

Some relations between $t(a, b, c, d; n)$ and $N(a, b, c, d; n)$

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1. Introduction. Let \mathbb{Z} and \mathbb{N} be the set of integers and the set of positive integers, respectively. Let $\mathbb{Z}^4 = \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$ and $\mathbb{N}^4 = \mathbb{N} \times \mathbb{N} \times \mathbb{N} \times \mathbb{N}$. For $n \in \mathbb{N}$ let

$$\sigma(n) = \sum_{d|n, d \in \mathbb{N}} d.$$

For $a, b, c, d \in \mathbb{N}$ and $n \in \mathbb{N} \cup \{0\}$ set

$$N(a, b, c, d; n) = |\{(x, y, z, w) \in \mathbb{Z}^4 \mid n = ax^2 + by^2 + cz^2 + dw^2\}|$$

and

$$t(a, b, c, d; n) = \left| \left\{ (x, y, z, w) \in \mathbb{Z}^4 \mid n = a \frac{x(x-1)}{2} + b \frac{y(y-1)}{2} + c \frac{z(z-1)}{2} + d \frac{w(w-1)}{2} \right\} \right|.$$

The numbers $x(x-1)/2$ ($x \in \mathbb{Z}$) are called *triangular numbers*.

In 1828 Jacobi showed that

$$N(1, 1, 1, 1; n) = 8 \sum_{d|n, 4 \nmid d} d.$$

For $d \in \{3, 5\}$, in 1847 Eisenstein (see [13]) gave formulas for the number of *proper* representations of n by $x^2 + y^2 + z^2 + dw^2$ (assuming that $\gcd(x, y, z, w) = 1$). From 1859 to 1866 Liouville made about 90 conjectures on $N(a, b, c, d; n)$ in a series of papers. Most conjectures of Liouville have been proved. See [2–9], Cooper’s survey paper [12], Dickson’s historical comments [13] and Williams’ book [18].

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Let

$$t'(a, b, c, d; n) = \left| \left\{ (x, y, z, w) \in \mathbb{N}^4 \mid n = a \frac{x(x-1)}{2} + b \frac{y(y-1)}{2} + c \frac{z(z-1)}{2} + d \frac{w(w-1)}{2} \right\} \right|.$$

As $x(x-1)/2 = (-x+1)(-x)/2$ we have

$$(1.1) \quad t(a, b, c, d; n) = 16t'(a, b, c, d; n).$$

In [14] Legendre stated that

$$(1.2) \quad t'(1, 1, 1, 1; n) = \sigma(2n + 1).$$

In 2003, Williams [16] showed that

$$(1.3) \quad t'(1, 1, 2, 2; n) = \frac{1}{4} \sum_{d|4n+3} (d - (-1)^{d-1/2}).$$

For $a, b, c, d \in \mathbb{N}$ with $5 \leq a + b + c + d \leq 8$ let

$$C(a, b, c, d) = 16 + 4i_1(i_1 - 1)i_2 + 8i_1i_3,$$

where i_j is the number of elements in $\{a, b, c, d\}$ which are equal to j . When $a + b + c + d \in \{5, 6, 7\}$, in 2005 Adiga, Cooper and Han [1] showed that

$$(1.4) \quad C(a, b, c, d)t'(a, b, c, d; n) = N(a, b, c, d; 8n + a + b + c + d).$$

When $a + b + c + d = 8$, in 2008 Baruah, Cooper and Hirschhorn [10] proved that

$$(1.5) \quad C(a, b, c, d)t'(a, b, c, d; n) = N(a, b, c, d; 8n + 8) - N(a, b, c, d; 2n + 2).$$

In 2009, Cooper [12] determined $t'(a, b, c, d; n)$ for $(a, b, c, d) = (1, 1, 1, 3), (1, 3, 3, 3), (1, 2, 2, 3), (1, 3, 6, 6), (1, 3, 4, 4), (1, 1, 2, 6)$ and $(1, 3, 12, 12)$.

In [15], Wang and Sun obtained explicit formulas for $t(a, b, c, d; n)$ in the cases $(a, b, c, d) = (1, 2, 2, 4), (1, 2, 4, 4), (1, 1, 4, 4), (1, 4, 4, 4), (1, 3, 3, 9), (1, 1, 9, 9), (1, 9, 9, 9), (1, 1, 1, 9), (1, 3, 9, 9)$ and $(1, 1, 3, 9)$.

Ramanujan's theta functions $\varphi(q)$ and $\psi(q)$ are defined by

$$\varphi(q) = \sum_{n=-\infty}^{\infty} q^{n^2} = 1 + 2 \sum_{n=1}^{\infty} q^{n^2}, \quad \psi(q) = \sum_{n=0}^{\infty} q^{n(n+1)/2} \quad (|q| < 1).$$

It is evident that for $|q| < 1$,

$$(1.6) \quad \sum_{n=0}^{\infty} N(a, b, c, d; n)q^n = \varphi(q^a)\varphi(q^b)\varphi(q^c)\varphi(q^d),$$

$$(1.7) \quad \sum_{n=0}^{\infty} t'(a, b, c, d; n)q^n = \psi(q^a)\psi(q^b)\psi(q^c)\psi(q^d).$$

From [10, Lemma 4.1] or [11] we know that for $|q| < 1$,

$$(1.8) \quad \psi(q)^2 = \varphi(q)\psi(q^2),$$

$$(1.9) \quad \varphi(q) = \varphi(q^4) + 2q\psi(q^8),$$

$$(1.10) \quad \varphi(q)^2 = \varphi(q^2)^2 + 4q\psi(q^4)^2,$$

$$(1.11) \quad \psi(q)\psi(q^3) = \varphi(q^6)\psi(q^4) + q\varphi(q^2)\psi(q^{12}).$$

By (1.9), for $k \in \mathbb{N}$,

$$(1.12) \quad \varphi(q^k) = \varphi(q^{4k}) + 2q^k\psi(q^{8k}) = \varphi(q^{16k}) + 2q^{4k}\psi(q^{32k}) + 2q^k\psi(q^{8k}).$$

In this paper, using (1.6)–(1.12) we reveal some connections between $t(a, b, c, d; n)$ and $N(a, b, c, d; n)$. Suppose $k, m \in \{0, 1, 2, \dots\}$, $a, n \in \mathbb{N}$ and $2 \nmid a$. We show that

$$t(a, b, c, d; n) = \frac{2}{3} \left(N(a, b, c, d; 8n + a + b + c + d) - N(a, b, c, d; 2n + (a + b + c + d)/4) \right)$$

for $(a, b, c, d) = (a, a, 2a, 8m + 4)$ and $(a, 3a, 4k + 2, 4m + 2)$ with $k \equiv m \pmod{2}$. For $2 \nmid ak$ we show that

$$t(a, 3a, k, k; m) = \frac{2}{3} N(a, 3a, 2k, 2k; 8m + 4a + 2k).$$

For $n \equiv k + (a - 1)/2 \pmod{2}$, we prove that

$$t(a, 3a, 8k + 4, 4m + 2; n) = \frac{2}{3} N(a, 3a, 8k + 4, 4m + 2; 8n + 4m + 8k + 4a + 6).$$

Let $a, k, m \in \mathbb{N}$ with $2 \nmid a$. We also show that

$$t(a, a, 2a, 4k; 4m + 3a) = 4t(a, 2a, 4a, k; m),$$

$$t(a, a, 6a, 4k; 4m + 3a) = 2t(a, a, 6a, k; m),$$

$$t(a, a, 8a, 2k; 2m) = t(a, 2a, 2a, k; m),$$

$$t(a, a, 8a, 2k; 2m + a) = 2t(a, 4a, 4a, k; m).$$

In addition, we give explicit formulas for $t(1, 3, 3, 6; n)$, $t(1, 1, 8, 8; n)$ and $t(1, 1, 4, 8; n)$. We also pose many conjectures on the relations between $t(a, b, c, d; n)$ and $N(a, b, c, d; n)$.

