R^m$. Then $F$ can be identified with its graph, a subset of $R^n \times R^m$. If this subset is definable, we will call $F$ *definable.* $F$ is called *lower semicontinuous* if for each $a \in A$, each $u \in F(a)$ and any neighborhood $U$ of $u$, there exists a neighborhood $V$ of $a$ such that $U \cap F(x) \neq \emptyset$ for each $x \in V$.

The aim of the present article is to prove the following theorem.

THEOREM 1.1. *Let $F : A \rightrightarrows R^m$ be a definable lower semicontinuous multivalued mapping on a definable subset $A$ of $R^n$ such that every value $F(x)$ is a closed $M$-Lipschitz cell in $R^m$, where the constant $M > 0$ is independent of $x \in A$. Then $F$ admits a continuous definable selection $\varphi : A \to R^m$.*

The following generalization of Theorem 1.1 is immediate.

COROLLARY 1.2. *Let $F : A \rightrightarrows R^m$ be a definable lower semicontinuous multivalued mapping on a definable subset $A$ of $R^n$. If there is a continuous definable mapping $\Phi : A \to \mathrm{Aut}(R^m)$ with values in the space of linear automorphisms* (1) *of $R^m$ such that $\Phi(x)(F(x))$ is a closed $M$-Lipschitz cell in $R^m$, then $F$ admits a continuous definable selection $\varphi : A \to R^m$.*

Theorem 1.1 is true for the semilinear expansion of $R$ provided that $A$ is bounded (see Remark 2.6 below). Moreover, every closed semilinear cell is Lipschitz and for every semilinear family of semilinear cells they are $M$-Lipschitz with a common $M$ [vdD, Chapter 1, (7.4)]. In this way we obtain the following generalization of [AT2, Theorem 4.10].

COROLLARY 1.3. *Let $F : A \rightrightarrows R^m$ be a semilinear lower semicontinuous multivalued mapping on a semilinear bounded subset $A$ of $R^n$ such that every value $F(x)$ is a closed semilinear cell in $R^m$. Then $F$ admits a continuous semilinear selection $\varphi : A \to R^m$.*

For other results on multivalued mappings in connection with o-minimal geometry we refer the reader to [AT1], [AT2] and [DP].

**2. Proof of Theorem 1.1.** The proof will be by induction on $m$. Consider first the case $m = 1$. Then $F(x) = \{t \in R : f(x) \le t \le g(x)\}$ for each $x \in A$, where $f : A \to R \cup \{-\infty\}$ and $g : A \to R \cup \{+\infty\}$ are definable functions (2). It is easy to check that $F$ is lower semicontinuous if and only if $g$ is lower semicontinuous and $f$ is upper semicontinuous. Therefore, the problem reduces to the following.

PROPOSITION 2.1. *Let $f : A \to R \cup \{-\infty\}$ and $g : A \to R \cup \{+\infty\}$ be definable functions such that $f(x) \le g(x)$ for each $x \in A$, and $f$ is upper*

(1) The space $\mathrm{Aut}(R^m)$ is naturally identified with a subset of $R^{m^2}$.

(2) This means that $f|f^{-1}(R)$ and $g|g^{-1}(R)$ are definable.

*semicontinuous while $g$ is lower semicontinuous. Then there exists a definable continuous function $\varphi : A \to R$ such that $f \le \varphi \le g$.*

To prove Proposition 2.1, which is a definable version of the Katětov–Tong Insertion Theorem, we need the following definable version of the Tietze Theorem.

THEOREM 2.2 (Definable Tietze Theorem). *Let $X$ and $Y$ be definable subsets of $R^n$ such that $Y$ is closed in $X$. Then every definable continuous function $\psi : Y \to R$ has a continuous definable extension $\Psi : X \to R$.*

For a proof of Theorem 2.2 see [vdD, Chapter 8, (3.10)] (compare also [AF, Lemma 6.6]).

REMARK 2.3. According to [AT2, Theorem 3.3] Theorem 2.2 holds true in the semilinear expansion of $R$, provided that $Y$ is bounded.

*Proof of Proposition 2.1.* We use induction on $d := \dim A$. The case $d = 0$ is trivial. Assume that $d > 0$. Let

$$B := \{a \in A : f \text{ and } g \text{ are both continuous in a neighborhood of } a \text{ in } A\}.$$

Then $B$ is a definable, open and dense subset of $A$. Hence $A \setminus B$ is definable closed in $A$ and $\dim(A \setminus B) < d$. By induction hypothesis there exists a definable continuous function $\psi : A \setminus B \to R$ such that $f(x) \le \psi(x) \le g(x)$ for each $x \in A \setminus B$. By the Definable Tietze Theorem there exists a definable continuous extension $\Psi : A \to R$ of $\psi$. Now set $\varphi(x) := \min(\max(\Psi(x), f(x)), g(x))$ for $x \in A$. It is clear that $f \le \varphi \le g$. Continuity of $\varphi$ on $B$ is obvious, since $\Psi, f$ and $g$ are continuous on $B$.

