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Monoidal semifilters and arrays of prime ideals

by

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Abstract. Let R be a commutative ring. If $A \subseteq R$ is an ideal and $\mathcal{F}$ is a monoidal semifilter of ideals in R, we say that a prime ideal P is a realization of $(A, \mathcal{F})$ if $P \supseteq A$ and $P \notin \mathcal{F}$. We give "if and only if" conditions for the existence of a realization of a family $\{(A_t, \mathcal{F}_t)\}_{t\in T}$ of such pairs indexed by a finite rooted tree T. We also apply our results to trees of prime ideals outside a given monoidal semifilter in a tensor product of algebras.

1. Introduction. Let R be a commutative ring with identity. We know that multiplicatively closed sets are among the most basic tools in commutative algebra. Our purpose in this paper is to show that, in certain situations, the role of multiplicatively closed sets can be played by "monoidal semifilters" considered by Lam and Reyes [7, §1].

DEFINITION 1.1. A non-empty family $\mathcal{F}$ of ideals in R is said to be a *monoidal semifilter* if:

(a) $R \in \mathcal{F}$,
(b) $\mathcal{F}$ is monoidal, i.e., given $I_1, I_2 \in \mathcal{F}$, we have $I_1 I_2 \in \mathcal{F}$, and
(c) $\mathcal{F}$ is a semifilter, i.e., given $I \in \mathcal{F}$, we have $J \in \mathcal{F}$ for any ideal $J \supseteq I$.

We will mostly be concerned with prime ideals lying outside a given monoidal semifilter. In particular, if $A \subseteq R$ is an ideal and $\mathcal{F}$ is a family of ideals that is a monoidal semifilter, we will say that a prime ideal P *realizes the pair* $(A, \mathcal{F})$ if $P \supseteq A$ and $P \notin \mathcal{F}$. We prove (see Theorem 1) that every non-empty chain of ideals in the complement $\mathcal{F}^c$ of $\mathcal{F}$ has an upper bound in $\mathcal{F}^c$ if and only if every ideal $J \in \mathcal{F}$ contains a finitely generated ideal $I \subseteq J$ such that $I \in \mathcal{F}$. In that case, it follows from the above and the results of [7] that if A is any ideal such that $A \notin \mathcal{F}$, then there exists a prime ideal P realizing the pair $(A, \mathcal{F})$.

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We also consider systems $\{(A_t, \mathcal{F}_t)\}_{t\in T}$ of such pairs indexed by a finite directed rooted tree T. We define when such a system is realized by a system $\{P_t\}_{t\in T}$ of prime ideals (see Definition 2.2), and for any chosen node $t_0 \in T$, we prove that the existence of such a realization depends on the realizations of a certain single pair $(B_{t_0}, \mathcal{G}_{t_0})$. Here, the ideal B_{t_0} and the monoidal semifilter $\mathcal{G}_{t_0}$ are obtained by means of four simple operations involving the ideals $\{A_t\}_{t\in T}$ and the monoidal semifilters $\{\mathcal{F}_t\}_{t\in T}$ (see Theorem 2). Further, given monoidal semifilters $\mathcal{F}_{(0)}$ and $\mathcal{F}_{(1)}$ respectively in commutative algebras $R_{(0)}$ and $R_{(1)}$ over a field k, we define a product monoidal semifilter $\mathcal{F}_{(0)} \otimes \mathcal{F}_{(1)}$ of ideals in $R_{(0)} \otimes_k R_{(1)}$. Let $\{P_{0t}\}_{t\in T}$ and $\{P_{1t}\}_{t\in T}$ be finite descending trees of prime ideals in $R_{(0)}$ and $R_{(1)}$ respectively. Suppose that the tree $\{P_{0t}\}_{t\in T}$ lies outside the monoidal semifilter $\mathcal{F}_{(0)}$ and the tree $\{P_{1t}\}_{t\in T}$ lies outside $\mathcal{F}_{(1)}$. Then we show (see Theorem 3) that there is a tree of prime ideals $\{P_t\}_{t\in T}$ in $R_{(0)} \otimes_k R_{(1)}$ lying outside the monoidal semifilter $\mathcal{F}_{(0)} \otimes \mathcal{F}_{(1)}$ that lifts both trees $\{P_{0t}\}_{t\in T}$ and $\{P_{1t}\}_{t\in T}$. We also prove (see Theorem 4) an inequality giving a lower bound for the maximum possible length of a chain of prime ideals of $R_{(0)} \otimes_k R_{(1)}$ lying in the complement $(\mathcal{F}_{(0)} \otimes \mathcal{F}_{(1)})^c$ of the monoidal semifilter $\mathcal{F}_{(0)} \otimes \mathcal{F}_{(1)}$. This generalizes the well known inequality $\dim(R_{(0)} \otimes_k R_{(1)}) \geq \dim(R_{(0)}) + \dim(R_{(1)})$ for the Krull dimension of a tensor product of commutative algebras over a field k.

We now describe the paper in more detail. First, we recall the following well known result (see, for instance, [6, Proposition 7.3] or [9, Theorem 2.1]).

Remark 1.2. Let $S \subseteq R$ be a multiplicatively closed subset containing 1. Then:

(a) Any ideal I in R that is maximal with respect to being disjoint from S is prime.
(b) Given an ideal $A \subseteq R$ such that $A \cap S = \emptyset$, there exists an ideal I containing A such that I is maximal with respect to being disjoint from S.
(c) Hence, given an ideal $A \subseteq R$ such that $A \cap S = \emptyset$, there exists a prime ideal P in R such that $P \supseteq A$ and $P \cap S = \emptyset$.

There are several other results in commutative algebra of a nature similar to Remark 1.2(a). For instance, a famous result of Cohen [5] states that any ideal that is maximal with respect to not being finitely generated must be prime. In [7], Lam and Reyes unified such results in a "Prime Ideal Principle" by giving several conditions on a family $\mathcal{F}$ of ideals such that any ideal I that is maximal with respect to not being in $\mathcal{F}$ must be prime (see also further work in Lam and Reyes [8] and Reyes [10]–[12]). As part of their general Prime Ideal Principle, Lam and Reyes showed in [7, Theorem 2.7] that if $\mathcal{F}$ is a monoidal semifilter (see Definition 1.1) and I is any ideal that is maximal with respect to not being in $\mathcal{F}$, then I must be prime. In particular, for a multiplicatively closed subset $S \subseteq R$, the family

(1.1) $$\mathcal{G}_S := \{I \mid I \text{ is an ideal and } I \cap S \neq \emptyset\}$$

of ideals turns out to be a monoidal semifilter. Therefore, we consider the situation of Remark 1.2 with a monoidal semifilter $\mathcal{F}$ in place of a multiplicatively closed set and an ideal A such that $A \notin \mathcal{F}$. Because of the above-mentioned result of [7], our first aim in this paper is to give conditions on $\mathcal{F}$ such that for any ideal $A \notin \mathcal{F}$, there actually exists an ideal I containing A that is maximal with respect to not being in $\mathcal{F}$. We prove the following result.

THEOREM 1. *Let $\mathcal{F}$ be a monoidal semifilter of ideals in R and let $\mathcal{F}^c$ denote the set of all ideals in R which do not belong to $\mathcal{F}$. Then the following statements are equivalent:*

(1) *Every non-empty chain of ideals in $\mathcal{F}^c$ has an upper bound in $\mathcal{F}^c$.*
(2) *Given any ideal $J \in \mathcal{F}$, there exists a finitely generated ideal I such that $I \subseteq J$ and $I \in \mathcal{F}$.*

It now follows that if $\mathcal{F}$ is a monoidal semifilter satisfying the conditions in Theorem 1 and A is an ideal, then there exists a prime ideal P such that $P \supseteq A$ and $P \notin \mathcal{F}$ if and only if $A \notin \mathcal{F}$. As mentioned before, we will refer to such a prime ideal P as a "realization" of the pair $(A, \mathcal{F})$. This is similar to the terminology of Bergman [3]: if A is an ideal and $S \subseteq R$ is a multiplicatively closed subset, Bergman [3] refers to a prime ideal P such that $P \supseteq A$ and $P \cap S = \emptyset$ as a "realization" of the pair (A, S). Further, instead of a single pair $(A, \mathcal{F})$, we consider a template $\tau = \{(A_t, \mathcal{F}_t)\}_{t \in T}$ of pairs indexed by a partially ordered set $(T, \leq)$. In particular, when T is a finite directed rooted tree (ascending or descending), we adapt the methods of Bergman [3] to give "if and only if" conditions for the existence of a family $\{P_t\}_{t \in T}$ of prime ideals realizing $\tau = \{(A_t, \mathcal{F}_t)\}_{t \in T}$. In other words, each prime P_t realizes the pair $(A_t, \mathcal{F}_t)$ and $P_t \subseteq P_{t'}$ for each $t \leq t'$ in T.

