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Analytically principal part of polynomials at infinity

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Abstract. Let $f\colon \mathbb{K}^n \to \mathbb{K}$ be a polynomial function, where $\mathbb{K} := \mathbb{R}$ or $\mathbb{C}$. We give, in terms of the Newton boundary at infinity of f, a sufficient condition for a deformation of f to be analytically (smoothly in the case $\mathbb{K} := \mathbb{C}$) trivial at infinity.

1. Introduction. The problem of C^0-sufficiency of jets is one of the most interesting problems in singularity theory. Roughly speaking, it is the problem of determining a *topologically principal part* of the Taylor expansion of a given function at the origin in $\mathbb{K}^n$; here and in the following, $\mathbb{K} := \mathbb{R}$ or $\mathbb{C}$.

N. H. Kuiper [17], T. C. Kuo [18], and J. Bochnak and S. Łojasiewicz [2] proved the following theorem:

> *Let $f\colon \mathbb{R}^n \to \mathbb{R}$ be a C^k function defined in a neighborhood of the origin $0 \in \mathbb{R}^n$ with $f(0) = 0$. Then the following conditions are equivalent:*
>
> (i) *There are positive constants C and r such that*
> $$\|\operatorname{grad} f(x)\| \geq C\|x\|^{k-1} \quad \text{for } \|x\| \leq r.$$
> (ii) *The k-jet of f is C^0-sufficient in the class C^k.*

Analogous results in the case of complex analytic functions were proved by S. H. Chang and Y. C. Lu [7], B. Teissier [28], and J. Bochnak and W. Kucharz [1]. Similar considerations were also carried out for polynomial mappings in a neighborhood of infinity by P. Cassou-Noguès and H. V. Hà [6], L. Fourrier [10], G. Skalski [27], and T. Rodak and S. Spodzieja [26].

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On the other hand, except for certain degenerate cases the topological type of an analytic function is expected to depend only on its Newton polyhedron. This has been confirmed together with a precise definition of non-degeneracy at the origin of $\mathbb{K}^n$ (see, for example, [8, 9, 11, 16, 22, 31]).

The purpose of this paper is to show that polynomial functions which are convenient and non-degenerate at infinity, are determined up to analytical type by their Newton polyhedra at infinity. More precisely, with the definitions in the next section, the main result of this paper is as follows.

THEOREM 1.1. *Let $f\colon \mathbb{K}^n \to \mathbb{K}$ be a polynomial function. Suppose that f is convenient and non-degenerate at infinity. Then, for any polynomial function $g\colon \mathbb{K}^n \to \mathbb{K}$ satisfying the condition $\Gamma(g) \subset \operatorname{Int}(\Gamma(f))$, the family $f + tg$ is analytically* (*resp., smoothly*) *trivial at infinity along the interval* $[0, 1]$ *in the case* $\mathbb{K} = \mathbb{R}$ (*resp.,* $\mathbb{K} = \mathbb{C}$).

REMARK. (i) The proof of Theorem 1.1 uses only the Curve Selection Lemma at infinity.

(ii) Related results concerning the behavior of complex polynomials under deformations have been established in [3, 4, 5, 12, 13, 24, 25, 29].

The paper is structured as follows. Section 2 presents some definitions and notation. Theorem 1.1 is proved in Section 3.

2. Definitions. Throughout this paper, the scalar product (resp., norm) in $\mathbb{R}^n$ is defined by $\langle x, y\rangle$ for any $x, y \in \mathbb{R}^n$ (resp., $\|x\| := \sqrt{\langle x, x\rangle}$ for any $x \in \mathbb{R}^n$). We denote by $\mathbb{Z}_+$ the set of non-negative integers.