2. Main results

LEMMA 2.1. *Suppose $a, k, m, n \in \mathbb{N}$ and $2 \nmid a$. Then*

$$N(a, a, 2k, 2m; 2n) = N(a, a, k, m; n).$$

Proof. Suppose $|q| < 1$. Using (1.10) we see that

$$\begin{aligned} \sum_{n=0}^{\infty} N(a, a, 2k, 2m; n)q^n &= \varphi(q^a)^2 \varphi(q^{2k}) \varphi(q^{2m}) \\ &= (\varphi(q^{2a})^2 + 4q^a \psi(q^{4a})^2) \varphi(q^{2k}) \varphi(q^{2m}). \end{aligned}$$

Extracting the even powers we obtain

$$\sum_{n=0}^{\infty} N(a, a, 2k, 2m; 2n)q^{2n} = \varphi(q^{2a})^2\varphi(q^{2k})\varphi(q^{2m}).$$

Replacing q with $q^{1/2}$ in the above formula we get

$$\begin{aligned} \sum_{n=0}^{\infty} N(a, a, 2k, 2m; 2n)q^n &= \varphi(q^a)^2\varphi(q^k)\varphi(q^m) \\ &= \sum_{n=0}^{\infty} N(a, a, k, m; n)q^n. \end{aligned}$$

Comparing the coefficients of q^n on both sides we obtain the result.

REMARK 2.1. In the case $a = 1$, Lemma 2.1 has been given in [5, Proposition 4.1(iii)].

LEMMA 2.2. For $|q| < 1$ we have

$$\varphi(q)^3 = \varphi(q^4)^3 + 6q\varphi(q^4)\psi(q^4)^2 + 12q^2\psi(q^4)^2\psi(q^8) + 8q^3\psi(q^8)^3.$$

Proof. By (1.8) and (1.9),

$$\begin{aligned} \varphi(q)^3 &= (\varphi(q^4) + 2q\psi(q^8))^3 \\ &= \varphi(q^4)^3 + 6q\varphi(q^4)\psi(q^8)(\varphi(q^4) + 2q\psi(q^8)) + 8q^3\psi(q^8)^3 \\ &= \varphi(q^4)^3 + 6q\psi(q^4)^2(\varphi(q^4) + 2q\psi(q^8)) + 8q^3\psi(q^8)^3. \end{aligned}$$

This yields the result.

THEOREM 2.1. Let $a \in \{1, 3, 5, \dots\}$ and $m \in \{0, 1, 2, \dots\}$. For $n \in \mathbb{N}$ we have

$$\begin{aligned} t(a, a, 2a, 8m + 4; n) &= \frac{2}{3} (N(a, a, a, 4m + 2; 4n + 4m + 2a + 2) \\ &\quad - N(a, a, a, 4m + 2; n + m + (a + 1)/2)) \\ &= \frac{2}{3} (N(a, a, 2a, 8m + 4; 8n + 8m + 4a + 4) \\ &\quad - N(a, a, 2a, 8m + 4; 2n + 2m + a + 1)). \end{aligned}$$

Proof. Suppose $|q| < 1$. By (1.6), (1.9) and Lemma 2.2, we find that

$$\begin{aligned} (2.1) \quad \sum_{n=0}^{\infty} N(a, a, a, 4m + 2; n)q^n &= \varphi(q^a)^3\varphi(q^{4m+2}) \\ &= (\varphi(q^{4a})^3 + 6q^a\varphi(q^{4a})\psi(q^{4a})^2 + 12q^{2a}\psi(q^{4a})^2\psi(q^{8a}) + 8q^{3a}\psi(q^{8a})^3) \\ &\quad \times (\varphi(q^{4(4m+2)}) + 2q^{4m+2}\psi(q^{8(4m+2)})). \end{aligned}$$

For any $r, s \in \mathbb{N}$ the power series expansions of $\varphi(q^{8s})^r$ and $\psi(q^{8s})^r$ are of the form $\sum_{n=0}^{\infty} b_n q^{8n}$. Hence from (2.1) we deduce that

$$\begin{aligned} \sum_{n=0}^{\infty} N(a, a, a, 4m + 2; 4n) q^{4n} \\ = \varphi(q^{4a})^3 \varphi(q^{16m+8}) + 12q^{2a} \psi(q^{4a})^2 \psi(q^{8a}) \cdot 2q^{4m+2} \psi(q^{32m+16}). \end{aligned}$$

Replacing q with $q^{1/4}$ in the above formula we obtain

$$\begin{aligned} \sum_{n=0}^{\infty} N(a, a, a, 4m + 2; 4n) q^n \\ = \varphi(q^a)^3 \varphi(q^{4m+2}) + 24q^{m+(a+1)/2} \psi(q^a)^2 \psi(q^{2a}) \psi(q^{8m+4}) \\ = \sum_{n=0}^{\infty} N(a, a, a, 4m + 2; n) q^n + 24q^{m+(a+1)/2} \sum_{n=0}^{\infty} t'(a, a, 2a, 8m + 4; n) q^n. \end{aligned}$$

Now comparing the coefficients of $q^{n+m+(a+1)/2}$ on both sides and then applying (1.1) and Lemma 2.1 we obtain

$$\begin{aligned} \frac{3}{2} t(a, a, 2a, 8m + 4; n) &= 24t'(a, a, 2a, 8m + 4; n) \\ &= N(a, a, a, 4m + 2; 4n + 4m + 2a + 2) \\ &\quad - N(a, a, a, 4m + 2; n + m + (a + 1)/2) \\ &= N(a, a, 2a, 8m + 4; 8n + 8m + 4a + 4) \\ &\quad - N(a, a, 2a, 8m + 4; 2n + 2m + a + 1). \end{aligned}$$

This is the result.

LEMMA 2.3. For $|q| < 1$ we have

$$\begin{aligned} \varphi(q)\varphi(q^3) &= \varphi(q^{16})\varphi(q^{48}) + 4q^{16}\psi(q^{32})\psi(q^{96}) + 2q\varphi(q^{48})\psi(q^8) \\ &\quad + 2q^3\varphi(q^{16})\psi(q^{24}) + 6q^4\psi(q^8)\psi(q^{24}) \\ &\quad + 4q^{13}\psi(q^8)\psi(q^{96}) + 4q^7\psi(q^{24})\psi(q^{32}). \end{aligned}$$

Proof. By (1.12),

$$\begin{aligned} \varphi(q)\varphi(q^3) &= (\varphi(q^{16}) + 2q^4\psi(q^{32}) + 2q\psi(q^8)) \\ &\quad \times (\varphi(q^{48}) + 2q^{12}\psi(q^{96}) + 2q^3\psi(q^{24})) \\ &= \varphi(q^{16})\varphi(q^{48}) + 4q^{16}\psi(q^{32})\psi(q^{96}) + 2q\varphi(q^{48})\psi(q^8) + 2q^3\varphi(q^{16})\psi(q^{24}) \\ &\quad + 2q^4(\varphi(q^{48})\psi(q^{32}) + q^8\varphi(q^{16})\psi(q^{96}) + 2\psi(q^8)\psi(q^{24})) \\ &\quad + 4q^{13}\psi(q^8)\psi(q^{96}) + 4q^7\psi(q^{24})\psi(q^{32}). \end{aligned}$$

Note that $\varphi(q^{48})\psi(q^{32}) + q^8\varphi(q^{16})\psi(q^{96}) = \psi(q^8)\psi(q^{24})$ by (1.11). Hence we obtain the result.

THEOREM 2.2. *Let $a \in \{1, 3, 5, \dots\}$, $k, m \in \{0, 1, 2, \dots\}$ and $k \equiv m \pmod{2}$. For $n \in \mathbb{N}$ we have*

$$t(a, 3a, 4k + 2, 4m + 2; n) = \frac{2}{3}(N(a, 3a, 4k + 2, 4m + 2; 8n + 4m + 4k + 4a + 4) - N(a, 3a, 4k + 2, 4m + 2; 2n + m + k + a + 1)).$$

Proof. Suppose $|q| < 1$. Using Lemma 2.3 and (1.12) we see that

$$\begin{aligned} \sum_{n=0}^{\infty} N(a, 3a, 4k + 2, 4m + 2; n)q^n &= \varphi(q^a)\varphi(q^{3a})\varphi(q^{4k+2})\varphi(q^{4m+2}) \\ &= \varphi(q^a)\varphi(q^{3a})(\varphi(q^{4(4k+2)}) + 2q^{4k+2}\psi(q^{8(4k+2)})) \\ &\quad \times (\varphi(q^{4(4m+2)}) + 2q^{4m+2}\psi(q^{8(4m+2)})) \\ &= (\varphi(q^{16a})\varphi(q^{48a}) + 4q^{16a}\psi(q^{32a})\psi(q^{96a}) + 2q^a\varphi(q^{48a})\psi(q^{8a}) \\ &\quad + 2q^{3a}\varphi(q^{16a})\psi(q^{24a}) + 6q^{4a}\psi(q^{8a})\psi(q^{24a}) \\ &\quad + 4q^{13a}\psi(q^{8a})\psi(q^{96a}) + 4q^{7a}\psi(q^{24a})\psi(q^{32a})) \\ &\quad \times (\varphi(q^{16k+8})\varphi(q^{16m+8}) + 2q^{4m+2}\varphi(q^{16k+8})\psi(q^{32m+16}) \\ &\quad + 2q^{4k+2}\psi(q^{32k+16})\varphi(q^{16m+8}) + 4q^{4k+4m+4}\psi(q^{32k+16})\psi(q^{32m+16})). \end{aligned}$$