More precisely, throughout we consider pairs $(A, \mathcal{F})$ where A is an ideal and $\mathcal{F}$ is a monoidal semifilter such that any non-empty chain of ideals in $\mathcal{F}^c$ has an upper bound in $\mathcal{F}^c$. Given two such monoidal semifilters $\mathcal{F}_1$ and $\mathcal{F}_2$, we define a product $\mathcal{F}_1 \cdot \mathcal{F}_2$ and show that any non-empty chain of ideals in $(\mathcal{F}_1 \cdot \mathcal{F}_2)^c$ has an upper bound in $(\mathcal{F}_1 \cdot \mathcal{F}_2)^c$. Thereafter, we make two further constructions; the first is that of an ideal:

(1.2) $$A \div \mathcal{F} := \{r \in R \mid rI \subseteq A \text{ for some } I \in \mathcal{F}\}.$$

For the second construction, we take the quotient map $q_A : R \to R/A$ and consider the family of ideals

(1.3)
$$\mathcal{T}(A, \mathcal{F}) := \{J \mid J \text{ an ideal such that } q_A(J) = q_A(I) \text{ for some } I \in \mathcal{F}\}.$$

We then show that $\mathcal{T}(A,\mathcal{F})$ is also a monoidal semifilter and that any nonempty chain of ideals in $\mathcal{T}(A,\mathcal{F})^c$ has an upper bound in $\mathcal{T}(A,\mathcal{F})^c$. Given a finite chain template $\tau = \{(A_i,\mathcal{F}_i)\}_{1\le i\le n}$, we use the constructions in (1.2) and (1.3) to build a template $D(\tau) = \{(B_i,\mathcal{G}_i)\}_{1\le i\le n}$ with the following property: for any chosen $j \in \{1,\dots,n\}$ and given a prime P realizing $(B_j,\mathcal{G}_j)$, there exists a realization $P_1 \subseteq \cdots \subseteq P_n$ of τ such that $P_j = P$. Further, a chain $P_1 \subseteq \cdots \subseteq P_n$ of prime ideals realizes $\tau = \{(A_i,\mathcal{F}_i)\}_{1\le i\le n}$ if and only if it realizes $D(\tau) = \{(B_i,\mathcal{G}_i)\}_{1\le i\le n}$. It follows that a necessary and sufficient condition for the existence of a realization of $\tau = \{(A_i,\mathcal{F}_i)\}_{1\le i\le n}$ is that we can find some $j \in \{1,\dots,n\}$ with $B_j \notin \mathcal{G}_j$. Thereafter, we prove the following analogous result for templates indexed by finite trees T.

THEOREM 2. *Let T be a finite directed rooted tree and let $\tau = \{(A_t,\mathcal{F}_t)\}_{t\in T}$ be a collection of pairs indexed by T. Then, for any chosen $t_0 \in T$, we can form a pair $(B_{t_0},\mathcal{G}_{t_0})$ having the following property: if there exists a prime ideal P realizing the pair $(B_{t_0},\mathcal{G}_{t_0})$, then there exists a family $\{P_t\}_{t\in T}$ of prime ideals realizing the template $\tau = \{(A_t,\mathcal{F}_t)\}_{t\in T}$ such that $P_{t_0} = P$. Further, B_{t_0} and $\mathcal{G}_{t_0}$ can be expressed in terms of the ideals $\{A_t\}_{t\in T}$ and the monoidal semifilters $\{\mathcal{F}_t\}_{t\in T}$ using the following four operations:*

(a) *Forming the sum of ideals in R.*
(b) *Forming the product $\mathcal{F}_1\cdot\mathcal{F}_2$ of two monoidal semifilters $\mathcal{F}_1$ and $\mathcal{F}_2$.*
(c) *Forming the ideal $A \div \mathcal{F}$ from an ideal A and a monoidal semifilter $\mathcal{F}$.*
(d) *Forming the monoidal semifilter $\mathcal{T}(A,\mathcal{F})$ from an ideal A and a monoidal semifilter $\mathcal{F}$.*

In [3], Bergman has applied the study of realizations of pairs (A,S) to prime ideals in tensor products of algebras. If $R_{(0)}$ and $R_{(1)}$ are algebras over a field k, this includes a new proof of the fact that the Krull dimensions satisfy $\dim(R_{(0)}) + \dim(R_{(1)}) \le \dim(R_{(0)} \otimes_k R_{(1)})$ (see also related work by Brenner [4] and Sharma [13], [14]). In Section 3, we apply our formalism of realizations of pairs $(A,\mathcal{F})$ and adapt the methods of Bergman [3, §7] to study prime ideals in tensor products outside a given monoidal semifilter. More precisely, if $\mathcal{F}_{(0)}$ is a monoidal semifilter in $R_{(0)}$, and $\mathcal{F}_{(1)}$ is a monoidal semifilter in $R_{(1)}$, then we construct the following monoidal semifilter $\mathcal{F} = \mathcal{F}_{(0)} \otimes \mathcal{F}_{(1)}$ of ideals in $R := R_{(0)} \otimes_k R_{(1)}$:

$$(1.4)\qquad \mathcal{F}_{(0)} \otimes \mathcal{F}_{(1)} = \{I \mid I \supseteq I_{(0)} \otimes_k I_{(1)} \text{ for some } I_{(0)} \in \mathcal{F}_{(0)},\ I_{(1)} \in \mathcal{F}_{(1)}\}.$$

We show that if $\mathcal{F}_{(0)}$ and $\mathcal{F}_{(1)}$ satisfy the equivalent conditions in Theorem 1, so does $\mathcal{F} = \mathcal{F}_{(0)} \otimes \mathcal{F}_{(1)}$. Then we prove the following result.

THEOREM 3. *Let T be a finite descending tree and let $\{P_{0t}\}_{t\in T}$ and $\{P_{1t}\}_{t\in T}$ be trees of prime ideals in $R_{(0)}$ and $R_{(1)}$ respectively such that $P_{0t} \notin \mathcal{F}_{(0)}$ and $P_{1t} \notin \mathcal{F}_{(1)}$ for all $t \in T$. Then there exists a finite descending tree*

$\{P_t\}_{t\in T}$ *of prime ideals in* $R = R_{(0)} \otimes_k R_{(1)}$ *such that* $P_t \notin \mathcal{F} = \mathcal{F}_{(0)} \otimes \mathcal{F}_{(1)}$ *and* $P_t \cap R_{(0)} = P_{0t}$, $P_t \cap R_{(1)} = P_{1t}$ *for all* t.

Finally, we apply these methods with descending trees to finite chains of prime ideals. For any subset $X \subseteq \mathrm{Spec}(R)$, we denote by $\dim(X)$ the supremum of the lengths of chains of prime ideals in X. Then we conclude with the following result.

THEOREM 4. *Let* $\mathcal{F}_{(0)}$ *and* $\mathcal{F}_{(1)}$ *be monoidal semifilters in* $R_{(0)}$ *and* $R_{(1)}$ *respectively and define* $\mathrm{Spec}(\mathcal{F}^c_{(0)}) := \mathcal{F}^c_{(0)} \cap \mathrm{Spec}(R)$ *and* $\mathrm{Spec}(\mathcal{F}^c_{(1)}) := \mathcal{F}^c_{(1)} \cap \mathrm{Spec}(R)$. *Then*

$$\dim(\mathrm{Spec}((\mathcal{F}_{(0)} \otimes \mathcal{F}_{(1)})^c)) \geq \dim(\mathrm{Spec}(\mathcal{F}^c_{(0)})) + \dim(\mathrm{Spec}(\mathcal{F}^c_{(1)})). \tag{1.5}$$

We mention here that the idea of developing realizations of templates with monoidal semifilters first came to us while doing the same with tensor triangulated ideals in [2]. The study of "prime ideals" in tensor triangulated categories as developed by Balmer [1] has certain formal similarities with commutative algebra, but the analogy can also be counter-intuitive at times. We also note that Reyes [10]–[12] has extended the Prime Ideal Principle of [7] to ideals (one-sided or two-sided) in non-commutative rings. Accordingly, we hope that we can further develop our results in the non-commutative setting in the future.