We now check the continuity of $\varphi$ at any $a \in A \setminus B$. We have $\varphi(a) = \psi(a) \in [f(a), g(a)]$. Fix any $\varepsilon > 0$. There exists a neighborhood $V$ of $a$ in $A$ such that $\psi(a) + \varepsilon > f(x)$, $\psi(a) - \varepsilon < g(x)$, $\psi(a) + \varepsilon > \Psi(x)$ and $\psi(a) - \varepsilon < \Psi(x)$ for each $x \in V$. Then

$$\varphi(a) - \varepsilon = \psi(a) - \varepsilon < \Psi(x) \le \max(\Psi(x), f(x)) < \psi(a) + \varepsilon = \varphi(a) + \varepsilon$$

and $\varphi(a) - \varepsilon < g(x)$. Hence

$$\varphi(a) - \varepsilon < \min\big(\max(\Psi(x), f(x)), g(x)\big) < \varphi(a) + \varepsilon. \ ■$$

REMARK 2.4. The proof of Proposition 2.1 holds true in the semilinear expansion of $R$ under the assumption that $A$ is semilinear bounded, provided we apply a semilinear version of Theorem 2.2 with $X = A$ (see Remark 2.3).

Assume now that $m > 1$ and our theorem is true for $m - 1$. To make the induction hypothesis work we prove the following.

PROPOSITION 2.5. *Under the assumptions of Theorem* 1.1, *let*

$$\pi : R^m \ni y = (y_1, \ldots, y_m) \mapsto y' = (y_1, \ldots, y_{m-1}) \in R^{m-1}$$

*be the natural projection. Let $\pi \circ F : A \rightrightarrows R^{m-1}$ be defined by $(\pi \circ F)(x) = \pi(F(x))$. Then, after identifying $\pi \circ F$ with its graph $\pi \circ F \subset R^n \times R^{m-1}$, the multivalued mapping*

$$G : \pi \circ F \ni (x, y') \mapsto \{y_m \in R : (y', y_m) \in F(x)\} \subset R$$

*is lower semicontinuous.*

*Proof.* By the definition of a closed cell, one can write, for each $x \in A$,

$$F(x) = \{(y', y_m) : y' \in \pi(F(x)),\ y_m \in R,\ f_x(y') \le y_m \le g_x(y')\},$$

where $f_x, g_x : \pi(F(x)) \to R$ are $M$-Lipschitz (or maybe $f_x = -\infty$, or $g_x = \infty$; these cases will follow by a simple modification of the argument below). Fix any $(a, b') \in \pi \circ F$, $u \in G(a, b') = \{y_m \in R : f_a(b') \le y_m \le g_a(b')\}$ and any open interval $(u - \varepsilon, u + \varepsilon)$. Let $W$ be the open ball $\{y' \in R^{m-1} : |y' - b'| < \varepsilon/(4M)\}$, where $|\cdot|$ is defined by $|y'| = |(y_1, \ldots, y_{m-1})| = \max_j |y_j|$. By lower semicontinuity of $F$ there exists a neighborhood $V$ of $a$ in $A$ such that $F(x) \cap (W \times (u - \varepsilon/2, u + \varepsilon/2)) \neq \emptyset$ whenever $x \in V$.

Let $(x, y') \in (\pi \circ F) \cap (V \times W)$. There exists $(z', v) \in F(x) \cap (W \times (u - \varepsilon/2, u + \varepsilon/2))$. Then $y' \in \pi(F(x))$ and $z' \in \pi(F(x))$; hence $|y' - z'| < \varepsilon/(2M)$ and $f_x(z') \le v \le g_x(z')$. Thus, $|f_x(y') - f_x(z')| \le M|y' - z'| < \frac{1}{2}\varepsilon$. Hence $f_x(y') \le f_x(z') + \frac{1}{2}\varepsilon \le v + \frac{1}{2}\varepsilon < u + \varepsilon$. Similarly, $|g_x(y') - g_x(z')| < \frac{1}{2}\varepsilon$ and so $g_x(y') \ge g_x(z') - \frac{1}{2}\varepsilon \ge v > u - \varepsilon$. Finally, $[f_x(y'), g_x(y')] \cap (u - \varepsilon, u + \varepsilon) \neq \emptyset$, which ends the proof. ∎

To finish the proof of Theorem 1.1, observe that the mapping $\pi \circ F$ is lower semicontinuous as the composition of a lower semicontinuous mapping with a continuous one, so by the induction hypothesis there exists a continuous definable selection $\varphi'$ for $\pi \circ F$. We identify $\varphi'$ with its graph. By Proposition 2.5, $G|\varphi' : \varphi' \rightrightarrows R$ is lower semicontinuous; hence, by Proposition 2.1, it admits a continuous definable selection $\sigma : \varphi' \to R$, which gives the required selection $\varphi = (\varphi', \sigma \circ (\mathrm{id}_A, \varphi'))$.