2.1. Non-degeneracy at infinity. If $\alpha := (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}_+^n$ and $x := (x_1, \ldots, x_n) \in \mathbb{R}^n$, we write x^α for the monomial $x_1^{\alpha_1} \cdots x_n^{\alpha_n}$. Let $f\colon \mathbb{K}^n \to \mathbb{K}$ be a polynomial function. Suppose that f is written as $f = \sum_\alpha a_\alpha x^\alpha$. Then the *support* of f, denoted by $\operatorname{supp}(f)$, is defined as the set of those $\alpha \in \mathbb{Z}_+^n$ such that $a_\alpha \neq 0$. Let $\Gamma(f)$ be the convex hull of $\operatorname{supp}(f) \cup \{0\}$. The *Newton boundary at infinity* of f, denoted by $\Gamma_\infty(f)$, is defined as the union of the closed faces of $\Gamma(f)$ which do not contain the origin in $\mathbb{R}^n$. Here and below, by *face* we shall understand a face of any dimension. A vertex is a face of dimension 0. We also set $\operatorname{Int}(\Gamma(f)) := \Gamma(f) \setminus \Gamma_\infty(f)$.

The polynomial f is said to be *convenient* if the polyhedron $\Gamma(f)$ intersects each coordinate axis at a point different from the origin. For each closed face Δ of $\Gamma_\infty(f)$ we denote by f_Δ the polynomial $\sum_{\alpha \in \Delta} a_\alpha x^\alpha$.

The following notion plays an important role in this paper (see [15, 16, 23]).

Definition 2.1. We say that f is (*Kouchnirenko*) *non-degenerate at infinity* if

$$\left\{x \in \mathbb{K}^n \;\middle|\; \frac{\partial f_\Delta}{\partial x_1}(x) = \cdots = \frac{\partial f_\Delta}{\partial x_n}(x) = 0\right\} \subset \{x \in \mathbb{K}^n \mid x_1 \cdots x_n = 0\}$$

for all closed faces Δ of $\Gamma_\infty(f)$.

2.2. Triviality at infinity. Every open set of the form $\mathbb{K}^n \setminus K$, where $K \subset \mathbb{K}^n$ is a compact set, is called a *neighborhood of infinity.*

Definition 2.2. Let $f, g\colon \mathbb{K}^n \to \mathbb{K}$ be polynomial functions. We say that the family $f + tg$ is *analytically* (resp., *smoothly*) *trivial at infinity* along the interval $[0,1]$ if there exist a neighborhood of infinity $\Omega_0 \subset \mathbb{K}^n$ and a continuous mapping $\Phi\colon [0,1] \times \Omega_0 \to \mathbb{K}^n$, $(t,x) \mapsto \Phi(t,x)$, such that:

(a) $\Phi_0(x) = x$ for $x \in \Omega_0$;
(b) for any $t \in [0,1]$, the mapping $\Phi_t\colon \Omega_0 \to \Phi_t(\Omega_0)$ is a real analytic diffeomorphism (resp., a C^∞-diffeomorphism) and $\lim_{x\to\infty} \Phi_t(x) = \infty$;
(c) $f(\Phi_t(x)) + tg(\Phi_t(x)) = f(x)$ for $x \in \Omega_0$ and $t \in [0,1]$,

where $\Phi_t\colon \Omega_0 \to \mathbb{K}^n$ is defined by $\Phi_t(x) := \Phi(t,x)$ for $x \in \Omega_0$ and $t \in [0,1]$.

3. Proof of the main result. We will prove Theorem 1.1 for $\mathbb{K} = \mathbb{R}$; the proof in the complex case is quite similar. So, let $f, g\colon \mathbb{R}^n \to \mathbb{R}$ be polynomial functions satisfying the conditions of Theorem 1.1. Let us consider the polynomial function

$$F\colon \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}, \qquad (t,x) \mapsto f(x) + tg(x).$$

For any fixed $t \in \mathbb{R}$, let $f_t(x) := F(t,x)$. By the assumption $\Gamma(g) \subset \operatorname{Int}(\Gamma(f))$, we have $\Gamma(f) \equiv \Gamma(f_t)$ and $\Gamma_\infty(f) \equiv \Gamma_\infty(f_t)$. Furthermore, f_t is convenient and non-degenerate at infinity.