2. Monoidal semifilters. Throughout, we will use the notation $I \triangleleft R$ to mean that I is an ideal in R (this includes the possibility that $I = R$). If $S \subseteq R$ is a multiplicatively closed subset containing 1 but not 0, we notice that the monoidal semifilter $\mathcal{G}_S := \{I \triangleleft R \mid I \cap S \neq \emptyset\}$ has an additional property, i.e., any ideal $A \notin \mathcal{G}_S$ must be contained in some ideal that is maximal with respect to not being in $\mathcal{G}_S$. We will now characterize monoidal semifilters $\mathcal{F}$ such that every non-empty chain of ideals in $\mathcal{F}^c$ has an upper bound in $\mathcal{F}^c$. For any ideal $I \triangleleft R$, we will denote by $\mu(I)$ the smallest possible cardinality for a generating set of the ideal I.

THEOREM 1. *Let* $\mathcal{F}$ *be a monoidal semifilter of ideals in* R *and let* $\mathcal{F}^c$ *denote the set of all ideals in* R *which do not belong to* $\mathcal{F}$. *Then the following statements are equivalent:*

(1) *Every non-empty chain of ideals in* $\mathcal{F}^c$ *has an upper bound in* $\mathcal{F}^c$.
(2) *Given any ideal* $J \in \mathcal{F}$, *there exists a finitely generated ideal* I *such that* $I \subseteq J$ *and* $I \in \mathcal{F}$.

Proof. (1)⇒(2): We choose an ideal $J \in \mathcal{F}$ and consider the following collection of ideals:

$$\mathcal{F}(J) := \{I \triangleleft R \mid I \subseteq J \text{ and } I \in \mathcal{F}\}. \tag{2.1}$$

We now choose an ideal $K \in \mathcal{F}(J)$ such that $\mu(K)$ is least in the set of cardinalities $\{\mu(I) \mid I \in \mathcal{F}(J)\}$. Let S be a generating set for the ideal K such that $|S| = \mu(K)$. If $\mu(K)$ is already finite, we are done.

Otherwise, set $\kappa := \mu(K)$. Then we have a bijection $f : \kappa \to S$. We know that $\kappa = \bigcup_{\alpha<\kappa} \alpha$, and for each ordinal $\alpha < \kappa$, we set $S_\alpha := \mathrm{Im}(f|\alpha) \subseteq S$. Clearly, $S = \bigcup_{\alpha<\kappa} S_\alpha$. For each $\alpha < \kappa$, we let $K_\alpha \subseteq K$ be the ideal generated by S_α. Since $S_\alpha \subseteq S_\beta$ for any $\alpha \le \beta < \kappa$, it is clear that $\{K_\alpha\}_{\alpha<\kappa}$ is a chain of ideals and $K = \bigcup_{\alpha<\kappa} K_\alpha$.

On the other hand, since each K_α is generated by S_α, it follows that $\mu(K_\alpha) \le |S_\alpha| < \kappa = \mu(K)$. Since $K_\alpha \subseteq K \subseteq J$, it follows from the definition of K that K_α cannot lie in $\mathcal{F}$, i.e., $K_\alpha \in \mathcal{F}^c$. Since $\{K_\alpha\}_{\alpha<\kappa}$ is a chain of ideals in $\mathcal{F}^c$, it has an upper bound in $\mathcal{F}^c$, say $L \in \mathcal{F}^c$. Then $L \supseteq \bigcup_{\alpha<\kappa} K_\alpha = K$. However, $\mathcal{F}$ being a semifilter, $K \in \mathcal{F}$ implies that $L \in \mathcal{F}$, which is a contradiction.

(2)⇒(1): Let $\{K_t\}_{t\in T}$ be a chain of ideals in $\mathcal{F}^c$ indexed by the totally ordered set T. We claim that $K := \bigcup_{t\in T} K_t$ lies in $\mathcal{F}^c$. Indeed, if $K \in \mathcal{F}$, we can find some finitely generated ideal $I \in \mathcal{F}$ such that $\bigcup_{t\in T} K_t = K \supseteq I$. However, since I is finitely generated, we must have some $t' \in T$ such that $K_{t'} \supseteq I$. Then, $\mathcal{F}$ being a semifilter, we have $K_{t'} \in \mathcal{F}$, which is a contradiction. ■

If the ring R itself is Noetherian, it is clear that the property in Theorem 1 is satisfied. It is also easy to see that if every ideal in $\mathcal{F}$ is finitely generated, then every non-empty chain in $\mathcal{F}^c$ has an upper bound in $\mathcal{F}^c$. More generally, if $\mathcal{J} = \{J_t\}_{t\in T}$ is a family of finitely generated ideals and we let $\mathcal{F}$ denote the family of all ideals containing a finite product of the ideals in $\mathcal{J}$, then the monoidal semifilter $\mathcal{F}$ has this property.

Proposition 2.1. *Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be monoidal semifilters in R. Consider the following family of ideals:*

$$\mathcal{F}_1 \cdot \mathcal{F}_2 := \{I \lhd R \mid I \supseteq I_1 I_2 \text{ for some } I_1 \in \mathcal{F}_1 \text{ and some } I_2 \in \mathcal{F}_2\}. \tag{2.2}$$

Then:

(a) *The family $\mathcal{F}_1 \cdot \mathcal{F}_2$ is a monoidal semifilter. Further, $\mathcal{F}_1 \cdot \mathcal{F}_2 \supseteq \mathcal{F}_1$ and $\mathcal{F}_1 \cdot \mathcal{F}_2 \supseteq \mathcal{F}_2$.*

(b) *Suppose that any non-empty chain of ideals in $\mathcal{F}_1^c$ (resp. in $\mathcal{F}_2^c$) has an upper bound in $\mathcal{F}_1^c$ (resp. in $\mathcal{F}_2^c$). Then any non-empty chain of ideals in $(\mathcal{F}_1 \cdot \mathcal{F}_2)^c$ has an upper bound in $(\mathcal{F}_1 \cdot \mathcal{F}_2)^c$.*

Proof. (a) We set $\mathcal{F} := \mathcal{F}_1 \cdot \mathcal{F}_2$. From (2.2), it is immediately clear that $\mathcal{F}$ is a semifilter. Further, if $I \supseteq I_1 I_2$ and $J \supseteq J_1 J_2$ with $I_1, J_1 \in \mathcal{F}_1$ and $I_2, J_2 \in \mathcal{F}_2$, we have $IJ \supseteq (I_1 J_1)(I_2 J_2)$. Since $\mathcal{F}_1$ and $\mathcal{F}_2$ are monoidal, it is now clear that so is $\mathcal{F}$. Finally, since $R \in \mathcal{F}_1$ and $R \in \mathcal{F}_2$, we must have $\mathcal{F}_1 \cdot \mathcal{F}_2 \supseteq \mathcal{F}_1$ and $\mathcal{F}_1 \cdot \mathcal{F}_2 \supseteq \mathcal{F}_2$.

(b) We consider $I \in \mathcal{F}$ with $I \supseteq I_1 I_2$ where $I_1 \in \mathcal{F}_1$ and $I_2 \in \mathcal{F}_2$. From Theorem 1, we know that there exist finitely generated ideals $J_1 \subseteq I_1$ and $J_2 \subseteq I_2$ with $J_1 \in \mathcal{F}_1$ and $J_2 \in \mathcal{F}_2$. Then it is clear that $I \supseteq J_1 J_2$. Hence, in this case, we can replace (2.2) with the equivalent definition:

$$(2.3) \quad \mathcal{F}_1 \cdot \mathcal{F}_2 = \{I \triangleleft R \mid I \supseteq I_1' I_2' \text{ for some f.g. ideals } I_1' \in \mathcal{F}_1,\ I_2' \in \mathcal{F}_2\}.$$

Since $I_1' I_2' \in \mathcal{F}$ for any $I_1' \in \mathcal{F}_1$ and $I_2' \in \mathcal{F}_2$ and the product of finitely generated ideals is finitely generated, it now follows from Theorem 1 that any non-empty chain of ideals in $(\mathcal{F}_1 \cdot \mathcal{F}_2)^c$ has an upper bound in $(\mathcal{F}_1 \cdot \mathcal{F}_2)^c$. ■

We now want to study conditions for the existence of a family $\{P_t\}_{t\in T}$ of prime ideals that "realizes" a template $\tau = \{(A_t, \mathcal{F}_t)\}_{t\in T}$ of pairs indexed by a partially ordered set T.

