

## Avoiding equal distances

by

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**Abstract.** We show that it is consistent that there is a nonmeager set of reals each of whose nonmeager subsets contains equal distances.

**1. Introduction.** In [1], Erdős and Kakutani showed that the continuum hypothesis (CH) is equivalent to the following statement: There is a partition of the set  $\mathbb{R}$  of reals into countably many rationally independent sets. It follows that, under CH, every nonmeager set of reals contains a nonmeager (in fact, everywhere nonmeager) subset avoiding equal distances. The aim of this note is to show that CH is needed here.

**THEOREM 1.1.** *It is consistent that there is a nonmeager  $X \subseteq \mathbb{R}$  such that for every nonmeager  $Y \subseteq X$ , there are  $a < b < c < d \in Y$  such that  $b - a = d - c$ .*

Note that, by a result of Rado (see [2, Theorem 3.2]), we cannot require  $Y$  to avoid arithmetic progressions of length 3. Also by [1],  $X$  cannot have size  $\aleph_1$ . So we start by adding  $\aleph_2$  Cohen reals and consider the set  $X$  of their pairwise sums. We then make every small subset  $Y$  of  $X$  meager using a finite support product where  $Y$  is small if it avoids equal distances. To capture the new subsets of  $X$  that may appear later, we use a  $\sigma$ -ideal  $\mathcal{I}$  (see below). The rest of the work is to show that  $X$  remains nonmeager in the final model. The dual problem for the null ideal will be dealt with in a forthcoming work.

## 2. Proof

**NOTATION.** We sometimes identify  $x \in 2^\omega$  with a real whose binary expansion is  $x$ . Addition is always the usual addition in  $\mathbb{R}$ . We also sometimes

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2010 *Mathematics Subject Classification:* Primary 03E35; Secondary 03E75.

*Key words and phrases:* forcing, Baire category.

Received 1 November 2015; revised 6 April 2016.

Published online 5 August 2016.

interpret  $y \in \mathbb{R}$  as a member of  $2^\omega$  which is the binary expansion of the fractional part of  $y$ . The relevant point here is that these transformations preserve meager sets.

Assume CH. Let  $S_\star = [\omega_2]^2$ . Let  $\mathcal{I} = \{S \subseteq S_\star : (\exists c : S \rightarrow \omega)(\forall A \in [\omega_2]^{\aleph_1})([A]^2 \subseteq S \Rightarrow |c[[A]^2]| \geq 2)\}$ . Note that, by the Erdős–Rado theorem,  $\mathcal{I}$  is a proper  $\sigma$ -ideal over  $S_\star$ .

Let  $\langle S_\gamma : \gamma < \gamma_\star \rangle$  be a one-one listing of  $\mathcal{I}$ . Let  $\mathbb{P}$  add  $\aleph_2$  Cohen reals  $\langle x_\alpha : \alpha < \omega_2 \rangle$ . In  $V^\mathbb{P}$ , let  $\mathbb{Q} = \prod \{\mathbb{Q}_\gamma : \gamma < \gamma_\star\}$  be the finite support product where  $\mathbb{Q}_\gamma = \mathbb{Q}_{S_\gamma}$  is a  $\sigma$ -centered forcing making  $\{x_\alpha + x_\beta : \{\alpha, \beta\} \in S_\gamma\}$  meager. For  $S \subseteq S_\star$ ,  $\mathbb{Q}_S$  is defined as follows:  $p \in \mathbb{Q}_S$  iff

- $p = (F_p, \bar{n}_p, \bar{\sigma}_p, N_p) = (F, \bar{n}, \bar{\sigma}, N)$ ,
- $F \subseteq [S]^2$  is finite,
- $\bar{n} = \langle n_k : k \leq N \rangle$  is an increasing sequence of integers with  $n_0 = 0$  and  $n_{k+1} - n_k > 2^{n_k - n_{k-1}}$ ,
- $\bar{\sigma} = \langle \sigma_k : k < N \rangle$  where  $\sigma_k \in {}^{[n_k, n_{k+1})}2$  for each  $k$ .