REMARK 2.6. The proof of Proposition 2.5 holds true for the semilinear expansion of $R$, so in view of Remark 2.4, Theorem 1.1 holds true for the semilinear structure under the assumption that $A$ is semilinear and bounded.

**3. A counterexample.** We now give a semialgebraic example showing that in Theorem 1.1 the assumption of common boundedness of the Lipschitz constants of the Lipschitz cells $F(x)$ cannot be omitted, even if $F$ is continuous.

Let $A = T_1 \cup T_2 \subset R^2$, where

$$T_1 = \{(x_1, x_2) : x_1 \in [0, 1],\ -x_1 \le x_2 \le x_1\},$$
$$T_2 = \{(x_1, x_2) : x_1 \in [-1, 0],\ x_1 \le x_2 \le -x_1\}.$$

We define $F : A \rightrightarrows R^2$ by

$$F(x_1,x_2) = \begin{cases} \{0\}\times[0,1], & (x_1,x_2)=(0,0),\\ \{(y,|y|/|x_1|) : -x_1+x_2 \le y \le x_1\}, & x_1>0,\ x_2\ge 0,\\ \{(y,|y|/|x_1|) : -x_1 \le y < x_1+x_2\}, & x_1>0,\ x_2\le 0,\\ \{(y,1-|y|/|x_1|) : x_1+x_2 \le y < -x_1\}, & x_1<0,\ x_2\ge 0,\\ \{(y,1-|y|/|x_1|) : x_1 \le y \le -x_1+x_2\}, & x_1<0,\ x_2\le 0. \end{cases}$$

The graph of $F$ is shown in Fig 1.

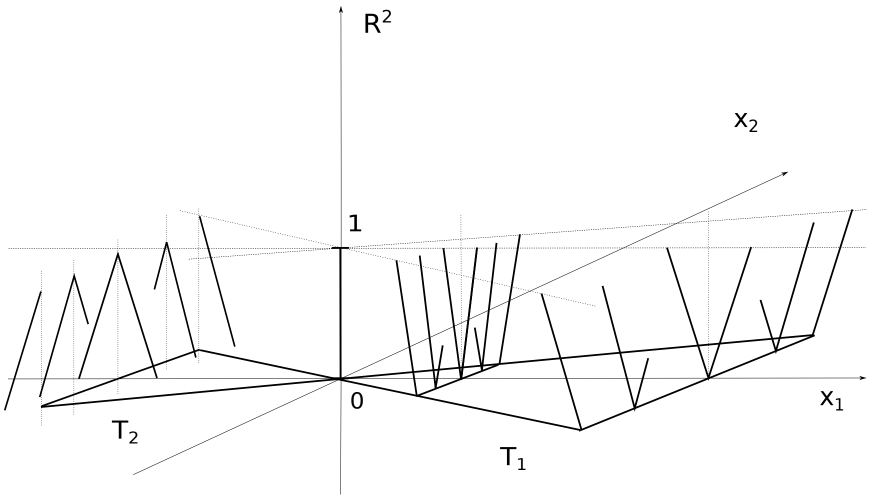


Fig. 1. The graph of $F$

Suppose that $F$ admits a continuous semialgebraic selection $\varphi=(\sigma,\rho) : A \to R^2$. Then, for $x_1>0$, $\sigma(x_1,x_1)\ge 0$ and $\sigma(x_1,-x_1)\le 0$; hence, there exists $\xi\in[-x_1,x_1]$ such that $\sigma(x_1,\xi)=0$, so $\rho(x_1,\xi)=|\sigma(x_1,\xi)|/|x_1|=0$ and $\varphi(x_1,\xi)=(0,0)$. Consequently, by continuity, $\varphi(0,0)=(0,0)$. Similarly, for any $x_1<0$, there exists $\xi\in[x_1,-x_1]$ such that $\varphi(x_1,\xi)=(0,1)$; hence $\varphi(0,0)=(0,1)$, a contradiction.