DEFINITION 2.2. (a) Let $\mathcal{F}$ be a monoidal semifilter of ideals in R satisfying condition (1) of Theorem 1, and A be an ideal in R. We say that a prime ideal P *realizes the pair* $(A, \mathcal{F})$ if $P \supseteq A$ and $P \notin \mathcal{F}$.

(b) Let $\tau = \{(A_t, \mathcal{F}_t)\}_{t\in T}$ be a family of pairs as in (a), indexed by a partially ordered set $(T, \leq)$. We say that a family $\{P_t\}_{t\in T}$ of prime ideals is a *realization of the template* $\tau = \{(A_t, \mathcal{F}_t)\}_{t\in T}$ if P_t realizes the pair $(A_t, \mathcal{F}_t)$ for each $t \in T$ and $\mathcal{P}_s \subseteq \mathcal{P}_t$ for each $s \leq t$ in T.

From now onward, we will only consider pairs $(A, \mathcal{F})$ as described in Definition 2.2(a).

COROLLARY 2.3. *A pair* $(A, \mathcal{F})$ *is realizable if and only if* $A \notin \mathcal{F}$.

Proof. Suppose there exists a prime ideal P such that $A \subseteq P$ and $P \notin \mathcal{F}$. Then, since $\mathcal{F}$ is a semifilter, we must have $A \notin \mathcal{F}$. Conversely, if $A \notin \mathcal{F}$, we consider the collection $\mathcal{F}^c_{\supseteq A}$ of all ideals I such that $I \supseteq A$ and $I \notin \mathcal{F}$. Since the pair $(A, \mathcal{F})$ satisfies condition (1) of Theorem 1, we can choose a maximal element in $\mathcal{F}^c_{\supseteq A}$, say P. Then, since $\mathcal{F}$ is a monoidal semifilter, it follows from [7, Theorem 2.7] that P must be prime, and hence P is a realization of the pair $(A, \mathcal{F})$. ■

PROPOSITION 2.4. *Let* $(A_1, \mathcal{F}_1)$ *and* $(A_2, \mathcal{F}_2)$ *be two pairs. Then a prime ideal* P *of* R *realizes both* $(A_1, \mathcal{F}_1)$ *and* $(A_2, \mathcal{F}_2)$ *if and only if it realizes* $(A_1 + A_2, \mathcal{F}_1 \cdot \mathcal{F}_2)$.

Proof. From Proposition 2.1, we know that any non-empty chain of ideals in $(\mathcal{F}_1 \cdot \mathcal{F}_2)^c$ has an upper bound in $(\mathcal{F}_1 \cdot \mathcal{F}_2)^c$. If a prime ideal P is in $\mathcal{F}_1 \cdot \mathcal{F}_2$, then it follows from (2.2) that there exist $I_1 \in \mathcal{F}_1$ and $I_2 \in \mathcal{F}_2$ such that $P \supseteq I_1 I_2$. Then either $P \supseteq I_1$ or $P \supseteq I_2$, and $\mathcal{F}_1$ and $\mathcal{F}_2$ being semifilters, we have either $P \in \mathcal{F}_1$ or $P \in \mathcal{F}_2$. Consequently, $P \notin \mathcal{F}_1 \cdot \mathcal{F}_2$ if and only if $P \notin \mathcal{F}_1$ and $P \notin \mathcal{F}_2$. This proves the result. ■

Given a pair $(A, \mathcal{F})$, we now consider

$$A \div \mathcal{F} := \{r \in R \mid rI \subseteq A \text{ for some } I \in \mathcal{F}\}. \tag{2.4}$$

Since $\mathcal{F}$ is monoidal, it is easy to see that $A \div \mathcal{F}$ is an ideal. We will also construct a monoidal semifilter $\mathcal{T}(A, \mathcal{F})$ from the pair $(A, \mathcal{F})$. For this, we consider the quotient map $q_A : R \to R/A$. It is clear that the image $q_A(I)$ of any ideal $I \triangleleft R$ is an ideal in R/A. Then we set

$$\mathcal{T}(A, \mathcal{F}) := \{J \triangleleft R \mid q_A(J) = q_A(I) \text{ for some } I \in \mathcal{F}\}, \tag{2.5}$$

which is a family of ideals containing $\mathcal{F}$. We note here that the family $\mathcal{T}(A, \mathcal{F})$ may equivalently be described as

$$\begin{aligned} \mathcal{T}(A, \mathcal{F}) &= \{J \triangleleft R \mid J + A = I + A \text{ for some } I \in \mathcal{F}\} \\ &= \{J \triangleleft R \mid J + A \in \mathcal{F}\}. \end{aligned} \tag{2.6}$$

We now show that $\mathcal{T}(A, \mathcal{F})$ is a monoidal semifilter that also satisfies condition (1) in Theorem 1.

LEMMA 2.5. *Let $(A, \mathcal{F})$ be a pair. Then the family $\mathcal{T}(A, \mathcal{F})$ of ideals is a monoidal semifilter. Further, any non-empty chain of ideals in $\mathcal{T}(A, \mathcal{F})^c$ has an upper bound in $\mathcal{T}(A, \mathcal{F})^c$.*

Proof. It is clear that $R \in \mathcal{T}(A, \mathcal{F})$. Also, since $q_A(I_1 I_2) = q_A(I_1) q_A(I_2)$ for any ideals I_1, $I_2 \triangleleft R$, it follows that $\mathcal{T}(A, \mathcal{F})$ is monoidal. Now, suppose that $J \triangleleft R$ and $I \in \mathcal{F}$ are such that $q_A(J) \supseteq q_A(I)$. Then $q_A(J) = q_A(I + J)$. Since $\mathcal{F}$ is a semifilter and $I \in \mathcal{F}$, we must have $I + J \in \mathcal{F}$. Hence, J is in $\mathcal{T}(A, \mathcal{F})$. Therefore, we can replace the expression in (2.5) with the following equivalent definition:

$$\mathcal{T}(A, \mathcal{F}) := \{J \triangleleft R \mid q_A(J) \supseteq q_A(I) \text{ for some } I \in \mathcal{F}\}. \tag{2.7}$$

From (2.7), it is immediate that $\mathcal{T}(A, \mathcal{F})$ is a semifilter.

Finally, we consider a chain $\{K_t\}_{t \in T}$ of ideals in $\mathcal{T}(A, \mathcal{F})^c$, but suppose that $K = \bigcup_{t \in T} K_t$ lies in $\mathcal{T}(A, \mathcal{F})$. From (2.7), there exists $I \in \mathcal{F}$ such that $q_A(K) \supseteq q_A(I)$. Further, from Theorem 1, there exists a finitely generated ideal $I' \subseteq I$ such that $I' \in \mathcal{F}$. Then $\bigcup_{t \in T} q_A(K_t) = q_A(K) \supseteq q_A(I) \supseteq q_A(I')$. Since I' is finitely generated, so is $q_A(I')$. Hence, there must exist $t_0 \in T$ such that $q_A(K_{t_0}) \supseteq q_A(I')$. Since $I' \in \mathcal{F}$, it now follows from (2.7) that $K_{t_0} \in \mathcal{T}(A, \mathcal{F})$, which is a contradiction. Hence, $K \in \mathcal{T}(A, \mathcal{F})^c$. ∎

If $S \subseteq R$ is a multiplicatively closed subset containing 1, we have mentioned before that $\mathcal{G}_S := \{I \triangleleft R \mid I \cap S \neq \emptyset\}$ is a monoidal semifilter and any non-empty chain of ideals in $\mathcal{G}_S^c$ has an upper bound in $\mathcal{G}_S^c$. Additionally, for any ideal A in R, it is easy to see that the ideal $A \div \mathcal{G}_S$ may be expressed as

$$A \div \mathcal{G}_S = A \div S := \{r \in R \mid rs \in A \text{ for some } s \in S\}. \tag{2.8}$$

Further, if $\mathcal{F}$ is any monoidal semifilter, we can check that

$$\mathcal{G}_S \cdot \mathcal{F} = \{I \lhd R \mid I \supseteq sJ \text{ for some } s \in S \text{ and } J \in \mathcal{F}\}. \tag{2.9}$$

The next result will be key in constructing "if and only if" conditions for the existence of realizations of tree templates of pairs.