For any $r, s \in \mathbb{N}$ the power series expansions of $\varphi(q^{8s})^r$ and $\psi(q^{8s})^r$ are of the form $\sum_{n=0}^{\infty} b_n q^{8n}$. Thus, from the above we deduce that

$$\begin{aligned} \sum_{n=0}^{\infty} N(a, 3a, 4k + 2, 4m + 2; 8n)q^{8n} \\ = (\varphi(q^{16a})\varphi(q^{48a}) + 4q^{16a}\psi(q^{32a})\psi(q^{96a}))\varphi(q^{16k+8})\varphi(q^{16m+8}) \\ + 24q^{4(k+m+a+1)}\psi(q^{8a})\psi(q^{24a})\psi(q^{32k+16})\psi(q^{32m+16}). \end{aligned}$$

Replacing q with $q^{1/8}$ in the above formula we obtain

$$\begin{aligned} \sum_{n=0}^{\infty} N(a, 3a, 4k + 2, 4m + 2; 8n)q^n \\ = (\varphi(q^{2a})\varphi(q^{6a}) + 4q^{2a}\psi(q^{4a})\psi(q^{12a}))\varphi(q^{2k+1})\varphi(q^{2m+1}) \\ + 24q^{(k+m+a+1)/2}\psi(q^a)\psi(q^{3a})\psi(q^{4k+2})\psi(q^{4m+2}). \end{aligned}$$

On the other hand, using (1.9) we see that

$$\begin{aligned} \sum_{n=0}^{\infty} N(a, 3a, 4k + 2, 4m + 2; n)q^n \\ = \varphi(q^a)\varphi(q^{3a})\varphi(q^{4k+2})\varphi(q^{4m+2}) \\ = (\varphi(q^{4a}) + 2q^a\psi(q^{8a}))(\varphi(q^{12a}) + 2q^{3a}\psi(q^{24a}))\varphi(q^{4k+2})\varphi(q^{4m+2}). \end{aligned}$$

Extracting the even powers leads to

$$\sum_{n=0}^{\infty} N(a, 3a, 4k + 2, 4m + 2; 2n)q^{2n} = (\varphi(q^{4a})\varphi(q^{12a}) + 4q^{4a}\psi(q^{8a})\psi(q^{24a}))\varphi(q^{4k+2})\varphi(q^{4m+2}).$$

Replacing q with $q^{1/2}$ we then obtain

$$\sum_{n=0}^{\infty} N(a, 3a, 4k + 2, 4m + 2; 2n)q^n = (\varphi(q^{2a})\varphi(q^{6a}) + 4q^{2a}\psi(q^{4a})\psi(q^{12a}))\varphi(q^{2k+1})\varphi(q^{2m+1}).$$

Hence

$$\begin{aligned} \sum_{n=0}^{\infty} (N(a, 3a, 4k + 2, 4m + 2; 8n) - N(a, 3a, 4k + 2, 4m + 2; 2n))q^n &= 24q^{(k+m+a+1)/2}\psi(q^a)\psi(q^{3a})\psi(q^{4k+2})\psi(q^{4m+2}) \\ &= 24q^{(k+m+a+1)/2} \sum_{n=0}^{\infty} t'(a, 3a, 4k + 2, 4m + 2; n)q^n \\ &= \frac{3}{2}q^{(k+m+a+1)/2} \sum_{n=0}^{\infty} t(a, 3a, 4k + 2, 4m + 2; n)q^n. \end{aligned}$$

Comparing the coefficients of $q^{n+(k+m+a+1)/2}$ yields the result.

THEOREM 2.3. *Let $a, k \in \mathbb{N}$ with $2 \nmid ak$. For $m \in \mathbb{N}$ we have*

$$t(a, 3a, k, k; m) = \frac{2}{3}N(a, 3a, 2k, 2k; 8m + 4a + 2k).$$

Proof. By (1.10),

$$\begin{aligned} \varphi(q^{2k})^2 &= \varphi(q^{4k})^2 + 4q^{2k}\psi(q^{8k})^2 \\ &= \varphi(q^{8k})^2 + 4q^{4k}\psi(q^{16k})^2 + 4q^{2k}\psi(q^{8k})^2. \end{aligned}$$

Thus, applying Lemma 2.3 we see that for $|q| < 1$,

$$\begin{aligned} \sum_{n=0}^{\infty} N(a, 3a, 2k, 2k; n)q^n &= \varphi(q^a)\varphi(q^{3a})\varphi(q^{2k})^2 \\ &= (\varphi(q^{16a})\varphi(q^{48a}) + 4q^{16a}\psi(q^{32a})\psi(q^{96a}) \\ &\quad + 2q^a\varphi(q^{48a})\psi(q^{8a}) + 2q^{3a}\varphi(q^{16a})\psi(q^{24a}) \\ &\quad + 6q^{4a}\psi(q^{8a})\psi(q^{24a}) + 4q^{13a}\psi(q^{8a})\psi(q^{96a}) + 4q^{7a}\psi(q^{24a})\psi(q^{32a})) \\ &\quad \times (\varphi(q^{8k})^2 + 4q^{4k}\psi(q^{16k})^2 + 4q^{2k}\psi(q^{8k})^2). \end{aligned}$$

For any $r, s \in \mathbb{N}$ the power series expansions of $\varphi(q^s)^r$ and $\psi(q^s)^r$ are of the form $\sum_{n=0}^{\infty} b_n q^{8n}$. Thus, from the above we deduce that

$$\begin{aligned} & \sum_{m=0}^{\infty} N(a, 3a, 2k, 2k; 8m + 4a + 2k) q^{8m+4a+2k} \\ &= 6q^{4a} \psi(q^{8a}) \psi(q^{24a}) \cdot 4q^{2k} \psi(q^{8k})^2 = 24q^{4a+2k} \psi(q^{8a}) \psi(q^{24a}) \psi(q^{8k})^2. \end{aligned}$$

Replacing q with $q^{1/8}$ we get

$$\begin{aligned} & \sum_{m=0}^{\infty} N(a, 3a, 2k, 2k; 8m + 4a + 2k) q^m = 24 \psi(q^a) \psi(q^{3a}) \psi(q^k)^2 \\ &= 24 \sum_{m=0}^{\infty} t'(a, 3a, k, k; m) q^m = \frac{24}{16} \sum_{m=0}^{\infty} t(a, 3a, k, k; m) q^m. \end{aligned}$$

Comparing the coefficients of q^m on both sides yields the result.

COROLLARY 2.1. *For $n \in \mathbb{N}$ we have*

$$N(2, 2, 3, 9; 8n + 6) = \frac{3}{5} N(1, 1, 3, 9; 8n + 6).$$

Proof. Taking $a = 3$ and $k = 1$ in Theorem 2.3 we see that $t(1, 1, 3, 9; m) = \frac{2}{3} N(2, 2, 3, 9; 8m + 14)$. On the other hand, from [15, proof of Theorem 2.3], we know that $t(1, 1, 3, 9; m) = \frac{2}{5} N(1, 1, 3, 9; 8m + 14)$. Thus, the result follows.