We write  $p \leq q$  iff  $F_p \subseteq F_q$ ,  $\bar{n}_p \preceq \bar{n}_q$ ,  $\bar{\sigma}_p \preceq \bar{\sigma}_q$  and for all  $N_p \leq k < N_q$  and  $\{\alpha, \beta\} \in F_p$ ,  $(x_\alpha + x_\beta) \upharpoonright [n_{q,k}, n_{q,k+1}) \neq \sigma_{q,k+1}$ . It is clear that  $\mathbb{Q}_S$  is a  $\sigma$ -centered forcing adding a meager set covering  $X_S = \{x_\alpha + x_\beta : \{\alpha, \beta\} \in S\}$ . We write  $X$  for  $X_{S_\star}$ .

Note that the set of conditions  $p = (p(0), p(1)) \in \mathbb{P} \star \mathbb{Q}$  such that for each  $\gamma \in \text{dom}(p(1))$ ,  $p(0)$  forces an actual value  $p(1)(\gamma)$ , and for every  $\{\alpha, \beta\} \in F_{p(1)(\gamma)}$ ,  $\{\alpha, \beta\} \subseteq \text{dom}(p(0))$ , is dense in  $\mathbb{P} \star \mathbb{Q}$ . We will always assume that our conditions have this form.

CLAIM 2.1. *In  $V^{\mathbb{P} \star \mathbb{Q}}$ , whenever  $Y \subseteq X$  is nonmeager, there are  $y_1 < y_2 < y_3 < y_4$  in  $Y$  such that  $y_2 - y_1 = y_4 - y_3$ .*

*Proof.* Choose  $S \subseteq S_\star$  such that  $Y = X_S$  is nonmeager and suppose  $p$  forces this. Let  $S_1 = \{\{\alpha, \beta\} : (\exists p_{\alpha, \beta} \geq p)(p_{\alpha, \beta} \Vdash \{\alpha, \beta\} \in \dot{S})\}$ . Then  $S_1 \in \mathcal{I}^+$ . Define an equivalence relation  $E$  on  $S_1$  as follows:  $\{\alpha_0, \beta_0\} E \{\alpha_1, \beta_1\}$  iff

- $|\text{dom}(p_{\alpha_0, \beta_0}(i))| = |\text{dom}(p_{\alpha_1, \beta_1}(i))| = l_i$  for  $i \in \{0, 1\}$ ; let  $\{\gamma_{\alpha_j, \beta_j, k}^i : k < l_i\}$  list  $\text{dom}(p_{\alpha_j, \beta_j}(i))$  in increasing order for  $i, j \in \{0, 1\}$ .
- $p_{\alpha_0, \beta_0}(0)(\gamma_{\alpha_0, \beta_0, k}^0) = p_{\alpha_1, \beta_1}(0)(\gamma_{\alpha_1, \beta_1, k}^0)$  for each  $k < l_0$ .
- $p_{\alpha_0, \beta_0}(1)(\gamma_{\alpha_0, \beta_0, k}^1)$  and  $p_{\alpha_1, \beta_1}(0)(\gamma_{\alpha_1, \beta_1, k}^1)$  have the same  $\bar{n}, \bar{\sigma}, N$  (but not necessarily  $F$ ), for each  $k < l_1$ .