Proposition 2.6. *Let $(A, \mathcal{F})$ be a pair and let Q be a prime ideal. Then:*

(a) *Q contains a prime ideal P realizing $(A, \mathcal{F})$ if and only if Q contains the ideal $A \div \mathcal{F}$.*
(b) *Q is contained in a prime ideal P realizing $(A, \mathcal{F})$ if and only if $Q \notin \mathcal{T}(A, \mathcal{F})$.*

Proof. (a) Suppose that $Q \supseteq A \div \mathcal{F}$. We set $S := R - Q$ and consider the monoidal semifilter $\mathcal{G}_S$ as defined in (1.1). Then, from (1.1), (2.4) and (2.9), it follows that $A \div \mathcal{F} \subseteq Q$ implies $A \notin \mathcal{G}_S \cdot \mathcal{F}$. Accordingly, we can choose a prime ideal P realizing $(A, \mathcal{G}_S \cdot \mathcal{F})$. Then, in particular, P realizes $(A, \mathcal{F})$ and $P \cap S = \emptyset$, i.e., $P \subseteq Q$. Conversely, suppose that Q contains a prime ideal P realizing $(A, \mathcal{F})$ and choose some $r \in A \div \mathcal{F}$. Then $rI \subseteq A \subseteq P$ for some $I \in \mathcal{F}$. Since $\mathcal{F}$ is a semifilter and $P \notin \mathcal{F}$, we must have $I \not\subseteq P$, and thus $r \in P$. Hence, $A \div \mathcal{F} \subseteq P \subseteq Q$.

(b) Suppose that $Q \notin \mathcal{T}(A, \mathcal{F})$. We claim that $Q + A \notin \mathcal{T}(A, \mathcal{F})$. Indeed, if $q_A : R \to R/A$ is the quotient map, it is clear that $q_A(Q + A) = q_A(Q)$, and hence $q_A(Q + A) \neq q_A(I)$ for any $I \in \mathcal{F}$. We now choose a prime ideal P realizing $(Q + A, \mathcal{T}(A, \mathcal{F}))$. In particular, $P \supseteq Q$ and P realizes $(A, \mathcal{F})$.

Conversely, suppose that $Q \subseteq P$ for some prime ideal P realizing $(A, \mathcal{F})$. Suppose that $Q \in \mathcal{T}(A, \mathcal{F})$. Then, $\mathcal{T}(A, \mathcal{F})$ being a semifilter, we must have $P \in \mathcal{T}(A, \mathcal{F})$. Using (2.5) we can find $I \in \mathcal{F}$ such that $q_A(I) = q_A(P)$. Now consider any $x \in I$. Since $q_A(I) = q_A(P)$, we must have some $a \in A$ such that $x + a \in P$. But $A \subseteq P$, and hence $x \in P$, i.e., we must have $I \subseteq P$. Since $\mathcal{F}$ is a semifilter, it now follows that $P \in \mathcal{F}$, which contradicts P realizing $(A, \mathcal{F})$. We conclude that $Q \notin \mathcal{T}(A, \mathcal{F})$. ■

We will now describe conditions for the existence of a realization of a template that is a finite chain.

Proposition 2.7. *Let $n \geq 1$ be an integer and let $\tau = \{(A_i, \mathcal{F}_i)\}_{1 \leq i \leq n}$ be a template indexed by the finite chain $\{1 < \cdots < n\}$. Define a template $D(\tau) = \{(B_i, \mathcal{G}_i)\}_{1 \leq i \leq n}$ by setting $B_1 := A_1$, $\mathcal{G}_n := \mathcal{F}_n$ and inductively*

$$\begin{aligned} B_{i+1} &:= A_{i+1} + (B_i \div \mathcal{F}_i), \quad && 1 \leq i < n, \\ \mathcal{G}_{i-1} &:= \mathcal{F}_{i-1} \cdot \mathcal{T}(A_i, \mathcal{G}_i), \quad && n \geq i > 1. \end{aligned} \tag{2.10}$$

Then:

(a) *A chain $P_1 \subseteq \cdots \subseteq P_n$ of prime ideals realizes the template τ if and only if it realizes the template $D(\tau)$.*

(b) *Choose any* $j \in \{1, \ldots, n\}$ *and suppose there exists a prime ideal* P *realizing* $(B_j, \mathcal{G}_j)$. *Then there exists a realization* $P_1 \subseteq \cdots \subseteq P_n$ *of* $\tau = \{(A_i, \mathcal{F}_i)\}_{1 \le i \le n}$ *such that* $P_j = P$.

(c) *If we choose any* $j \in \{1, \ldots, n\}$, *a necessary and sufficient condition for the existence of a realization of* $\tau = \{(A_i, \mathcal{F}_i)\}_{1 \le i \le n}$ *is that* $B_j \notin \mathcal{G}_j$.

Proof. To begin with, we must verify that each $(B_i, \mathcal{G}_i)$ appearing in the template $D(\tau)$ is actually a pair as described in Definition 2.2(a), i.e., each $\mathcal{G}_i$ is a monoidal semifilter satisfying condition (1) of Theorem 1. By definition, we have $\mathcal{G}_n = \mathcal{F}_n$. The required fact now follows from the formula $\mathcal{G}_{i-1} = \mathcal{F}_{i-1} \cdot \mathcal{T}(A_i, \mathcal{G}_i)$ by repeatedly applying Proposition 2.1 and Lemma 2.5.

(a) From (2.10), it is clear that $A_i \subseteq B_i$ and $\mathcal{F}_i \subseteq \mathcal{G}_i$ for all i. Hence, any realization of $D(\tau) = \{(B_i, \mathcal{G}_i)\}_{1 \le i \le n}$ must be a realization of $\tau = \{(A_i, \mathcal{F}_i)\}_{1 \le i \le n}$. Conversely, suppose that $P_1 \subseteq \cdots \subseteq P_n$ is a realization of τ. We now claim that P_i realizes $(A_i, \mathcal{G}_i)$ for all i. Since $\mathcal{G}_n = \mathcal{F}_n$, this is certainly true for $i = n$. We suppose that it is true for all $i \ge j$ for some given $j \in \{1, \ldots, n\}$. Then $P_{j-1} \subseteq P_j$ and P_j realizes $(A_j, \mathcal{G}_j)$. From Proposition 2.6(b), it follows that $P_{j-1} \notin \mathcal{T}(A_j, \mathcal{G}_j)$, i.e., P_{j-1} realizes $(0, \mathcal{T}(A_j, \mathcal{G}_j)$. Combining this with the fact that P_{j-1} already realizes $(A_{j-1}, \mathcal{F}_{j-1})$, it now follows from Proposition 2.4 that P_{j-1} realizes $(A_{j-1}, \mathcal{F}_{j-1} \cdot \mathcal{T}(A_j, \mathcal{G}_j)) = (A_{j-1}, \mathcal{G}_{j-1})$. Hence, P_i realizes $(A_i, \mathcal{G}_i)$ for each $1 \le i \le n$.

Finally, we claim that P_i realizes $(B_i, \mathcal{G}_i)$ for each i. Since $B_1 = A_1$, this is certainly true for $i = 1$. We suppose that it holds for all $i \le j$ for some given $j \in \{1, \ldots, n\}$. Then $P_{j+1} \supseteq P_j$ and we know that P_j realizes $(B_j, \mathcal{G}_j)$. It follows from Proposition 2.6(a) that $P_{j+1} \supseteq B_j \div \mathcal{G}_j \supseteq B_j \div \mathcal{F}_j$. Combining this with the fact that P_{j+1} already realizes $(A_{j+1}, \mathcal{G}_{j+1})$, it follows from Proposition 2.4 that P_{j+1} realizes $(A_{j+1} + (B_j \div \mathcal{F}_j), \mathcal{G}_{j+1}) = (B_{j+1}, \mathcal{G}_{j+1})$. Hence, the chain $P_1 \subseteq \cdots \subseteq P_n$ is also a realization of $D(\tau) = \{(B_i, \mathcal{G}_i)\}_{1 \le i \le n}$.