THEOREM 2.4. *Let $a \in \{1, 3, 5, \dots\}$ and $k, m \in \{0, 1, 2, \dots\}$. If $n \in \mathbb{N}$ and $n \equiv k + (a - 1)/2 \pmod{2}$, then*

$$t(a, 3a, 8k + 4, 4m + 2; n) = \frac{2}{3} N(a, 3a, 8k + 4, 4m + 2; 8n + 4m + 8k + 4a + 6).$$

Proof. Suppose $|q| < 1$. Using Lemma 2.3 and (1.12) we see that

$$\begin{aligned} & \sum_{n=0}^{\infty} N(a, 3a, 8k + 4, 4m + 2; n) q^n = \varphi(q^a) \varphi(q^{3a}) \varphi(q^{8k+4}) \varphi(q^{4m+2}) \\ &= \varphi(q^a) \varphi(q^{3a}) (\varphi(q^{4(8k+4)}) + 2q^{8k+4} \psi(q^{8(8k+4)})) \\ &\quad \times (\varphi(q^{4(4m+2)}) + 2q^{4m+2} \psi(q^{8(4m+2)})) \\ &= (\varphi(q^{16a}) \varphi(q^{48a}) + 4q^{16a} \psi(q^{32a}) \psi(q^{96a}) \\ &\quad + 2q^a \varphi(q^{48a}) \psi(q^{8a}) + 2q^{3a} \varphi(q^{16a}) \psi(q^{24a}) \\ &\quad + 6q^{4a} \psi(q^{8a}) \psi(q^{24a}) + 4q^{13a} \psi(q^{8a}) \psi(q^{96a}) + 4q^{7a} \psi(q^{24a}) \psi(q^{32a})) \\ &\quad \times (\varphi(q^{32k+16}) \varphi(q^{16m+8}) + 2q^{4m+2} \varphi(q^{32k+16}) \psi(q^{32m+16}) \\ &\quad + 2q^{8k+4} \psi(q^{64k+32}) \varphi(q^{16m+8}) + 4q^{8k+4m+6} \psi(q^{64k+32}) \psi(q^{32m+16})). \end{aligned}$$

For any $r, s \in \mathbb{N}$ the power series expansions of $\varphi(q^{8s})^r$ and $\psi(q^{8s})^r$ are of the form $\sum_{n=0}^{\infty} b_n q^{8n}$. Thus, from the above and the fact that $4m + 2 \equiv$

$4 - 2(-1)^m \pmod{8}$ we deduce that

$$\begin{aligned} \sum_{n=0}^{\infty} N(a, 3a, 8k + 4, 4m + 2; 8n + 4 - 2(-1)^m) q^{8n+4-2(-1)^m} \\ = (\varphi(q^{16a})\varphi(q^{48a}) + 4q^{16a}\psi(q^{32a})\psi(q^{96a}))2q^{4m+2}\varphi(q^{32k+16})\psi(q^{32m+16}) \\ + 6q^{4a}\psi(q^{8a})\psi(q^{24a}) \cdot 4q^{8k+4m+6}\psi(q^{64k+32})\psi(q^{32m+16}) \end{aligned}$$

and so

$$\begin{aligned} \sum_{n=0}^{\infty} N(a, 3a, 8k + 4, 4m + 2; 8n + 4 - 2(-1)^m) q^{8n} \\ = 2q^{8\lceil m/2 \rceil} \varphi(q^{32k+16})\psi(q^{32m+16}) (\varphi(q^{16a})\varphi(q^{48a}) + 4q^{16a}\psi(q^{32a})\psi(q^{96a})) \\ + 24q^{8(k+\lceil m/2 \rceil+(a+1)/2)} \psi(q^{8a})\psi(q^{24a})\psi(q^{64k+32})\psi(q^{32m+16}), \end{aligned}$$

where $\lceil x \rceil$ is the greatest integer not exceeding x . Replacing q with $q^{1/8}$ in the above formula we obtain

$$\begin{aligned} \sum_{n=0}^{\infty} N(a, 3a, 8k + 4, 4m + 2; 8n + 4 - 2(-1)^m) q^n \\ = 2q^{\lceil m/2 \rceil} \varphi(q^{4k+2})\psi(q^{4m+2}) (\varphi(q^{2a})\varphi(q^{6a}) + 4q^{2a}\psi(q^{4a})\psi(q^{12a})) \\ + 24q^{k+\lceil m/2 \rceil+(a+1)/2} \psi(q^a)\psi(q^{3a})\psi(q^{8k+4})\psi(q^{4m+2}). \end{aligned}$$

Suppose that $n \equiv k + (a - 1)/2 \pmod{2}$. Then $n + k + \lceil m/2 \rceil + (a + 1)/2 \equiv \lceil m/2 \rceil + 1 \pmod{2}$. Now comparing the coefficients of $q^{n+k+\lceil m/2 \rceil+(a+1)/2}$ in the above expansion we obtain

$$\begin{aligned} N(a, 3a, 8k + 4, 4m + 2; 8(n + k + \lceil m/2 \rceil + (a + 1)/2) + 4 - 2(-1)^m) \\ = 24t'(a, 3a, 8k + 4, 4m + 2; n) = \frac{3}{2}t(a, 3a, 8k + 4, 4m + 2; n). \end{aligned}$$

This yields the result.

THEOREM 2.5. *Let $a, k \in \mathbb{N}$ with $2 \nmid a$. For $m \in \mathbb{N}$ we have*

$$t(a, a, 6a, 4k; 4m + 3a) = 2t(a, a, 6a, k; m).$$

Proof. Suppose $|q| < 1$. Using (1.8)–(1.12) we see that

$$\begin{aligned} (2.2) \quad \sum_{n=0}^{\infty} t'(a, a, 6a, 4k; n) q^n \\ = \psi(q^a)^2\psi(q^{6a})\psi(q^{4k}) = \varphi(q^a)\psi(q^{2a})\psi(q^{6a})\psi(q^{4k}) \\ = (\varphi(q^{4a}) + 2q^a\psi(q^{8a})) (\varphi(q^{12a})\psi(q^{8a}) + q^{2a}\varphi(q^{4a})\psi(q^{24a}))\psi(q^{4k}) \\ = (\varphi(q^{4a})\varphi(q^{12a})\psi(q^{8a}) + 2q^a\psi(q^{8a})^2\varphi(q^{12a}) + q^{2a}\varphi(q^{4a})^2\psi(q^{24a}) \\ + 2q^{3a}\varphi(q^{4a})\psi(q^{8a})\psi(q^{24a}))\psi(q^{4k}). \end{aligned}$$

Collecting the terms of the form q^{4m+3a} we get

$$\begin{aligned} \sum_{m=0}^{\infty} t'(a, a, 6a, 4k; 4m + 3a)q^{4m+3a} &= 2q^{3a}\varphi(q^{4a})\psi(q^{8a})\psi(q^{24a})\psi(q^{4k}) \\ &= 2q^{3a}\psi(q^{4a})^2\psi(q^{24a})\psi(q^{4k}). \end{aligned}$$

Replacing q with $q^{1/4}$ we deduce that

$$\begin{aligned} \sum_{m=0}^{\infty} t'(a, a, 6a, 4k; 4m + 3a)q^m &= 2\psi(q^a)^2\psi(q^{6a})\psi(q^k) \\ &= 2 \sum_{m=0}^{\infty} t'(a, a, 6a, k; m)q^m. \end{aligned}$$

Hence

$$\begin{aligned} t(a, a, 6a, 4k; 4m + 3a) &= 16t'(a, a, 6a, 4k; 4m + 3a) \\ &= 32t'(a, a, 6a, k; m) = 2t(a, a, 6a, k; m) \end{aligned}$$

as asserted.

THEOREM 2.6. *Let $a, k \in \mathbb{N}$ with $2 \nmid a$. For $m \in \mathbb{N}$ we have*

$$t(a, a, 2a, 4k; 4m + 3a) = 4t(a, 2a, 4a, k; m).$$

Proof. Suppose $|q| < 1$. Using (1.8) and (1.12) we see that

$$\begin{aligned} \sum_{n=0}^{\infty} t'(a, a, 2a, 4k; n)q^n &= \psi(q^a)^2\psi(q^{2a})\psi(q^{4k}) = \varphi(q^a)\psi(q^{2a})^2\psi(q^{4k}) \\ &= \varphi(q^a)\varphi(q^{2a})\psi(q^{4a})\psi(q^{4k}) \\ &= (\varphi(q^{4a}) + 2q^a\psi(q^{8a}))(\varphi(q^{8a}) + 2q^{2a}\psi(q^{16a}))\psi(q^{4a})\psi(q^{4k}) \\ &= (\varphi(q^{4a})\varphi(q^{8a}) + 2q^a\varphi(q^{8a})\psi(q^{8a}) + 2q^{2a}\varphi(q^{4a})\psi(q^{16a}) \\ &\quad + 4q^{3a}\psi(q^{8a})\psi(q^{16a}))\psi(q^{4a})\psi(q^{4k}). \end{aligned}$$

Collecting the terms of the form q^{4m+3a} we get

$$\sum_{m=0}^{\infty} t'(a, a, 2a, 4k; 4m + 3a)q^{4m+3a} = 4q^{3a}\psi(q^{8a})\psi(q^{16a})\psi(q^{4a})\psi(q^{4k}).$$

Replacing q with $q^{1/4}$ we see that

$$\begin{aligned} \sum_{m=0}^{\infty} t'(a, a, 2a, 4k; 4m + 3a)q^m &= 4\psi(q^a)\psi(q^{2a})\psi(q^{4a})\psi(q^k) \\ &= 4 \sum_{m=0}^{\infty} t'(a, 2a, 4a, k; m)q^m. \end{aligned}$$

Hence

$$\begin{aligned} t(a, a, 2a, 4k; 4m + 3a) &= 16t'(a, a, 2a, 4k; 4m + 3a) \\ &= 64t'(a, 2a, 4a, k; m) = 4t(a, 2a, 4a, k; m), \end{aligned}$$

which completes the proof.