For simplicity, we let

$$\operatorname{Grad}_x F(t,x) := \left(x_1 \frac{\partial F}{\partial x_1}(t,x), \ldots, x_n \frac{\partial F}{\partial x_n}(t,x)\right).$$

Lemma 3.1. *There exist positive constants C and R such that*

$$\|\operatorname{Grad}_x F(t,x)\| \geq C|g(x)|, \qquad \forall t \in (-2,2),\ \forall \|x\| > R.$$

Proof. By contradiction and using the Curve Selection Lemma at infinity [19, 21], we can find an analytic function $t(s)$ and an analytic curve $\varphi(s) = (\varphi_1(s), \ldots, \varphi_n(s))$ for $s \in (0, \epsilon)$ satisfying:

(a) $-2 < t(s) < 2$;
(b) $\|\varphi(s)\| \to \infty$ as $s \to 0^+$; and

$$\text{(c)}\quad \|\operatorname{Grad}_x F(t(s),\varphi(s))\|^2 = \sum_{i=1}^{n}\Big(\varphi_i(s)\frac{\partial F}{\partial x_i}(t(s),\varphi(s))\Big)^2 \ll |g(\varphi(s))|^2.$$

Let $I := \{i \mid \varphi_i \not\equiv 0\} \subset \{1,\dots,n\}$. By (b), $I \neq \emptyset$. For $i \in I$, we can expand the coordinate φ_i in terms of the parameter: say

$$\varphi_i(s) = x_i^0 s^{a_i} + \text{higher order terms in } s,$$

where $x_i^0 \neq 0$ and $a_i \in \mathbb{Q}$. From (b), we obtain $\min_{i\in I} a_i < 0$.

Let $\mathbb{R}^I := \{\alpha = (\alpha_1,\dots,\alpha_n) \in \mathbb{R}^n \mid \alpha_i = 0 \text{ for } i \notin I\}$. Since f is convenient, $\Gamma(f)\cap\mathbb{R}^I \neq \emptyset$. We set $a := (a_1,\dots,a_n) \in \mathbb{R}^n$, where $a_i := 0$ for $i \notin I$. Let d be the minimal value of the linear function $\langle a,\alpha\rangle = \sum_{i\in I} a_i\alpha_i$ on $\Gamma(f)\cap\mathbb{R}^I$, and let Δ be the (unique) maximal face ([1]) of $\Gamma(f)$ where the linear function $\sum_{i\in I} a_i\alpha_i$ takes its minimal value d, i.e.,

$$\Delta := \{\alpha \in \Gamma(f) \mid \langle a,\alpha\rangle = d\}.$$

We have $d < 0$ and Δ is a closed face of $\Gamma_\infty(f)$ because f is convenient and $\min_{i=1,\dots,n} a_i < 0$. Since $\Gamma(g) \subset \operatorname{Int}(\Gamma(f))$ and $t(s)$ is bounded, we can see that

$$\varphi_i(s)\frac{\partial F}{\partial x_i}(t(s),\varphi(s)) = x_i^0\frac{\partial f_\Delta}{\partial x_i}(x^0)s^d + \text{higher order terms in } s \quad \text{for } i \in I,$$

where $x^0 := (x_1^0,\dots,x_n^0)$ and $x_i^0 := 1$ for $i \notin I$. Consequently,

$$\begin{aligned}
\|\operatorname{Grad}_x F(t(s),\varphi(s))\|^2 &= \sum_{i=1}^{n}\Big(\varphi_i(s)\frac{\partial F}{\partial x_i}(t(s),\varphi(s))\Big)^2\\
&= \sum_{i\in I}\Big(\varphi_i(s)\frac{\partial F}{\partial x_i}(t(s),\varphi(s))\Big)^2\\
&= s^{2d}\sum_{i\in I}\Big(x_i^0\frac{\partial f_\Delta}{\partial x_i}(x^0)\Big)^2 + \text{higher order terms in } s\\
&= s^{2d}\sum_{i=1}^{n}\Big(x_i^0\frac{\partial f_\Delta}{\partial x_i}(x^0)\Big)^2 + \text{higher order terms in } s.
\end{aligned}$$

(The last equality follows from the fact that f_Δ does not depend on x_i for $i \notin I$.) Since the polynomial f is non-degenerate at infinity,

$$\sum_{i=1}^{n}\Big(x_i^0\frac{\partial f_\Delta}{\partial x_i}(x^0)\Big)^2 \neq 0.$$

Therefore

$$\|\operatorname{Grad}_x F(t(s),\varphi(s))\| \simeq s^d \quad \text{as } s \to 0^+. \tag{3.1}$$

([1]) Maximal with respect to inclusion.