(b) We choose some $j \in \{1, \ldots, n\}$ and consider a prime ideal $P_j := P$ realizing $(B_j, \mathcal{G}_j)$. If $j < n$, we know from (2.10) that $\mathcal{G}_j = \mathcal{F}_j \cdot \mathcal{T}(A_{j+1}, \mathcal{G}_{j+1})$. Hence, $P_j \notin \mathcal{T}(A_{j+1}, \mathcal{G}_{j+1})$ and Proposition 2.6(b) shows that we can choose a prime ideal $P_{j+1} \supseteq P_j$ such that P_{j+1} realizes $(A_{j+1}, \mathcal{G}_{j+1})$. Moreover, since $P_j \subseteq P_{j+1}$ and P_j realizes $(B_j, \mathcal{G}_j)$, it follows from Proposition 2.6(a) that $P_{j+1} \supseteq B_j \div \mathcal{G}_j \supseteq B_j \div \mathcal{F}_j$. Using Proposition 2.4, we now see that P_{j+1} realizes $(A_{j+1} + (B_j \div \mathcal{F}_j), \mathcal{G}_{j+1}) = (B_{j+1}, \mathcal{G}_{j+1})$.

On the other hand, if also $j > 1$, we see that $P_j \supseteq B_j \supseteq B_{j-1} \div \mathcal{F}_{j-1}$, so Proposition 2.6(a) shows that we can choose a prime ideal $P_{j-1} \subseteq P_j$ realizing $(B_{j-1}, \mathcal{F}_{j-1})$. On the other hand, since $P_{j-1} \subseteq P_j$ and P_j realizes $(A_j, \mathcal{G}_j)$ (note that $A_j \subseteq B_j$), it follows from Proposition 2.6(b) that $P_{j-1} \notin \mathcal{T}(A_j, \mathcal{G}_j)$. Combining this with the fact that $P_{j-1} \notin \mathcal{F}_{j-1}$, we find that

$P_{j-1} \notin \mathcal{F}_{j-1} \cdot \mathcal{T}(A_j, \mathcal{G}_j) = \mathcal{G}_{j-1}$. We have already seen that $P_{j-1} \supseteq B_{j-1}$, and hence P_{j-1} realizes $(B_{j-1}, \mathcal{G}_{j-1})$. Accordingly, starting with any realization P_j of $(B_j, \mathcal{G}_j)$, we can proceed in both directions to obtain a realization $P_1 \subseteq \cdots \subseteq P_n$ of the template $D(\tau) = \{(B_i, \mathcal{G}_i)\}_{1 \leq i \leq n}$. From part (a), it follows that this is also a realization of $\tau = \{(A_i, \mathcal{F}_i)\}_{1 \leq i \leq n}$. The result of (c) follows by combining (a) and (b). ∎

We will now generalize the method of Proposition 2.7 to templates $\tau = \{(A_t, \mathcal{F}_t)\}_{t \in T}$ indexed by a finite directed rooted tree T. When T is a finite descending tree (resp. a finite ascending tree), the root node will be the unique maximal element (resp. the unique minimal element) of the tree T. Further, we notice that if T is a finite descending tree (resp. a finite ascending tree) and $t_0 \in T$ is any chosen node, the subset $T_{\leq t_0} := \{t \in T \mid t \leq t_0\}$ (resp. the subset $T^{\geq t_0} := \{t \in T \mid t \geq t_0\}$) is also a finite descending tree (resp. a finite ascending tree).

THEOREM 2. *Let T be a finite directed rooted tree and let $\tau = \{(A_t, \mathcal{F}_t)\}_{t \in T}$ be a collection of pairs indexed by T. Then, for any chosen $t_0 \in T$, we can form a pair $(B_{t_0}, \mathcal{G}_{t_0})$ having the following property: if there exists a prime ideal P realizing $(B_{t_0}, \mathcal{G}_{t_0})$, then there exists a family $\{P_t\}_{t \in T}$ of prime ideals realizing the template $\tau = \{(A_t, \mathcal{F}_t)\}_{t \in T}$ such that $P_{t_0} = P$. Further, B_{t_0} and $\mathcal{G}_{t_0}$ can be expressed in terms of the ideals $\{A_t\}_{t \in T}$ and the monoidal semifilters $\{\mathcal{F}_t\}_{t \in T}$ using the following four operations:*

(a) *Forming the sum of ideals in R.*
(b) *Forming the product $\mathcal{F}_1 \cdot \mathcal{F}_2$ of two monoidal semifilters $\mathcal{F}_1$ and $\mathcal{F}_2$.*
(c) *Forming the ideal $A \div \mathcal{F}$ from an ideal A and a monoidal semifilter $\mathcal{F}$.*
(d) *Forming the monoidal semifilter $\mathcal{T}(A, \mathcal{F})$ from an ideal A and a monoidal semifilter $\mathcal{F}$.*

Proof. For definiteness, we will assume that T is a finite descending tree. From the proof, it will be clear that the same method works in the case of T being a finite ascending tree.

We will proceed by induction on $|T|$, the number of nodes in the tree T. The result is obvious if $|T| = 1$. Suppose that the result holds for all trees with less than n nodes and let T be a finite descending tree with n nodes. We choose some $t_0 \in T$ and let $t_1, \ldots, t_k$ be the list of nodes immediately below t_0. In other words, given any $i \in \{1, \ldots, k\}$, we have $t_i < t_0$ and there is no node $t \in T$ such that $t_i < t < t_0$. We now notice that each

$$T_{\leq t_i} := \{t \in T \mid t \leq t_i\}, \quad 1 \leq i \leq k, \tag{2.11}$$

is a finite and rooted descending tree with less than n nodes. We also consider $T_{\leq t_0} := \{t \in T \mid t \leq t_0\}$ and notice that $T \setminus T_{\leq t_0}$ is still a finite and rooted descending tree but with less than n nodes.

On the other hand, let s_0 be the unique node that is immediately above t_0, i.e., $t_0 < s_0$ and there is no node $t \in T$ such that $t_0 < t < s_0$. Since $s_0 \in T \setminus T_{\leq t_0}$, by induction assumption, we can form a pair $(B_0, \mathcal{G}_0)$ by using operations (a)–(d) such that any realization of $(B_0, \mathcal{G}_0)$ can be expanded into a realization for the template $\{(A_t, \mathcal{F}_t)\}_{t \in T \setminus T_{\leq t_0}}$. Similarly, for any $1 \leq i \leq k$, by considering the root t_i for each tree $T_{\leq t_i}$, we can obtain a pair $(B_i, \mathcal{G}_i)$ by using operations (a)–(d) such that any realization of $(B_i, \mathcal{G}_i)$ can be expanded into a realization for the template $\{(A_t, \mathcal{F}_t)\}_{t \in T_{\leq t_i}}$. Finally, considering the pair $(A_{t_0}, \mathcal{F}_{t_0})$ at the node $t_0 \in T$, we define

$$B_{t_0} := A_{t_0} + \sum_{i=1}^{k} (B_i \div \mathcal{G}_i), \qquad \mathcal{G}_{t_0} := \mathcal{F}_{t_0} \cdot \mathcal{T}(B_0, \mathcal{G}_0). \tag{2.12}$$

Proposition 2.1 and Lemma 2.5 imply that the monoidal semifilter $\mathcal{G}_{t_0} = \mathcal{F}_{t_0} \cdot \mathcal{T}(B_0, \mathcal{G}_0)$ satisfies condition (1) in Theorem 1. Now, suppose that P is a prime ideal realizing $(B_{t_0}, \mathcal{G}_{t_0})$. In particular, P realizes $(A_{t_0}, \mathcal{F}_{t_0})$. Further, since $P \supseteq B_i \div \mathcal{G}_i$, P must contain a prime P_i realizing $(B_i, \mathcal{G}_i)$ for each $1 \leq i \leq k$. Then the realization P_i of $(B_i, \mathcal{G}_i)$ can be expanded into a realization for $\{(A_t, \mathcal{F}_t)\}_{t \in T_{\leq t_i}}$. On the other hand, $P \notin \mathcal{T}(B_0, \mathcal{G}_0)$ and it follows from Proposition 2.6(b) that we can choose a prime ideal $P_0 \supseteq P$ that realizes $(B_0, \mathcal{G}_0)$. Again, the realization P_0 of $(B_0, \mathcal{G}_0)$ can be expanded into a realization for $\{(A_t, \mathcal{F}_t)\}_{t \in T \setminus T_{\leq t_0}}$. Thus we obtain a realization of the whole template $\tau = \{(A_t, \mathcal{F}_t)\}_{t \in T}$ with P appearing at position t_0. ∎