Notice that in the above example the dimensions of both the domain and the target space are minimal (see [CzP]).

**4. $M$-Lipschitz cells as a uniformly equi-$LC^p$ family, with arbitrary $p$.** Let $\mathcal{S}_M^m$ denote the set of all closed $M$-Lipschitz cells in $R^m$, where $M>0$ is a constant. We will show that $\mathcal{S}_M^m$ is uniformly equi-$LC^p$, with arbitrary $p$, in the sense of Michael [M]. This follows immediately from the fact that every closed cell is contractible, together with the following proposition.

PROPOSITION 4.1. *Let $M,\varepsilon\in R$, $M\ge 1$, $\varepsilon>0$. Set $k_1=l_1=1$ and $k_m=2^{2m-2}$ and $l_m=2^{2m-3}$ for $m\ge 2$. Endow $R^m$ with the metric*

$|(a_1,\dots,a_m)-(b_1,\dots,b_m)| = \max_j |a_j-b_j|$. *For any* $a=(a_1,\dots,a_m)\in R^m$, *consider the following cuboids with center* $a$:

$$P(a,\varepsilon) := [a_1-k_1\varepsilon, a_1+k_1\varepsilon]\times\cdots\times[a_m-k_mM^{m-1}\varepsilon, a_m+k_mM^{m-1}\varepsilon],$$
$$Q(a,\varepsilon) := [a_1-l_1\varepsilon, a_1+l_1\varepsilon]\times\cdots\times[a_m-l_mM^{m-1}\varepsilon, a_m+l_mM^{m-1}\varepsilon].$$

*Then, for any* $S\in\mathcal{S}_M^m$, *if* $S\cap Q(a,\varepsilon)\neq\emptyset$, *then* $S\cap P(a,\varepsilon)\in\mathcal{S}_M^m$.

*Proof.* The assertion is trivial for $m=1$, so assume that $m\geq 2$ and the assertion is true for $m-1$. Let $\pi : R^m \ni (a_1,\dots,a_m)\mapsto(a_1,\dots,a_{m-1})\in R^{m-1}$. Then $\pi(P(a,\varepsilon)) = P(\pi(a),\varepsilon)$ and $\pi(Q(a,\varepsilon)) = Q(\pi(a),\varepsilon)$. Let $S\in\mathcal{S}_M^m$ and $S\cap Q(a,\varepsilon)\neq\emptyset$. Then $\pi(S)\cap Q(\pi(a),\varepsilon)\neq\emptyset$. Hence, by the induction hypothesis, $T := \pi(P(a,\varepsilon))\cap\pi(S)$ is an $M$-Lipschitz cell in $R^{m-1}$. We distinguish three cases.

(I) $S=[f_1,f_2]=\{y=(y',y_m) : y'\in\pi(S), f_1(y')\leq y_m\leq f_2(y')\}$, where $y'=(y_1,\dots,y_{m-1})$ and $f_i:\pi(S)\to R$ $(i=1,2)$ are $M$-Lipschitz. By assumption, there exists $u=(u',u_m)\in S\cap Q(a,\varepsilon)$. Then $u'\in T$, $f_1(u')\leq u_m\leq f_2(u')$ and $a_m-l_mM^{m-1}\varepsilon\leq u_m\leq a_m+l_mM^{m-1}\varepsilon$. Thus, for any $y'\in T$,

$$\begin{aligned}f_1(y') &\leq f_1(u')+M|y'-u'|\leq u_m+M\operatorname{diam}\pi(P(a,\varepsilon))\\ &\leq a_m+l_mM^{m-1}\varepsilon+M\cdot 2k_{m-1}M^{m-2}\varepsilon = a_m+k_mM^{m-1}\varepsilon,\end{aligned}$$

and similarly

$$\begin{aligned}f_2(y') &\geq f_2(u')-M|y'-u'|\geq u_m-M\operatorname{diam}\pi(P(a,\varepsilon))\\ &\geq a_m-l_mM^{m-1}\varepsilon-M\cdot 2k_{m-1}M^{m-2}\varepsilon = a_m-k_mM^{m-1}\varepsilon.\end{aligned}$$

Consequently,

$$\begin{aligned}S\cap P(a,\varepsilon) = \{(y',y_m) : y'\in T,&\\ \max(f_1(y'), a_m-k_mM^{m-1}\varepsilon)\leq y_m&\leq\min(f_2(y'), a_m+k_mM^{m-1}\varepsilon)\}\end{aligned}$$

is an $M$-Lipschitz cell.

In the cases (II) $S=[f,+\infty)$ and (III) $S=(-\infty,f]$ we argue in a similar way. ∎

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