THEOREM 2.7. *Let $a, k \in \mathbb{N}$ with $2 \nmid a$. For $n \in \mathbb{N}$ we have*

$$\begin{aligned} t(a, a, 8a, 2k; 2n) &= t(a, 2a, 2a, k; n), \\ t(a, a, 8a, 2k; 2n + a) &= 2t(a, 4a, 4a, k; n). \end{aligned}$$

Proof. Suppose $|q| < 1$. Using (1.8) and (1.9) we see that

$$\begin{aligned} (2.3) \quad \sum_{n=0}^{\infty} t'(a, a, 8a, 2k; n)q^n &= \psi(q^a)^2 \psi(q^{8a}) \psi(q^{2k}) = \varphi(q^a) \psi(q^{2a}) \psi(q^{8a}) \psi(q^{2k}) \\ &= \psi(q^{2a}) (\varphi(q^{4a}) + 2q^a \psi(q^{8a})) \psi(q^{8a}) \psi(q^{2k}). \end{aligned}$$

Extracting the even powers and then applying (1.8) we see that

$$\begin{aligned} \sum_{n=0}^{\infty} t'(a, a, 8a, 2k; 2n)q^{2n} &= \psi(q^{2a}) \varphi(q^{4a}) \psi(q^{8a}) \psi(q^{2k}) \\ &= \psi(q^{2a}) \psi(q^{4a})^2 \psi(q^{2k}). \end{aligned}$$

Replacing q with $q^{1/2}$ we then get

$$\sum_{n=0}^{\infty} t'(a, a, 8a, 2k; 2n)q^n = \psi(q^a) \psi(q^{2a})^2 \psi(q^k) = \sum_{n=0}^{\infty} t'(a, 2a, 2a, k; n)q^n.$$

Hence

$$\begin{aligned} t(a, a, 8a, 2k; 2n) &= 16t'(a, a, 8a, 2k; 2n) \\ &= 16t'(a, 2a, 2a, k; n) = t(a, 2a, 2a, k; n). \end{aligned}$$

On the other hand, extracting the odd powers in (2.3) we see that

$$\sum_{n=0}^{\infty} t'(a, a, 8a, 2k; 2n + a)q^{2n+a} = \psi(q^{2a}) \cdot 2q^a \psi(q^{8a}) \psi(q^{8a}) \psi(q^{2k}).$$

Replacing q with $q^{1/2}$ we then get

$$\begin{aligned} \sum_{n=0}^{\infty} t'(a, a, 8a, 2k; 2n + a)q^n &= 2\psi(q^a) \psi(q^{4a})^2 \psi(q^k) \\ &= \sum_{n=0}^{\infty} 2t'(a, 4a, 4a, k; n)q^n. \end{aligned}$$

Hence

$$\begin{aligned}
 t(a, a, 8a, 2k; 2n + a) &= 16t'(a, a, 8a, 2k; 2n + a) \\
 &= 32t'(a, 4a, 4a, k; n) = 2t(a, 4a, 4a, k; n).
 \end{aligned}$$

This completes the proof.

THEOREM 2.8. *For $n \in \mathbb{N}$ we have*

$$t(1, 1, 8, 8; n) = \sigma(4n + 9) - (2 - (-1)^n) \sum_{\substack{(x,y) \in \mathbb{Z} \times \mathbb{Z}, x \equiv 1 \pmod{4} \\ 4n+9=x^2+4y^2}} x.$$

Proof. By Theorem 2.7, $t(1, 1, 8, 8; n) = t(1, 2, 2, 4; n/2)$ for even n , and $t(1, 1, 8, 8; n) = 2t(1, 4, 4, 4; (n-1)/2)$ for odd n . Now applying [15, Theorems 3.2 and 3.3] yields the result.

THEOREM 2.9. *For $n \in \mathbb{N}$ we have*

$$t(1, 1, 4, 8; n) = 2(-1)^n \sum_{d|4n+7} d \left(\frac{2}{d} \right) - (1 - (-1)^n) \sum_{\substack{(x,y) \in \mathbb{Z} \times \mathbb{Z}, x \equiv 1 \pmod{4} \\ 4n+7=x^2+2y^2}} x.$$

Proof. By Theorem 2.7, $t(1, 1, 4, 8; n) = t(1, 2, 2, 2; n/2)$ for even n , and $t(1, 1, 4, 8; n) = 2t(1, 2, 4, 4; (n-1)/2)$ for odd n . It is easily seen that

$$\begin{aligned}
 &t(1, 2, 2, 2; m) \\
 &= |\{(x, y, z, w) \in \mathbb{Z}^4 \mid 8m + 7 = x^2 + 2y^2 + 2z^2 + 2w^2, 2 \nmid xyzw\}| \\
 &= |\{(x, y, z, w) \in \mathbb{Z}^4 \mid 8m + 7 = x^2 + 2y^2 + 2z^2 + 2w^2\}| \\
 &= N(1, 2, 2, 2; 8m + 7).
 \end{aligned}$$

Now applying [17, (1.4)] and [15, Theorem 3.4] yields the result.

LEMMA 2.4. *For $|q| < 1$ we have*

$$\begin{aligned}
 \varphi(q)^2 &= \varphi(q^8)^2 + 4q^4\psi(q^{16})^2 + 4q^2\psi(q^8)^2 \\
 &\quad + 4q\varphi(q^{16})\psi(q^8) + 8q^5\psi(q^8)\psi(q^{32}).
 \end{aligned}$$

Proof. By (1.10),

$$\begin{aligned}
 \varphi(q)^2 - \varphi(q^8)^2 &= (\varphi(q)^2 - \varphi(q^2)^2) + (\varphi(q^2)^2 - \varphi(q^4)^2) + (\varphi(q^4)^2 - \varphi(q^8)^2) \\
 &= 4q\psi(q^4)^2 + 4q^2\psi(q^8)^2 + 4q^4\psi(q^{16})^2.
 \end{aligned}$$

By (1.8) and (1.9),

$$\psi(q^4)^2 = \varphi(q^4)\psi(q^8) = (\varphi(q^{16}) + 2q^4\psi(q^{32}))\psi(q^8).$$

Now combining the above we deduce the result.

THEOREM 2.10. *If $n \in \mathbb{N}$ and $8n + 13 = 3^\beta n_1$ with $n_1 \in \mathbb{N}$ and $3 \nmid n_1$, then*

$$\begin{aligned} t(1, 3, 3, 6; n) &= \frac{2}{3} N(1, 3, 3, 6; 8n + 13) \\ &= \frac{2}{3} \left(3^\beta + \binom{n_1}{3} \right) \prod_{p|n_1} \frac{p^{\text{ord}_p n_1 + 1} - \left(\frac{6}{p}\right)^{\text{ord}_p n_1 + 1}}{p - \left(\frac{6}{p}\right)}, \end{aligned}$$

where p runs through all distinct prime divisors of n_1 , $\left(\frac{a}{p}\right)$ is the Legendre symbol and $\text{ord}_p n_1$ is the unique nonnegative integer r such that $p^r \mid n_1$ but $p^{r+1} \nmid n_1$.