3. Monoidal semifilters and ideals in tensor products. In this section, we consider monoidal semifilters and partially ordered systems of prime ideals in the tensor products of algebras over a given field k. If R is a commutative ring, we let the dimension $\dim(X)$ of any subset $X \subseteq \operatorname{Spec}(R)$ be the supremum of the lengths n of chains $P_0 \supsetneq \cdots \supsetneq P_n$ of prime ideals with each P_i in X. As before, for any monoidal semifilter $\mathcal{F}$, we denote by $\mathcal{F}^c$ the collection of ideals in R which do not belong to $\mathcal{F}$. We notice that the collection $\operatorname{Spec}(\mathcal{F}^c) := \mathcal{F}^c \cap \operatorname{Spec}(R)$ of all prime ideals in $\mathcal{F}^c$ can be expressed as the complement:

$$\operatorname{Spec}(\mathcal{F}^c) = \operatorname{Spec}(R) \setminus \bigcup_{I \in \mathcal{F}} V(I) = \bigcap_{I \in \mathcal{F}} (\operatorname{Spec}(R) \setminus V(I)), \tag{3.1}$$

where for any ideal I in R, $V(I) \subseteq \operatorname{Spec}(R)$ denotes the Zariski closed subset of prime ideals containing I. Then (3.1) shows that in general, $\operatorname{Spec}(\mathcal{F}^c)$ is not even a Zariski open subset of $\operatorname{Spec}(R)$. However, $\operatorname{Spec}(\mathcal{F}^c)$ can be expressed as the intersection of the Zariski open sets $\operatorname{Spec}(R) \setminus V(I)$ as I varies over all the ideals in $\mathcal{F}$.

Throughout this section, we let $R_{(0)}$ and $R_{(1)}$ be given commutative algebras over a field k and we set $R := R_{(0)} \otimes_k R_{(1)}$. We will often write

$R = R_{(0)} \otimes_k R_{(1)}$ simply as $R = R_{(0)} \otimes R_{(1)}$. We also suppose that $\mathcal{F}_{(0)}$ (resp. $\mathcal{F}_{(1)}$) is a monoidal semifilter of ideals in $R_{(0)}$ (resp. $R_{(1)}$) such that any non-empty chain of ideals in $\mathcal{F}^c_{(0)}$ has an upper bound in $\mathcal{F}^c_{(0)}$ (resp. any non-empty chain of ideals in $\mathcal{F}^c_{(1)}$ has an upper bound in $\mathcal{F}^c_{(1)}$).

LEMMA 3.1. *Consider the following family $\mathcal{F} = \mathcal{F}_{(0)} \otimes \mathcal{F}_{(1)}$ of ideals in $R = R_{(0)} \otimes R_{(1)}$:*

$$\mathcal{F} := \{I \triangleleft R \mid I \supseteq I_{(0)} \otimes I_{(1)} \textit{ for some } I_{(0)} \in \mathcal{F}_{(0)},\ I_{(1)} \in \mathcal{F}_{(1)}\}. \tag{3.2}$$

Then $\mathcal{F}$ is a monoidal semifilter and any non-empty chain of ideals in $\mathcal{F}^c$ has an upper bound in $\mathcal{F}^c$.

Proof. Since $\mathcal{F}_{(0)}$ and $\mathcal{F}_{(1)}$ are monoidal semifilters, it is clear from (3.2) that $\mathcal{F}$ is a monoidal semifilter. We now consider some $I \in \mathcal{F}$, and choose $I_{(0)} \in \mathcal{F}_{(0)}$ and $I_{(1)} \in \mathcal{F}_{(1)}$ such that $I \supseteq I_{(0)} \otimes I_{(1)}$. Since non-empty chains in $\mathcal{F}^c_{(0)}$ and $\mathcal{F}^c_{(1)}$ have upper bounds in $\mathcal{F}^c_{(0)}$ and $\mathcal{F}^c_{(1)}$ respectively, Theorem 1 yields finitely generated ideals $I'_{(0)} \in \mathcal{F}_{(0)}$ and $I'_{(1)} \in \mathcal{F}_{(1)}$ such that $I_{(0)} \supseteq I'_{(0)}$ and $I_{(1)} \supseteq I'_{(1)}$. It is clear that the ideal $I'_{(0)} \otimes I'_{(1)}$ is finitely generated and $I'_{(0)} \otimes I'_{(1)} \in \mathcal{F}$. Further, we see that $I \supseteq I_{(0)} \otimes I_{(1)} \supseteq I'_{(0)} \otimes I'_{(1)} \in \mathcal{F}$. It now follows from Theorem 1 that any non-empty chain of ideals in $\mathcal{F}^c$ has an upper bound in $\mathcal{F}^c$. ∎

As noted in Section 2, for a multiplicatively closed subset $S \subseteq R$, we can consider a monoidal semifilter $\mathcal{G}_S := \{I \triangleleft R \mid I \cap S \neq \emptyset\}$. Further, as in (2.9), we have

$$\mathcal{G}_S \cdot \mathcal{F} = \{I \triangleleft R \mid I \supseteq sJ \text{ for some } s \in S \text{ and } J \in \mathcal{F}\} \tag{3.3}$$

for any monoidal semifilter $\mathcal{F}$ in R.

LEMMA 3.2. *Let $P_{(0)}$ and $P_{(1)}$ be prime ideals in $R_{(0)}$ and $R_{(1)}$ respectively such that $P_{(0)} \notin \mathcal{F}_{(0)}$ and $P_{(1)} \notin \mathcal{F}_{(1)}$. Set*

$$A := P_{(0)} \otimes R_{(1)} + R_{(0)} \otimes P_{(1)}, \qquad S := (R_{(0)} - P_{(0)}) \otimes (R_{(1)} - P_{(1)}). \tag{3.4}$$

Then, if $\mathcal{F} = \mathcal{F}_{(0)} \otimes \mathcal{F}_{(1)}$, we have

$$A = A \div (\mathcal{G}_S \cdot \mathcal{F}) = A \div \mathcal{F} = A \div S = \{r \in R \mid rs \in A \textit{ for some } s \in S\}.$$

Proof. We consider any $I \in \mathcal{F}$. From (3.2), it follows that $I \supseteq I_{(0)} \otimes I_{(1)}$ for some $I_{(0)} \in \mathcal{F}_{(0)}$ and some $I_{(1)} \in \mathcal{F}_{(1)}$. Since $P_{(0)} \notin \mathcal{F}_{(0)}$ and $P_{(1)} \notin \mathcal{F}_{(1)}$, we know that $P_{(0)} \not\supseteq I_{(0)}$ and $P_{(1)} \not\supseteq I_{(1)}$. Accordingly, we choose $s_0 \in I_{(0)} - P_{(0)}$ and $s_1 \in I_{(1)} - P_{(1)}$, and consider $s_0 \otimes s_1 \in S = (R_{(0)} - P_{(0)}) \otimes (R_{(1)} - P_{(1)})$. But we also have $s_0 \otimes s_1 \in I_{(0)} \otimes I_{(1)} \subseteq I$, and hence $I \cap S \neq \emptyset$, i.e., $I \in \mathcal{G}_S$. Hence, $\mathcal{F} \subseteq \mathcal{G}_S$ and (3.3) yields $\mathcal{G}_S \cdot \mathcal{F} = \mathcal{G}_S$. From the definitions, it is also clear that $A \div \mathcal{G}_S = A \div S$. We now have

$$A \div S = A \div \mathcal{G}_S = A \div (\mathcal{G}_S \cdot \mathcal{F}) \supseteq A \div \mathcal{F} \supseteq A. \tag{3.5}$$

Finally, it follows from [3, Lemma 15(ii)] that $A \div S = A$. This proves the result. ∎

Proposition 3.3. *Let $P_{(0)}$ and $P_{(1)}$ be prime ideals in $R_{(0)}$ and $R_{(1)}$ respectively such that $P_{(0)} \notin \mathcal{F}_{(0)}$ and $P_{(1)} \notin \mathcal{F}_{(1)}$. Then there exists a prime ideal P in $R = R_{(0)} \otimes R_{(1)}$ such that $P \notin \mathcal{F} = \mathcal{F}_{(0)} \otimes \mathcal{F}_{(1)}$ and $P \cap R_{(0)} = P_{(0)}$, $P \cap R_{(1)} = P_{(1)}$.*