On the other hand, it follows easily from $\Gamma(g) \subset \operatorname{Int}(\Gamma(f))$ that

$$g(\varphi(s)) = o(s^d) \quad \text{as } s \to 0^+.$$

This, together with (3.1), contradicts (c). ∎

LEMMA 3.2. *There exist positive constants C and R such that*

$$\|\operatorname{Grad}_x F(t,x)\| \geq C, \qquad \forall t \in (-2,2), \forall \|x\| > R.$$

Proof. By contradiction and using the Curve Selection Lemma at infinity [19, 21], we can find an analytic function $t(s)$ and an analytic curve $\varphi(s) = (\varphi_1(s), \dots, \varphi_n(s))$ for $s \in (0,\epsilon)$ satisfying:

(a) $-2 < t(s) < 2$;
(b) $\|\varphi(s)\| \to \infty$ as $s \to 0^+$; and
(c) $\|\operatorname{Grad}_x F(t(s),\varphi(s))\|^2 = \sum_{i=1}^n \big(\varphi_i(s)\frac{\partial F}{\partial x_i}(t(s),\varphi(s))\big)^2 \to 0$ as $s \to 0^+$.

Let $I := \{i \mid \varphi_i \not\equiv 0\} \subset \{1,\dots,n\}$. By (b), $I \neq \emptyset$. For $i \in I$, we can expand the coordinate φ_i in terms of the parameter: say

$$\varphi_i(s) = x_i^0 s^{a_i} + \text{higher order terms in } s,$$

where $x_i^0 \neq 0$ and $a_i \in \mathbb{Q}$. From (b), we obtain $\min_{i\in I} a_i < 0$.

Let $\mathbb{R}^I := \{\alpha = (\alpha_1,\dots,\alpha_n) \in \mathbb{R}^n \mid \alpha_i = 0 \text{ for } i \notin I\}$. Since the polynomial f is convenient, we have $\Gamma(f) \cap \mathbb{R}^I \neq \emptyset$. We set $a := (a_1,\dots,a_n) \in \mathbb{R}^n$, where $a_i = 0$ for $i \notin I$. Let d be the minimal value of the linear function $\sum_{i\in I} a_i\alpha_i$ on $\Gamma(f) \cap \mathbb{R}^I$, and let Δ be the (unique) maximal face of $\Gamma(f)$ where the linear function $\sum_{i\in I} a_i\alpha_i$ takes its minimal value d. Then it is easy to see that $d < 0$ and Δ is a closed face of $\Gamma_\infty(f)$. Since $\Gamma(g) \subset \operatorname{Int}(\Gamma(f))$ and $t(s)$ is bounded, we can see that

$$\varphi_i(s)\frac{\partial F}{\partial x_i}(t(s),\varphi(s)) = x_i^0\frac{\partial f_\Delta}{\partial x_i}(x^0)s^d + \text{higher order terms in } s \quad \text{for } i \in I,$$

where $x^0 := (x_1^0,\dots,x_n^0)$ and $x_i^0 := 1$ for $i \notin I$. Then (c) implies that

$$x_i^0\frac{\partial f_\Delta}{\partial x_i}(x_i^0) = 0 \quad \text{for } i \in I.$$

Note that f_Δ does not depend on x_i for $i \notin I$. Therefore,

$$x_i^0\frac{\partial f_\Delta}{\partial x_i}(x_i^0) = 0 \quad \text{for } i = 1,\dots,n.$$

This contradicts our assumption that f is non-degenerate at infinity. ∎

LEMMA 3.3. *For each $i = 1,\dots,n$, let*

$$W_i(t,x) := \begin{cases} \dfrac{-g(x)}{\|\operatorname{Grad}_x F(t,x)\|^2}\, x_i^2\dfrac{\partial F}{\partial x_i}(t,x) & \textit{if } \operatorname{Grad}_x F(t,x) \neq 0, \\ 0 & \textit{otherwise.} \end{cases}$$

Then there exist positive constants C and R such that the mapping

$$W\colon (-2,2)\times\{x\in\mathbb{R}^n \mid \|x\|>R\}\to\mathbb{R}^n, \qquad (t,x)\mapsto (W_1(t,x),\dots,W_n(t,x)),$$

is analytic and satisfies

$$\|W(t,x)\|\le C\|x\|, \qquad d_xF(t,x)W(t,x)=-g(x),$$

where d_xF stands for the derivative of F with respect to x.