Proof. Suppose $|q| < 1$. By (1.12) and Lemma 2.4,

$$\begin{aligned} \sum_{n=0}^{\infty} N(1, 3, 3, 6; n)q^n &= \varphi(q)\varphi(q^6)\varphi(q^3)^2 \\ &= (\varphi(q^{16}) + 2q^4\psi(q^{32}) + 2q\psi(q^8))(\varphi(q^{24}) + 2q^6\psi(q^{48})) \\ &\quad \times (\varphi(q^{24})^2 + 4q^{12}\psi(q^{48})^2 + 4q^6\psi(q^{24})^2 \\ &\quad + 4q^3\varphi(q^{48})\psi(q^{24}) + 8q^{15}\psi(q^{24})\psi(q^{96})) \\ &= (\varphi(q^{16})\varphi(q^{24}) + 2q^4\psi(q^{32})\varphi(q^{24}) + 2q\psi(q^8)\varphi(q^{24}) \\ &\quad + 2q^6\varphi(q^{16})\psi(q^{48}) + 4q^{10}\psi(q^{32})\psi(q^{48}) + 4q^7\psi(q^8)\psi(q^{48})) \\ &\quad \times (\varphi(q^{24})^2 + 4q^{12}\psi(q^{48})^2 + 4q^6\psi(q^{24})^2 \\ &\quad + 4q^3\varphi(q^{48})\psi(q^{24}) + 8q^{15}\psi(q^{24})\psi(q^{96})). \end{aligned}$$

For any $r, s \in \mathbb{N}$ the power series expansions of $\varphi(q^s)^r$ and $\psi(q^s)^r$ are of the form $\sum_{n=0}^{\infty} b_n q^{8n}$. Thus, from the above and (1.8) we deduce that

$$\begin{aligned} \sum_{n=0}^{\infty} N(1, 3, 3, 6; 8n + 13)q^{8n+13} &= \sum_{n=0}^{\infty} N(1, 3, 3, 6; 8n + 5)q^{8n+5} \\ &= 2q\psi(q^8)\varphi(q^{24}) \cdot 4q^{12}\psi(q^{48})^2 + 2q^6\varphi(q^{16})\psi(q^{48}) \cdot 8q^{15}\psi(q^{24})\psi(q^{96}) \\ &\quad + 4q^{10}\psi(q^{32})\psi(q^{48}) \cdot 4q^3\psi(q^{24})\varphi(q^{48}) + 4q^7\psi(q^8)\psi(q^{48}) \cdot 4q^6\psi(q^{24})^2 \end{aligned}$$

and so

$$\begin{aligned} \sum_{n=0}^{\infty} N(1, 3, 3, 6; 8n + 13)q^{8n} &= 8\psi(q^8)\psi(q^{24})^2\psi(q^{48}) + 16q^8\varphi(q^{16})\psi(q^{48})\psi(q^{24})\psi(q^{96}) \\ &\quad + 16\psi(q^{24})\psi(q^{32})\psi(q^{48})\varphi(q^{48}) + 16\psi(q^8)\psi(q^{48})\psi(q^{24})^2. \end{aligned}$$

By (1.11),

$$\psi(q^{32})\varphi(q^{48}) + q^8\varphi(q^{16})\psi(q^{96}) = \psi(q^8)\psi(q^{24}).$$

Hence,

$$\begin{aligned} & \sum_{n=0}^{\infty} N(1, 3, 3, 6; 8n + 13)q^{8n} \\ &= 8\psi(q^8)\psi(q^{24})^2\psi(q^{48}) + 16\psi(q^{24})\psi(q^{48})\psi(q^8)\psi(q^{24}) + 16\psi(q^8)\psi(q^{48})\psi(q^{24})^2 \\ &= 40\psi(q^8)\psi(q^{24})^2\psi(q^{48}). \end{aligned}$$

Replacing q with $q^{1/8}$ we see that

$$\begin{aligned} \sum_{n=0}^{\infty} N(1, 3, 3, 6; 8n + 13)q^n &= 40\psi(q)\psi(q^3)^2\psi(q^6) = 40 \sum_{n=0}^{\infty} t(1, 3, 3, 6; n)q^n \\ &= \frac{5}{2} \sum_{n=0}^{\infty} t(1, 3, 3, 6; n)q^n. \end{aligned}$$

Therefore

$$t(1, 3, 3, 6; n) = \frac{2}{5}N(1, 3, 3, 6; 8n + 13).$$

Now applying [9, Theorem 4.1] yields the remaining part.

REMARK 2.2. Using Maple we find

$$t(a, b, c, d; n) = \frac{2}{5}N(a, b, c, d; 8n + a + b + c + d)$$

for $(a, b, c, d) = (1, 1, 1, 2), (1, 1, 1, 3), (1, 1, 2, 3), (1, 1, 3, 9), (1, 3, 3, 3), (1, 3, 3, 6)$ and $(1, 3, 9, 9)$. When $(a, b, c, d) = (1, 1, 1, 2), (1, 1, 1, 3)$ or $(1, 1, 2, 3)$ the result follows from (1.4). The formula for $t(1, 3, 3, 3; n)$ was given in [12, Theorem 5.5], and the cases $(1, 1, 3, 9)$ and $(1, 3, 9, 9)$ were solved by Wang and Sun in [15].

THEOREM 2.11. *If $n \in \mathbb{N}$ and $n \equiv 1, 2 \pmod{4}$, then*

$$t(1, 1, 4, 6; n) = 2N(1, 1, 4, 6; 2n + 3).$$

Proof. By (1.12) and Lemma 2.4,

$$\begin{aligned} (2.4) \quad & \sum_{n=0}^{\infty} N(1, 1, 4, 6; n)q^n = \varphi(q)^2\varphi(q^4)\varphi(q^6) \\ &= \varphi(q)^2(\varphi(q^{16}) + 2q^4\psi(q^{32}))(\varphi(q^{24}) + 2q^6\psi(q^{48})) \\ &= (\varphi(q^8)^2 + 4q^4\psi(q^{16})^2 + 4q^2\psi(q^8)^2 + 4q\varphi(q^{16})\psi(q^8) + 8q^5\psi(q^8)\psi(q^{32})) \\ & \quad \times (\varphi(q^{16})\varphi(q^{24}) + 4q^{10}\psi(q^{32})\psi(q^{48}) + 2q^4\varphi(q^{24})\psi(q^{32}) + 2q^6\varphi(q^{16})\psi(q^{48})). \end{aligned}$$

Thus,

$$\sum_{m=0}^{\infty} N(1, 1, 4, 6; 8m + 5)q^{8m+5} = 16q^5\psi(q^8)\psi(q^{32})\varphi(q^{16})\varphi(q^{24}),$$

and

$$\sum_{m=0}^{\infty} N(1, 1, 4, 6; 8m + 7)q^{8m+7} = 8q^7\psi(q^8)\psi(q^{48})(\varphi(q^{16})^2 + 4q^8\psi(q^{32})^2).$$

From the above, (1.8) and (1.10) we deduce that

$$\begin{aligned} \sum_{m=0}^{\infty} N(1, 1, 4, 6; 8m + 5)q^m &= 16\psi(q)\psi(q^4)\varphi(q^2)\varphi(q^3) \\ &= 16\psi(q)\psi(q^2)^2\varphi(q^3), \\ (2.5) \quad \sum_{m=0}^{\infty} N(1, 1, 4, 6; 8m + 7)q^m &= 8\psi(q)\psi(q^6)(\varphi(q^2)^2 + 4q\psi(q^4)^2) \\ &= 8\varphi(q)^2\psi(q)\psi(q^6). \end{aligned}$$

On the other hand, from (2.2) we know that

$$\begin{aligned} \sum_{m=0}^{\infty} t'(1, 1, 4, 6; 4m + 1)q^{4m+1} &= 2q\psi(q^4)\psi(q^8)^2\varphi(q^{12}), \\ \sum_{m=0}^{\infty} t'(1, 1, 4, 6; 4m + 2)q^{4m+2} &= q^2\varphi(q^4)^2\psi(q^4)\psi(q^{24}). \end{aligned}$$

Therefore,

$$\begin{aligned} \sum_{m=0}^{\infty} t'(1, 1, 4, 6; 4m + 1)q^m &= 2\psi(q)\psi(q^2)^2\varphi(q^3), \\ \sum_{m=0}^{\infty} t'(1, 1, 4, 6; 4m + 2)q^m &= \varphi(q)^2\psi(q)\psi(q^6). \end{aligned}$$

This together with (2.5) gives $N(1, 1, 4, 6; 8m + 5) = 8t'(1, 1, 4, 6; 4m + 1)$ and $N(1, 1, 4, 6; 8m + 7) = 8t'(1, 1, 4, 6; 4m + 2)$. To complete the proof, we recall that $t(a, b, c, d; n) = 16t'(a, b, c, d; n)$.