It is clear that  $E$  is an equivalence relation on  $S_1$  with countably many equivalence classes. Since  $S_1 \in \mathcal{I}^+$ , we can choose  $A \in [\omega_2]^{\aleph_1}$  such that  $[A]^2 \subseteq S_1$  and  $\{\alpha_0, \beta_0\} E \{\alpha_1, \beta_1\}$  for all  $\{\alpha_0, \beta_0\}, \{\alpha_1, \beta_1\} \in [A]^2$ . Let  $l_0, l_1$  be the corresponding domain sizes. By the Ramsey theorem, there is an infinite  $A_1 \subseteq A$  such that whenever  $\alpha_0 < \beta_0$  and  $\alpha_1 < \beta_1$  are from  $A_1$ , for all  $i \in \{0, 1\}$  and  $k_0, k_1 < l_i$  the truth value of  $\gamma_{\alpha_0, \beta_0, k_0}^i = \gamma_{\alpha_1, \beta_1, k_1}^i$  depends

only on the order type of  $\langle \alpha_0, \beta_0, \alpha_1, \beta_1 \rangle$ . Choose  $\alpha_1 < \alpha_2 < \alpha_3 < \alpha_4 \in A_1$  such that  $A_1$  has at least two members between any two  $\alpha_i$ 's. It is easy to check that the conditions  $p_{\alpha_1, \alpha_3}, p_{\alpha_1, \alpha_4}, p_{\alpha_2, \alpha_3}, p_{\alpha_2, \alpha_4}$  have a least common extension  $q$ . For example, to see that  $p_{\alpha_1, \alpha_3}$  and  $p_{\alpha_2, \alpha_3}$  have a least common extension, choose some  $\beta \in A_1 \cap (\alpha_2, \alpha_3)$  and use the fact that  $\langle \alpha_1, \alpha_3, \beta, \alpha_3 \rangle$ ,  $\langle \alpha_2, \alpha_3, \beta, \alpha_3 \rangle$  and  $\langle \alpha_1, \alpha_3, \alpha_2, \alpha_3 \rangle$  have the same order type. Similarly, for  $p_{\alpha_2, \alpha_3}$  and  $p_{\alpha_1, \alpha_4}$ , choose  $\beta_1 < \beta_2$  from  $(\alpha_2, \alpha_3) \cap A_1$  and use the fact that  $\langle \alpha_1, \alpha_4, \beta_1, \beta_2 \rangle$ ,  $\langle \alpha_2, \alpha_3, \beta_1, \beta_2 \rangle$  and  $\langle \alpha_1, \alpha_4, \alpha_2, \alpha_3 \rangle$  have the same order type. Now  $q$  forces that  $x_{\alpha_1} + x_{\alpha_3}$ ,  $x_{\alpha_1} + x_{\alpha_4}$ ,  $x_{\alpha_2} + x_{\alpha_3}$  and  $x_{\alpha_1} + x_{\alpha_4}$  are in  $Y$  and  $(x_{\alpha_1} + x_{\alpha_3}) + (x_{\alpha_2} + x_{\alpha_4}) = (x_{\alpha_1} + x_{\alpha_4}) + (x_{\alpha_2} + x_{\alpha_3})$ . ■

CLAIM 2.2.  $X$  is nonmeager in  $V^{\mathbb{P} \star \mathbb{Q}}$ .

*Proof.* Suppose not. Let  $p_\star$  and  $\langle \dot{T}_m : m < \omega \rangle$  be such that  $p_\star \Vdash (\forall m)(\dot{T}_m \subseteq {}^{<\omega}2$  is a nowhere dense subtree)  $\wedge \dot{X} \subseteq \bigcup_m \dot{T}_m$ . Since  $\mathbb{P} \star \mathbb{Q}$  is ccc, we can assume that each  $\dot{T}_m$  is in  $V^{\mathbb{P} \star \prod_{k \geq 1} \mathbb{Q}_k}$  where  $\mathbb{Q}_k = \mathbb{Q}_{S_k}$  for some  $S_k \in \mathcal{I}$ . For each  $\{\alpha, \beta\} \in S_\star$ , choose  $p_{\alpha, \beta}$ ,  $m(\alpha, \beta)$ ,  $k(\alpha, \beta)$ ,  $v(\alpha, \beta)$ ,  $l(\alpha, \beta)$ ,  $n(\alpha, \beta)$  etc. such that:

- $p_{\alpha, \beta} \geq p_\star$  and  $p_{\alpha, \beta} \Vdash (\dot{x}_\alpha + \dot{x}_\beta) \in [\dot{T}_{m(\alpha, \beta)}]$ .
- $p_{\alpha, \beta} = \langle p_{\alpha, \beta}(k) : k \leq k(\alpha, \beta) \rangle$  where  $p_{\alpha, \beta}(0)$  is the Cohen part and  $p_{\alpha, \beta}(k) \in \mathbb{Q}_k$ .
- $\text{dom}(p_{\alpha, \beta}(0)) = v(\alpha, \beta)$ ,  $|v(\alpha, \beta)| = l(\alpha, \beta)$  and  $\alpha, \beta \in v(\alpha, \beta)$ .
- For each  $\gamma \in v(\alpha, \beta)$ ,  $p_{\alpha, \beta}(0)(\gamma) \in {}^{n(\alpha, \beta)}2$ .
- For each  $1 \leq k \leq k(\alpha, \beta)$ ,  $p_{\alpha, \beta}(k) = (F_{\alpha, \beta, k}, \bar{n}_{\alpha, \beta, k}, \bar{\sigma}_{\alpha, \beta, k}, N_{\alpha, \beta, k})$  where  $F_{\alpha, \beta, k} \subseteq [v_{\alpha, \beta}]^2 \cap [S_k]^2$  and  $n_{N_{\alpha, \beta, k}} = n(\alpha, \beta)$  does not depend on  $k$ .

Let  $\{\gamma_{\alpha, \beta, l} : l < l(\alpha, \beta)\}$  list  $v(\alpha, \beta)$  in increasing order. Since  $S_\star \in \mathcal{I}^+$ , as before, we can choose  $A \in [\omega_2]^{\aleph_0}$  such that for every  $\alpha < \beta$  from  $A$  the following hold:

- $\{\alpha, \beta\} \notin \bigcup_{k \geq 1} S_k$ .
- $m(\alpha, \beta) = m_\star$ ,  $k(\alpha, \beta) = k_\star$ ,  $l(\alpha, \beta) = l_\star$ ,  $n(\alpha, \beta) = n_\star$ .
- For each  $l < l_\star$ ,  $p_{\alpha, \beta}(0)(\gamma_{\alpha, \beta, l}) = \eta_\star^l \in {}^{n_\star}2$ .
- For each  $1 \leq k \leq k_\star$ ,  $\bar{n}_{\alpha, \beta, k} = \bar{n}_\star^k$ ,  $\bar{\sigma}_{\alpha, \beta, k} = \bar{\sigma}_\star^k$ ,  $N_{\alpha, \beta, k} = N_\star^k$ .
- For all  $\alpha_1 < \alpha_2$  and  $\beta_1 < \beta_2$  from  $A$  and  $l_1, l_2 < l_\star$ , the truth value of  $\gamma_{\alpha_1, \beta_1, l_1} = \gamma_{\alpha_2, \beta_2, l_2}$  depends only on the order type of  $\langle \alpha_1, \alpha_2, \beta_1, \beta_2 \rangle$ .

Let  $\langle \alpha_i : i < \omega \rangle$  be increasing members of  $A$ . Choose  $n_{\star\star} > n_\star + (k_\star + l_\star + 10)!$ . Choose  $q_1 \geq p_{\alpha_0, \alpha_1}$  such that

- $q_1 = \langle q_1(k) : k \leq k_\star \rangle$ .
- $\text{dom}(q_1(0)) = \text{dom}(p_{\alpha_0, \alpha_1})$  and  $q_1(0)(l) = \eta_\star^l \frown 0^{n_{\star\star} - n_\star}$  for each  $l < l_\star$ .
- $q_1(k) = (F_{\alpha_\star, \beta_\star, k}, \bar{n}_\star^k \frown n_{\star\star}, \bar{\sigma}_\star^k \frown 01^{n_{\star\star} - n_\star - 2}0, N_\star^k + 1)$  for each  $1 \leq k \leq k_\star$ .