Proof. Let C and R be positive constants for which Lemmas 3.1 and 3.2 hold true. Then W is well-defined and analytic. Furthermore, for all $t\in(-2,2)$ and $\|x\|>R$ we have

$$\begin{aligned}\|W(t,x)\| &\le \frac{|g(x)|}{\|\operatorname{Grad}_x F(t,x)\|^2}\sum_{i=1}^n |x_i|\left|x_i\frac{\partial F}{\partial x_i}(t,x)\right|\\ &\le \frac{|g(x)|}{\|\operatorname{Grad}_x F(t,x)\|^2}\sqrt{\sum_{i=1}^n x_i^2}\sqrt{\sum_{i=1}^n\Big(x_i\frac{\partial F}{\partial x_i}(t,x)\Big)^2}\\ &= \frac{|g(x)|}{\|\operatorname{Grad}_x F(t,x)\|}\|x\|.\end{aligned}$$

By Lemma 3.1, we conclude that

$$\|W(t,x)\|\le\|x\|/C.$$

Finally, for $t\in(-2,2)$ and $\|x\|>R$ we have

$$\begin{aligned}d_xF(t,x)W(t,x) &= \sum_{i=1}^n\frac{\partial F}{\partial x_i}(t,x)W_i(t,x)\\ &= \sum_{i=1}^n\frac{-g(x)}{\sum_{k=1}^n\big(x_k\frac{\partial F}{\partial x_k}(t,x)\big)^2}\Big(x_i\frac{\partial F}{\partial x_i}(t,x)\Big)^2=-g(x). \quad\blacksquare\end{aligned}$$

LEMMA 3.4 (see [27, Lemmas 1 and 2]). *Let $D:=\{x\in\mathbb{R}^n\mid\|x\|>R\}$, $R>0$, and let $W\colon(-2,2)\times D\to\mathbb{R}^n$ be a continuous mapping such that for some $C>0$ we have*

$$\|W(t,x)\|\le C\|x\| \quad \textit{for all } (t,x)\in(-2,2)\times D.$$

Assume that $h\colon(\alpha,\beta)\to D$ is a maximal solution of the system of differential equations

$$y'(t)=W(t,y(t)).$$

If $0\in(\alpha,\beta)$, $h(0)=x$, and $\|x\|>Re^C$, then $\beta>1$ and

$$e^{-C}\|x\|\le\|h(1)\|\le e^C\|x\|.$$

If $1\in(\alpha,\beta)$, $h(1)=x$, and $\|x\|>Re^C$, then $\alpha<0$ and

$$e^{-C}\|x\|\le\|h(0)\|\le e^C\|x\|.$$

Proof of Theorem 1.1. We first consider the real case. There exist positive constants C and R such that the mapping

$$W\colon (-2,2) \times \{x \in \mathbb{R}^n \mid \|x\| > R\} \to \mathbb{R}^n$$

defined in Lemma 3.3 is analytic. Consider the system of differential equations

$$y'(t) = W(t, y(t)). \tag{3.2}$$

Let $D := \{x \in \mathbb{R}^n \mid \|x\| > R\}$. Since W is analytic in $(-2,2) \times D$, it follows from [20, Theorem 1.8.12] (see also [14]) that for any $(s,x) \in (-2,2) \times D$, there exists a unique solution $\phi_{(s,x)}$ of (3.2) defined on an open interval $I(s,x)$ of $\mathbb{R}$ and satisfying the initial condition $\phi_{(s,x)}(s) = x$. Moreover, the mapping

$$\{(s,x,t) \in \mathbb{R} \times \mathbb{R}^n \times \mathbb{R} \mid s \in (-2,2),\ \|x\| > R,\ t \in I(s,x)\} \to \mathbb{R}^n,$$
$$(s,x,t) \mapsto \phi_{(s,x)}(t),$$

is analytic.