THEOREM 2.12. For $n \in \mathbb{N}$ with $n \equiv 1 \pmod{4}$ we have

$$t(2, 2, 3, 9; n) = \frac{4}{3}N(2, 2, 3, 9; 2n + 4).$$

Proof. Using (1.8)–(1.12) we see that for $|q| < 1$,

$$\begin{aligned} \sum_{n=0}^{\infty} t'(2, 2, 3, 9; n)q^n &= \psi(q^2)^2\psi(q^3)\psi(q^9) \\ &= \varphi(q^2)\psi(q^4)(\varphi(q^{18})\psi(q^{12}) + q^3\varphi(q^6)\psi(q^{36})) \\ &= (\varphi(q^8) + 2q^2\psi(q^{16}))\psi(q^4) \\ &\quad \times ((\varphi(q^{72}) + 2q^{18}\psi(q^{144}))\psi(q^{12}) + q^3(\varphi(q^{24}) + 2q^6\psi(q^{48}))\psi(q^{36})). \end{aligned}$$

Hence

$$\begin{aligned} \sum_{m=0}^{\infty} t'(2, 2, 3, 9; 4m + 5)q^{4m+5} &= (2q^5\varphi(q^{24})\psi(q^{16}) + 2q^9\varphi(q^8)\psi(q^{48}))\psi(q^4)\psi(q^{36}) \\ &= 2q^5\psi(q^4)\psi(q^{12})\psi(q^4)\psi(q^{36}) \end{aligned}$$

and so

$$\sum_{m=0}^{\infty} t'(2, 2, 3, 9; 4m + 5)q^m = 2\psi(q)^2\psi(q^3)\psi(q^9) = 2 \sum_{m=0}^{\infty} t'(1, 1, 3, 9; m)q^m.$$

Therefore,

$$\begin{aligned} t(2, 2, 3, 9; 4m + 5) &= 16t'(2, 2, 3, 9; 4m + 5) \\ &= 32t'(1, 1, 3, 9; m) = 2t(1, 1, 3, 9; m). \end{aligned}$$

Taking $a = 3$ and $k = 1$ in Theorem 2.3 we see that $t(1, 1, 3, 9; m) = \frac{2}{3}N(2, 2, 3, 9; 8m + 14)$. Thus, $t(2, 2, 3, 9; 4m + 5) = 2t(1, 1, 3, 9; m) = \frac{4}{3}N(2, 2, 3, 9; 8m + 14)$. This yields the result.

THEOREM 2.13. *For $n \in \mathbb{N}$ we have*

$$\begin{aligned} t(1, 2, 2, 6; n) &= \frac{1}{2}N(1, 1, 4, 6; 8n + 11), \\ t(1, 1, 8, 12; 2n) &= \frac{1}{2}N(1, 1, 8, 12; 16n + 22). \end{aligned}$$

Proof. From (2.4) and (1.8) we know that

$$\begin{aligned} \sum_{m=0}^{\infty} N(1, 1, 4, 6; 8m + 3)q^{8m+3} &= 32q^{11}\varphi(q^{16})\psi(q^8)\psi(q^{32})\psi(q^{48}) \\ &= 32q^{11}\psi(q^8)\psi(q^{16})^2\psi(q^{48}). \end{aligned}$$

Hence

$$\begin{aligned} \sum_{m=0}^{\infty} N(1, 1, 4, 6; 8m + 3)q^m &= 32q\psi(q)\psi(q^2)^2\psi(q^6) \\ &= 32 \sum_{n=0}^{\infty} t'(1, 2, 2, 6; n)q^{n+1} = 2 \sum_{n=0}^{\infty} t(1, 2, 2, 6; n)q^{n+1}. \end{aligned}$$

Now comparing the coefficients of q^{n+1} we obtain $N(1, 1, 4, 6; 8n + 11) = 2t(1, 2, 2, 6; n)$. Applying Lemma 2.1 and Theorem 2.7 we see that

$$\begin{aligned} t(1, 1, 8, 12; 2n) &= t(1, 2, 2, 6; n) = \frac{1}{2}N(1, 1, 4, 6; 8n + 11) \\ &= \frac{1}{2}N(1, 1, 8, 12; 16n + 22). \end{aligned}$$

The proof is complete.

THEOREM 2.14. For $m \in \mathbb{N}$ we have

$$t(1, 1, 6, 24; 4m + 1) = 2t(2, 2, 3, 3; m),$$

$$t(1, 1, 6, 24; 4m) = t(1, 1, 3, 3; m) = 2^{\alpha+4}\sigma(m_1),$$

where α and m_1 are given by $m + 1 = 2^\alpha 3^\beta m_1$ and $\gcd(m_1, 6) = 1$.

Proof. Taking $a = 1$ and $k = 6$ in (2.2) we see that

$$\sum_{n=0}^{\infty} t'(1, 1, 6, 24; n)q^n$$

$$= (\varphi(q^4)\varphi(q^{12})\psi(q^8) + 2q\psi(q^8)^2\varphi(q^{12}) + q^2\varphi(q^4)^2\psi(q^{24})$$

$$+ 2q^3\varphi(q^4)\psi(q^8)\psi(q^{24}))\psi(q^{24}).$$

Hence, using (1.8) we see that

$$\sum_{m=0}^{\infty} t'(1, 1, 6, 24; 4m)q^{4m} = \varphi(q^4)\psi(q^8)\varphi(q^{12})\psi(q^{24}) = \psi(q^4)^2\psi(q^{12})^2,$$

$$\sum_{m=0}^{\infty} t'(1, 1, 6, 24; 4m + 1)q^{4m+1} = 2q\psi(q^8)^2\varphi(q^{12})\psi(q^{24}) = 2q\psi(q^8)^2\psi(q^{12})^2.$$

Therefore,

$$\sum_{m=0}^{\infty} t'(1, 1, 6, 24; 4m)q^m = \psi(q)^2\psi(q^3)^2 = \sum_{m=0}^{\infty} t'(1, 1, 3, 3; m)q^m,$$

$$\sum_{m=0}^{\infty} t'(1, 1, 6, 24; 4m + 1)q^m = 2\psi(q^2)^2\psi(q^3)^2 = 2 \sum_{m=0}^{\infty} t'(2, 2, 3, 3; m)q^m.$$

Since $t(a, b, c, d; n) = 16t'(a, b, c, d; n)$, we obtain $t(1, 1, 6, 24; 4m + 1) = 2t(2, 2, 3, 3; m)$ and $t(1, 1, 6, 24; 4m) = t(1, 1, 3, 3; m)$. Now applying [15, Lemma 4.1] we deduce the theorem.

THEOREM 2.15. Suppose $a, b, c, d \in \mathbb{N}$, $2 \nmid ab$, $a \equiv b \pmod{4}$ and $c \equiv 4 \pmod{8}$. Then

$$t(a, b, c, d; n) = N(a, b, c, d; 8n + a + b + c + d)$$

$$- N(a, b, c, 4d; 8n + a + b + c + d).$$

Proof. For $m \in \mathbb{N}$ we see that

$$N(a, b, c, d; m) = |\{(x, y, z, w) \in \mathbb{Z}^4 \mid m = ax^2 + by^2 + cz^2 + dw^2, 2 \mid w\}|$$

$$+ |\{(x, y, z, w) \in \mathbb{Z}^4 \mid m = ax^2 + by^2 + cz^2 + dw^2, 2 \nmid w\}|.$$

Thus,

$$N(a, b, c, d; 8n + a + b + c + d) - N(a, b, c, 4d; 8n + a + b + c + d)$$

$$= |\{(x, y, z, w) \in \mathbb{Z}^4 \mid 8n + a + b + c + d = ax^2 + by^2 + cz^2 + dw^2, 2 \nmid w\}|.$$

Suppose $8n + a + b + c + d = ax^2 + by^2 + cz^2 + dw^2$ for some $x, y, z, w \in \mathbb{Z}$ and $2 \nmid w$. Then $ax^2 + by^2 + cz^2 \equiv a + b + c \pmod{8}$. If $2 \mid z$, then $ax^2 + by^2 \equiv ax^2 + by^2 + cz^2 \equiv a + b + c \equiv a + b + 4 \pmod{8}$. Since $2 \nmid ab$ and $a + b \equiv 2 \pmod{4}$ we see that $2 \nmid xy$ and so $ax^2 + by^2 \equiv a + b \pmod{8}$. We get a contradiction. Hence $2 \nmid z$ and so $ax^2 + by^2 \equiv a + b \pmod{8}$. This implies that $2 \nmid xy$. Therefore,

$$\begin{aligned} & N(a, b, c, d; 8n + a + b + c + d) - N(a, b, c, 4d; 8n + a + b + c + d) \\ &= |\{(x, y, z, w) \in \mathbb{Z}^4 \mid 8n + a + b + c + d = ax^2 + by^2 + cz^2 + dw^2, 2 \nmid w\}| \\ &= |\{(x, y, z, w) \in \mathbb{Z}^4 \mid 8n + a + b + c + d = ax^2 + by^2 + cz^2 + dw^2, 2 \nmid xyzw\}| \\ &= |\{(x, y, z, w) \in \mathbb{Z}^4 \mid 8n + a + b + c + d = a(2x - 1)^2 \\ &\qquad\qquad\qquad + b(2y - 1)^2 + c(2z - 1)^2 + d(2w - 1)^2\}| \\ &= |\{(x, y, z, w) \in \mathbb{Z}^4 \mid 8n = a \cdot 4x(x - 1) + b \cdot 4y(y - 1) \\ &\qquad\qquad\qquad + c \cdot 4z(z - 1) + d \cdot 4w(w - 1)\}| \\ &= t(a, b, c, d; n). \end{aligned}$$

This proves the theorem.