Since  $p_\star$  forces that  $[T_{m_\star}^\circ]$  is nowhere dense, we can find  $n_{\star\star\star} > n_{\star\star}$ ,  $q_2 \geq q_1$  and  $\rho \in [n_{\star\star}, n_{\star\star\star})2$  such that:

- $q_2 = \langle q_2(k) : k \leq K \rangle$  for some  $k_\star \leq K < \omega$ .
- For  $1 \leq k \leq k_\star$ , if  $q_2(k) = (F_k, \bar{n}_k, \bar{\sigma}_k, N_k)$ , then  $n_{\star\star\star} < n_{k, N_k}$ .
- $q_2 \Vdash (\forall x \in [T_{m_\star}^\circ])(x \upharpoonright [n_{\star\star}, n_{\star\star\star}) \neq \rho$ .

For  $j \geq 2$ , consider the set  $s_j = \{l < l_\star : (\exists l' < l_\star)(\gamma_{\alpha_0, \alpha_1, l} = \gamma_{\alpha_j, \alpha_{j+1}, l'})\}$ . We claim that  $s_j = s_\star$  is constant. To see this, suppose  $2 \leq j_1 < j_2$ . Choose  $j_3$  much larger than  $j_2$  and use the fact that the order types of  $\langle \alpha_0, \alpha_1, \alpha_{j_i}, \alpha_{j_i+1} \rangle$ ,  $\langle \alpha_0, \alpha_1, \alpha_{j_3}, \alpha_{j_3+1} \rangle$  and  $\langle \alpha_{j_i}, \alpha_{j_i+1}, \alpha_{j_3}, \alpha_{j_3+1} \rangle$  are the same for  $i \in \{1, 2\}$ . It also follows that  $\{\gamma_{\alpha_{j_1}, \alpha_{j_1+1}, l} : l \in l_\star \setminus s_\star\} \cap \{\gamma_{\alpha_{j_2}, \alpha_{j_2+1}, l} : l \in l_\star \setminus s_\star\} = \emptyset$  whenever  $2 \leq j_1 < j_2 - 1$ . So we can choose  $j$  large enough such that  $(\text{dom}(q_2(0)) \cup \bigcup_{k \leq K} F_{q_2(k)}) \cap (\{\gamma_{\alpha_j, \alpha_{j+1}, l} : l \in l_\star \setminus s_\star\} \cup \{\alpha_j, \alpha_{j+1}\}) = \emptyset$ . The next claim gives us the desired contradiction.

CLAIM 2.3. For some  $q_3, q_3 \geq q_2, q_3 \geq p_{\alpha_j, \alpha_{j+1}}$  and  $q_3 \Vdash \rho \subseteq \dot{x}_{\alpha_j} + \dot{x}_{\alpha_{j+1}}$ .

*Proof.* Set  $q_3 = \langle q_3(k) : k \leq K \rangle$  and  $\text{dom}(q_3(0)) = \text{dom}(q_2(0)) \cup \{\gamma_{\alpha_j, \alpha_{j+1}, l} : l \in l_\star \setminus s_\star\}$ . For each  $l \in l_\star \setminus s_\star$ , we would like to find  $\eta_{\star\star}^l \succeq \eta_\star^l$  such that:

- (a) If  $l_1 < l_2 \in l_\star \setminus s_\star, 1 \leq k \leq k_\star, \{\gamma_{\alpha_j, \alpha_{j+1}, l_1}, \gamma_{\alpha_j, \alpha_{j+1}, l_2}\} \in S_k$  and  $N_\star^k \leq i < N_{q_2(k)}$ , then  $(\eta_{\star\star}^{l_1} + \eta_{\star\star}^{l_2}) \upharpoonright [n_{q_2(k), i}, n_{q_2(k), i+1}] \neq \sigma_{q_2(k), i}$ .
- (b) If  $l \in s_\star, l' \in l_\star \setminus s_\star, 1 \leq k \leq k_\star$  and  $N_\star^k \leq i < N_{q_2(k)}$ , then  $(q_2(0)(\gamma_{\alpha_0, \alpha_1, l}) + \eta_{\star\star}^{l'}) \upharpoonright [n_{q_2(k), i}, n_{q_2(k), i+1}] \neq \sigma_{q_2(k), i}$ .
- (c) If  $\gamma_{\alpha_j, \alpha_{j+1}, l_1} = \alpha_j, \gamma_{\alpha_j, \alpha_{j+1}, l_2} = \alpha_{j+1}$  (so that  $l_1, l_2 \in l_\star \setminus s_\star$ ), then  $(\eta_{\star\star}^{l_1} + \eta_{\star\star}^{l_2}) \upharpoonright [n_{\star\star}, n_{\star\star\star}) = \rho$ .

Here, for  $\sigma, \tau \in 2^{<\omega}, m < n < \omega$  and  $\rho : [m, n) \rightarrow 2$ , by  $(\sigma + \tau) \upharpoonright [m, n) \neq \rho$  we mean that  $(x + y) \upharpoonright [m, n) \neq \rho$  for all  $x \in [\sigma]$  and  $y \in [\tau]$ .

This suffices since then we can let  $q_3(0) = q_2(0) \cup \{(\gamma_{\alpha_j, \alpha_{j+1}, l}, \eta_{\star\star}^l) : l \in l_\star \setminus s_\star\}$  and  $q_3(k) = (F_{q_2(k)} \cup F_{p_{\alpha_j, \alpha_{j+1}}(k)}, \bar{n}_{q_2(k)}, \bar{\sigma}_{q_2(k)}, N_{q_2(k)})$  for  $1 \leq k \leq K$ .

First set  $\eta_{\star\star}^l \upharpoonright [n_\star, n_{\star\star}) = 0^{n_{\star\star} - n_\star}$  for every  $l \in l_\star \setminus s_\star$ . Next let  $W = \{(k, i) : 1 \leq k \leq k_\star, N_\star^k + 1 \leq i < N_{q_2(k)}\}$ . Note that for  $(k, i) \in W$ , we have  $n_{q_2(k), i+1} - n_{q_2(k), i} > 2^{i - N_\star^k} (n_{\star\star} - n_\star) > 2^{i - N_\star^k} (k_\star + l_\star + 10)!$ .

Inductively choose pairwise disjoint intervals  $\langle I_{k,i} : (k, i) \in W \rangle$  with  $I_{k,i} = [m_{k,i}, m_{k,i} + (l_\star + 5)!) \subseteq [n_{q_2(k), i}, n_{q_2(k), i+1})$ . We claim that for each  $(k, i) \in W$ , we can choose  $\langle \eta_{\star\star}^l \upharpoonright I_{k,i} : l \in l_\star \setminus s_\star \rangle$  such that the  $(k, i)$ th instance of requirements (a), (b) are met. To see this, note that we have at most  $\binom{l_\star - |s_\star|}{2} + |s_\star|(l_\star - |s_\star|)$  inequalities (coming from (a) and (b)) and one equality from (c) to satisfy, and since  $\{\alpha_j, \alpha_{j+1}\} \notin \bigcup \{S_k : k \geq 1\}$ , there is no conflict between requirements (a) and (c). ■

**Acknowledgments.** Research of A. Kumar was supported by a Post-doctoral Fellowship at the Einstein Institute of Mathematics funded by European Research Council grant 338821.

Research of S. Shelah was partially supported by European Research Council grant 338821; publication no. 1079.

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