Proof. We maintain the notation from the statement of Lemma 3.2. We choose a prime ideal $Q \subseteq R$ such that $Q \supseteq A = A \div \mathcal{F}$. From Lemma 3.2, we know that $A \div (\mathcal{G}_S \cdot \mathcal{F}) = A \div \mathcal{F}$ and hence $Q \supseteq A \div (\mathcal{G}_S \cdot \mathcal{F})$. By Proposition 2.6, we can choose a prime ideal $P \subseteq Q$ such that P realizes $(A, \mathcal{G}_S \cdot \mathcal{F})$. From (3.3), it is clear that $P \notin \mathcal{F}$ and P realizes $(A, \mathcal{G}_S)$, i.e., $P \supseteq A$ and $P \cap S = \emptyset$. From [3, Lemma 15(iii)], it follows that $P \cap R_{(0)} = P_{(0)}$ and $P \cap R_{(1)} = P_{(1)}$. This proves the result. ∎

For finite descending trees, we now show that we can even lift trees of prime ideals in $R_{(0)}$ and $R_{(1)}$ outside $\mathcal{F}_{(0)}$ and $\mathcal{F}_{(1)}$ respectively to a common descending tree of prime ideals in R outside $\mathcal{F} = \mathcal{F}_{(0)} \otimes \mathcal{F}_{(1)}$. As in Section 2, a finite descending tree T will have a root node t_0 which is the unique maximal element of T. For any node $t \in T$, we will denote by $|t|$ the largest non-negative integer n such that there exists a chain of distinct elements:

$$t_0 > t_1 > \cdots > t_n = t \tag{3.6}$$

in T starting with the root node t_0. For a given node $t \in T$, we note that the decreasing sequence $(t_0 > t_1 > \cdots > t_n = t)$ of maximal length from the root node t_0 to t is unique. It is clear that the root node t_0 satisfies $|t_0| = 0$.

Theorem 3. *Let T be a finite descending tree, and let $\{P_{0t}\}_{t\in T}$ and $\{P_{1t}\}_{t\in T}$ be trees of prime ideals in $R_{(0)}$ and $R_{(1)}$ respectively such that $P_{0t} \notin \mathcal{F}_{(0)}$ and $P_{1t} \notin \mathcal{F}_{(1)}$ for each $t \in T$. Then there exists a finite descending tree $\{P_t\}_{t\in T}$ of prime ideals in $R = R_{(0)} \otimes R_{(1)}$ such that $P_t \notin \mathcal{F} = \mathcal{F}_{(0)} \otimes \mathcal{F}_{(1)}$ and $P_t \cap R_{(0)} = P_{0t}$, $P_t \cap R_{(1)} = P_{1t}$ for all $t \in T$.*

Proof. We start with the root node $t_0 \in T$. By Proposition 3.3, we can choose a prime ideal P_{t_0} in $R = R_{(0)} \otimes R_{(1)}$ such that $P_{t_0} \notin \mathcal{F} = \mathcal{F}_{(0)} \otimes \mathcal{F}_{(1)}$ and satisfying $P_{t_0} \cap R_{(0)} = P_{0t_0}$ and $P_{t_0} \cap R_{(1)} = P_{1t_0}$. We now proceed by induction: suppose that we have already found prime ideals $P_t \notin \mathcal{F} = \mathcal{F}_{(0)} \otimes \mathcal{F}_{(1)}$ such that $P_t \cap R_{(0)} = P_{0t}$ and $P_t \cap R_{(1)} = P_{1t}$ for all nodes $t \in T$ with $|t| \leq k$ for some integer k. The result is already true for $k = 0$. Let $t \in T$ be a node with $|t| = k + 1$. Since T is a finite descending tree, we can consider the unique node $t' \in T$ with $|t'| = k$ and $t' > t$. Similar to (3.4), we now set

$$A_t := P_{0t} \otimes R_{(1)} + R_{(0)} \otimes P_{1t}, \qquad S_t := (R_{(0)} - P_{0t}) \otimes (R_{(1)} - P_{1t}). \tag{3.7}$$

Since $P_{t'} \cap R_{(0)} = P_{0t'}$ and $P_{t'} \cap R_{(1)} = P_{1t'}$, it is clear that $P_{t'} \supseteq P_{0t'} \otimes R_{(1)} + R_{(0)} \otimes P_{1t'}$. Further, since $P_{0t'} \supseteq P_{0t}$ and $P_{1t'} \supseteq P_{1t}$, we get $P_{t'} \supseteq A_t$. Then, as in the proof of Lemma 3.2, we have $A_t = A_t \div S_t$ and the prime ideal $P_{t'}$ containing $A_t = A_t \div S_t$ must contain a prime ideal P_t realizing (A_t, S_t). Further, as in the proof of Proposition 3.3, this implies that $P_t \cap R_{(0)} = P_{0t}$ and $P_t \cap R_{(1)} = P_{1t}$. Finally, since $P_t \subseteq P_{t'}$ and $P_{t'} \notin \mathcal{F}$, it follows that $P_t \notin \mathcal{F}$. This proves the result. ∎

Theorem 4. *Let $\mathcal{F}_{(0)}$ and $\mathcal{F}_{(1)}$ be monoidal semifilters in $R_{(0)}$ and $R_{(1)}$ respectively such that any non-empty chain of ideals in $\mathcal{F}^c_{(0)}$ (resp. $\mathcal{F}^c_{(1)}$) has an upper bound in $\mathcal{F}^c_{(0)}$ (resp. $\mathcal{F}^c_{(1)}$). Then*

$$\dim(\operatorname{Spec}((\mathcal{F}_{(0)} \otimes \mathcal{F}_{(1)})^c)) \geq \dim(\operatorname{Spec}(\mathcal{F}^c_{(0)})) + \dim(\operatorname{Spec}(\mathcal{F}^c_{(1)})). \tag{3.8}$$

Proof. Let $P_0 \supsetneq P_1 \supsetneq \cdots \supsetneq P_m$ be a chain of distinct prime ideals in $R_{(0)}$ outside $\mathcal{F}_{(0)}$, and let $P_{m+1} \supsetneq \cdots \supsetneq P_{m+n+1}$ be a chain of distinct prime ideals in $R_{(1)}$ outside $\mathcal{F}_{(1)}$. We now take the opposite T^{op} of the partially ordered set $T = \{0 < 1 < \cdots < m+n+1\}$. We can define the finite descending tree $\{P_{0t}\}_{t \in T^{\mathrm{op}}}$ (resp. $\{P_{1t}\}_{t \in T^{\mathrm{op}}}$) of prime ideals in $R_{(0)}$ (resp. in $R_{(1)}$) by setting

$$\begin{aligned} P_{0t} &= \begin{cases} P_t & \text{if } 0 \leq t \leq m, \\ P_m & \text{if } m+1 \leq t \leq m+n+1, \end{cases} \\ P_{1t} &= \begin{cases} P_{m+1} & \text{if } 0 \leq t \leq m, \\ P_t & \text{if } m+1 \leq t \leq m+n+1. \end{cases} \end{aligned} \tag{3.9}$$

From Theorem 3, we can find a descending chain $\{Q_t\}_{t \in T^{\mathrm{op}}}$ of prime ideals with $Q_t \notin \mathcal{F} = \mathcal{F}_{(0)} \otimes \mathcal{F}_{(1)}$ and such that $Q_t \cap R_{(0)} = P_{0t}$ and $Q_t \cap R_{(1)} = P_{1t}$. We claim that at least $m+n+1$ of the prime ideals in the chain

$$Q_0 \supseteq Q_1 \supseteq \cdots \supseteq Q_m \supseteq Q_{m+1} \supseteq \cdots \supseteq Q_{m+n+1} \tag{3.10}$$

are distinct. For this, we note that if $0 \leq t < m$, then $Q_t \cap R_{(0)} = P_{0t} = P_t \supsetneq P_{t+1} = P_{0,t+1} = Q_{t+1} \cap R_{(0)}$. Hence, $Q_0 \supsetneq Q_1 \supsetneq \cdots \supsetneq Q_m$. Similarly, $Q_t \cap R_{(1)} = P_{1t} = P_t \supsetneq P_{t+1} = P_{1,t+1} = Q_{t+1} \cap R_{(1)}$ for $m+1 \leq t < m+n+1$. This gives a chain

$$Q_0 \supsetneq Q_1 \supsetneq \cdots \supsetneq Q_m \supseteq Q_{m+1} \supsetneq \cdots \supsetneq Q_{m+n+1}, \tag{3.11}$$

from which the result of (3.8) follows. ∎

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