Set $\Omega := \{x \in \mathbb{R}^n \mid \|x\| > Re^C\}$ and define $\Phi, \Psi\colon [0,1] \times \Omega \to D$ by

$$\Phi(t,x) = \phi_{(0,x)}(t) \quad \text{and} \quad \Psi(t,y) = \phi_{(t,y)}(0).$$

By Lemmas 3.3 and 3.4, the mappings Φ, Ψ are well defined, analytic, and satisfy

$$\begin{aligned} e^{-C}\|x\| &\le \|\Phi(t,x)\| \le e^C\|x\| \quad \text{for } t \in [0,1],\ x \in \Omega, \\ e^{-C}\|y\| &\le \|\Psi(t,y)\| \le e^C\|y\| \quad \text{for } t \in [0,1],\ y \in \Omega. \end{aligned} \tag{3.3}$$

Let $\Omega_0 := \{x \in \mathbb{R}^n \mid \|x\| > Re^{2C}\}$ and $\Omega_t := \{y \in \mathbb{R}^n \mid \Psi(t,y) \in \Omega_0\}$ for $t \in (0,1]$. Then for each $t \in [0,1]$, Ω_t is an open subset of Ω and

$$\{y \in \mathbb{R}^n \mid \|y\| > Re^{3C}\} \subset \Omega_t,$$

and therefore Ω_t is a neighborhood of infinity.

On the other hand, from the global uniqueness of solutions of (3.2), it is easy to see that for any $t \in [0,1]$ we have

(a) $\Phi(0,x) = x$ for $x \in \Omega_0$;
(b) $\Psi(t, \Phi(t,x)) = x$ for $x \in \Omega_0$; and
(c) $\Phi(t, \Psi(t,y)) = y$ for $y \in \Omega_t$.

Summing up, for each $t \in [0,1]$ the mappings $\Phi_t\colon \Omega_0 \to \Omega_t$, $x \mapsto \Phi(t,x)$, and $\Psi_t\colon \Omega_t \to \Omega_0, y \mapsto \Psi(t,y)$, are analytic diffeomorphisms of neighborhoods of infinity and $\Psi_t = \Phi_t^{-1}$. Moreover, by (3.3), we have $\|\Phi_t(x)\| \to \infty$ if, and only if, $\|x\| \to \infty$.

Finally, thanks to Lemma 3.3, we obtain

$$\begin{aligned}\frac{dF}{dt}(t, \phi_{(0,x)}(t)) &= g(\phi_{(0,x)}(t)) + d_x F(t, \phi_{(0,x)}(t))\phi'_{(0,x)}(t)\\ &= g(\phi_{(0,x)}(t)) + d_x F(t, \phi_{(0,x)}(t)) W(t, \phi_{(0,x)}(t))\\ &= g(\phi_{(0,x)}(t)) - g(\phi_{(0,x)}(t)) = 0.\end{aligned}$$

Hence, $F(t, \phi_{(0,x)}(t)) = f(x)$ for all $t \in [0, 1]$, and so

$$f(\Phi_t(x)) + tg(\Phi_t(x)) = f(x) \quad \text{for } x \in \Omega_0.$$

This proves the theorem in the real case.

We next consider the complex case, i.e., $\mathbb{K} = \mathbb{C}$. For each $z \in \mathbb{C}$, $\bar{z}$ stands for the complex conjugate of z; the norm of $x := (x_1, \dots, x_n) \in \mathbb{C}^n$ is defined by $\|x\| := \sqrt{|x_1|^2 + \cdots + |x_n|^2}$.

Let $F(t, x) := f(x) + tg(x)$ as before and consider the system

$$y'(t) = W(t, y(t)),$$

where $W(t, x) := (W_1(t, x), \dots, W_n(t, x))$ with W_i, $i = 1, \dots, n$, defined by

$$W_i(t, x) := \begin{cases} \dfrac{-g(x)}{\|\mathrm{Grad}_x F(t, x)\|^2} |x_i|^2 \overline{\dfrac{\partial F}{\partial x_i}(t, x)} & \text{if } \mathrm{Grad}_x F(t, x) \neq 0, \\ 0 & \text{otherwise.} \end{cases}$$

Then the rest of the argument goes as in the real case. We leave the verification of the details to the reader as an exercise. ∎

Remark. The method used here to prove Theorem 1.1 modifies the method used in [17, 18] (see also [11, 26, 27, 30, 31]).

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