Calculations with Maple suggest the following conjectures on $t(a, b, c, d; n)$ (for $a \leq b \leq c \leq d$, $a \leq 10$, $b \leq 20$, $c \leq 30$ and $d \leq 40$).

CONJECTURE 2.1. *Let $n \in \mathbb{N}$ with $n \equiv 0, 3 \pmod{4}$. Then*

$$t(1, 1, 4, 6; n) = \frac{2}{3}N(1, 1, 4, 6; 8n + 12) - N(1, 1, 4, 6; 2n + 3).$$

CONJECTURE 2.2. *Let $n \in \mathbb{N}$ with $3 \mid n$. Then*

$$t(1, 1, 8, 12; n) = \frac{1}{2}N(1, 1, 8, 12; 8n + 22).$$

CONJECTURE 2.3. *Let $n \in \mathbb{N}$ with $3 \mid n$. Then*

$$t(1, 3, 8, 8; n) = \frac{1}{3}N(1, 3, 8, 8; 8n + 20) - 2N(1, 3, 8, 8; 2n + 5).$$

CONJECTURE 2.4. *Let $n \in \mathbb{N}$ with $n \equiv 0 \pmod{6}$. Then*

$$t(1, 2, 3, 8; n) = \frac{2}{3}N(1, 2, 3, 8; 8n + 14) - 2N(1, 2, 3, 8; 4n + 7).$$

CONJECTURE 2.5. *Let $n \in \mathbb{N}$ with $n \equiv 0, 2 \pmod{8}$. Then*

$$t(1, 2, 4, 17; n) = 4N(1, 2, 4, 17; n + 3).$$

CONJECTURE 2.6. *Let $n \in \mathbb{N}$. If $n \equiv 2, 3 \pmod{5}$, then*

$$t(1, 1, 5, 8; n) = \frac{1}{2}N(1, 1, 5, 8; 8n + 15).$$

CONJECTURE 2.7. *Let $n \in \mathbb{N}$. If $n \equiv 0, 3, 4, 6, 7 \pmod{9}$, then*

$$t(1, 1, 8, 9; n) = \frac{1}{2}N(1, 1, 8, 9; 8n + 19).$$

CONJECTURE 2.8. *Let $n \in \mathbb{N}$. If $n \equiv 0, 4, 7, 8, 9, 10 \pmod{13}$, then*

$$t(1, 1, 8, 13; n) = \frac{1}{2}N(1, 1, 8, 13; 8n + 23).$$

CONJECTURE 2.9. Let $n \in \mathbb{N}$. If $n \equiv 0, 3, 5, 6, 7 \pmod{11}$, then

$$t(1, 1, 4, 11; n) = \frac{1}{3}N(1, 1, 4, 11; 8n + 17).$$

CONJECTURE 2.10. Let $n \in \mathbb{N}$. If $n \equiv 0, 1, 2, 4, 7 \pmod{11}$, then

$$t(1, 1, 2, 22; n) = \frac{1}{3}N(1, 1, 2, 22; 8n + 26).$$

CONJECTURE 2.11. Let $n \in \mathbb{N}$ with $n \equiv 1 \pmod{3}$. Then

$$t(1, 3, 12, 36; n) = \frac{1}{2}N(1, 3, 12, 36; 8n + 52) - 2N(1, 3, 12, 36; 2n + 13).$$

CONJECTURE 2.12. Let $n \in \mathbb{N}$ with $n \equiv 1 \pmod{4}$. Then

$$t(3, 5, 20, 32; n) = \frac{1}{2}N(3, 5, 20, 32; 8n + 60) - 2N(3, 5, 20, 32; 2n + 15).$$

CONJECTURE 2.13. Let $n \in \mathbb{N}$ with $n \equiv 1 \pmod{4}$. Then

$$t(1, 6, 15, 18; n) = \frac{2}{3}N(1, 6, 15, 18; 8n + 40) - 2N(1, 6, 15, 18; 2n + 10).$$

CONJECTURE 2.14. Let $n \in \mathbb{N}$ with $n \equiv 1 \pmod{3}$. Then

$$t(1, 6, 18, 27; n) = \frac{2}{3}N(1, 6, 18, 27; 8n + 52) - 2N(1, 6, 18, 27; 2n + 13).$$

CONJECTURE 2.15. Let $n \in \mathbb{N}$ with $n \equiv 1 \pmod{3}$. Then

$$t(1, 8, 9, 18; n) = \frac{2}{3}N(1, 8, 9, 18; 8n + 36) - 2N(1, 8, 9, 18; 2n + 9).$$

CONJECTURE 2.16. Let $n \in \mathbb{N}$ with $4 \mid n$. Then

$$t(1, 7, 10, 30; n) = \frac{2}{3}N(1, 7, 10, 30; 8n + 48) - 2N(1, 7, 10, 30; 2n + 12).$$

CONJECTURE 2.17. Let $n \in \mathbb{N}$ with $n \equiv 3 \pmod{4}$. Then

$$t(1, 10, 15, 30; n) = \frac{2}{3}N(1, 10, 15, 30; 8n + 56) - 2N(1, 10, 15, 30; 2n + 14).$$

CONJECTURE 2.18. Let $n \in \mathbb{N}$ with $n \equiv 2 \pmod{8}$. Then

$$t(1, 7, 28, 28; n) = \frac{2}{3}N(1, 7, 28, 28; 8n + 64) - 2N(1, 7, 28, 28; 2n + 16).$$

CONJECTURE 2.19. Let $n \in \mathbb{N}$ with $n \equiv 8 \pmod{9}$. Then

$$t(1, 9, 16, 18; n) = \frac{2}{3}N(1, 9, 16, 18; 8n + 44) - 2N(1, 9, 16, 18; 2n + 11).$$

CONJECTURE 2.20. Let $n \in \mathbb{N}$ with $n \equiv 1, 7 \pmod{9}$. Then

$$t(1, 9, 18, 24; n) = \frac{2}{3}N(1, 9, 18, 24; 8n + 52) - 2N(1, 9, 18, 24; 2n + 13).$$

CONJECTURE 2.21. Let $n \in \mathbb{N}$ with $n \equiv 1, 4 \pmod{9}$. Then

$$t(1, 9, 18, 32; n) = \frac{2}{3}N(1, 9, 18, 32; 8n + 60) - 2N(1, 9, 18, 32; 2n + 15).$$

CONJECTURE 2.22. Let $n \in \mathbb{N}$ with $n \equiv 5 \pmod{9}$. Then

$$t(1, 9, 18, 40; n) = \frac{2}{3}N(1, 9, 18, 40; 8n + 68) - 2N(1, 9, 18, 40; 2n + 17).$$

CONJECTURE 2.23. Let $n \in \mathbb{N}$ with $n \equiv 2, 5 \pmod{9}$. Then

$$t(1, 10, 27, 30; n) = \frac{2}{3}N(1, 10, 27, 30; 8n + 68) - 2N(1, 10, 27, 30; 2n + 17).$$

We remark that Conjectures 2.1–2.5 have been checked for $n \leq 300$, Conjectures 2.6–2.17 have been checked for $n \leq 200$, Conjecture 2.18 has been checked for $n \leq 400$, and Conjectures 2.19–2.23 have been checked for $n \leq 500$.

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