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# Dyadic weights on $\mathbb{R}^n$ and reverse Hölder inequalities

by

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**Abstract.** We prove that for any weight $\phi$ defined on $[0,1]^n$ that satisfies a reverse Hölder inequality with exponent $p > 1$ and constant $c \geq 1$ on all dyadic subcubes of $[0,1]^n$, its non-increasing rearrangement $\phi^*$ satisfies a reverse Hölder inequality with the same exponent and constant not more than $2^n c - 2^n + 1$ on all subintervals of the form $[0,t]$, $0 < t \leq 1$. As a consequence, there is an interval $[p, p_0(p,c)) = I_{p,c}$ such that $\phi \in L^q$ for any $q \in I_{p,c}$.

**1. Introduction.** The theory of Muckenhoupt's weights has proved to be an important tool in analysis. One of the most important facts about these weights is their self-improving property. A way to express this is through the so called reverse Hölder inequalities (see [1], [3] and [7]).

Here we will study such inequalities in a dyadic setting. We will say that a measurable function $g : [0,1] \to \mathbb{R}^+$ satisfies a *reverse Hölder inequality* with exponent $p > 1$ and constant $c \geq 1$ if

$$\frac{1}{b-a}\int_a^b g(u)^p\,du \leq c\left(\frac{1}{b-a}\int_a^b g(u)\,du\right)^p \tag{1.1}$$

for every subinterval $[a,b]$ of $[0,1]$.

The following is proved in [2]:

THEOREM A. *Let $g$ be a nonincreasing function defined on $[0,1]$ which satisfies (1.1) for every interval $[a,b] \subseteq [0,1]$. Define $p_0 > p$ as the root of the equation*

$$\frac{p_0 - p}{p_0}\left(\frac{p_0}{p_0-1}\right)^p \cdot c = 1. \tag{1.2}$$

*Then $g \in L^q([0,1])$ for any $q \in [p, p_0)$. Additionally for every $q$ in the above*

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*range, g satisfies a reverse Hölder inequality with possibly another constant* $c' \geq 1$. *Moreover the result is sharp, that is, the value* $p_0$ *cannot be increased.*

In [4] and [5], the following is proved:

THEOREM B. *If* $\phi : [0,1] \to \mathbb{R}^+$ *is integrable satisfying* (1.1) *for every* $[a,b] \subseteq [0,1]$, *then its non-increasing rearrangement* $\phi^*$ *satisfies the same inequality with the same constant* $c$.

Here $\phi^*$ is defined on $(0,1]$ by

$$\phi^*(t) = \sup_{\substack{E \subseteq [0,1] \\ |E|=t}} \inf_{x \in E} |\phi(x)|, \qquad t \in (0,1].$$

It can also be defined as the unique left continuous, non-increasing function equimeasurable to $|\phi|$, that is, for every $\lambda > 0$,

$$|\{\phi > \lambda\}| = |\{\phi^* > \lambda\}|,$$

where $|\cdot|$ denotes the Lebesgue measure on $[0,1]$.

An immediate consequence of Theorem B is that Theorem A can be generalized by omitting the assumption of the monotonicity of $g$.

Recently, in [8], the following was proved:

THEOREM C. *Let* $g : (0,1] \to \mathbb{R}^+$ *be a non-increasing function which satisfies* (1.1) *on every interval* $(0,t]$, $0 < t \leq 1$, *that is,*

$$\frac{1}{t}\int_0^t g(u)^p\,du \leq c\left(\frac{1}{t}\int_0^t g(u)\,du\right)^p \tag{1.3}$$

*for every* $t \in (0,1]$. *Define* $p_0$ *by* (1.2). *Then for any* $q \in [p, p_0)$,

$$\frac{1}{t}\int_0^t g(u)^q\,du \leq c'\left(\frac{1}{t}\int_0^t g(u)\,du\right)^q \tag{1.4}$$

*for every* $t \in (0,1]$ *and some constant* $c' \geq c$. *Thus* $g \in L^q((0,1])$ *for any such* $q$. *Moreover the result is sharp, that is, we cannot increase* $p_0$.

A consequence of Theorem C is that under the assumption that $g$ is non-increasing, the hypothesis that (1.1) is satisfied only on all intervals $(0,t]$ is enough for the existence of a $p' > p$ for which $g \in L^{p'}([0,1])$.

In several dimensions, as far as we know, there does not exist any result similar to Theorems A, B and C. All we know is the following, which can be found in [3].

THEOREM D. *Let* $Q_0 \subseteq \mathbb{R}^n$ *be a cube and* $\phi : Q_0 \to \mathbb{R}^+$ *a measurable function that satisfies*

$$\frac{1}{|Q|}\int_Q \phi^p \leq c\left(\frac{1}{|Q|}\int_Q \phi\right)^p \tag{1.5}$$

*for fixed constants $p > 1$ and $c \geq 1$, and every cube $Q \subseteq Q_0$. Then there exists $\varepsilon = \varepsilon(n,p,c)$ such that*

$$\frac{1}{|Q|}\int_Q \phi^q \leq c'\bigg(\frac{1}{|Q|}\int_Q \phi\bigg)^q \tag{1.6}$$

*for every $q \in [p, p+\varepsilon)$, any cube $Q \subseteq Q_0$ and some constant $c' = c'(q,p,n,c)$.*

In several dimensions no estimate of the quantity $\varepsilon$ has been found. The purpose of this work is to study the multidimensional case in the dyadic setting. More precisely, we consider a measurable function $\phi$ defined on $[0,1]^n = Q_0$ which satisfies (1.5) for any dyadic subcube $Q$ of $Q_0$. These cubes can be realized by bisecting the sides of $Q_0$, then bisecting every side of the resulting dyadic cube and so on. We denote by $\mathcal{T}_{2^n}$ the tree consisting of the above mentioned dyadic subcubes of $[0,1]^n$. We will prove the following:

Theorem 1. *Let $\phi : Q_0 = [0,1]^n \to \mathbb{R}^+$ be such that*

$$\frac{1}{|Q|}\int_Q \phi^p \leq c\bigg(\frac{1}{|Q|}\int_Q \phi\bigg)^p \tag{1.7}$$

*for any $Q \in \mathcal{T}_{2^n}$ and some fixed constants $p > 1$ and $c \geq 1$. Let $h = \phi^*$ be the non-increasing rearrangement of $\phi$. Then*

$$\frac{1}{t}\int_0^t h(u)^p\,du \leq (2^n c - 2^n + 1)\bigg(\frac{1}{t}\int_0^t h(u)\,du\bigg)^p \tag{1.8}$$

*for any $t \in [0,1]$.*

As a consequence, $h = \phi^*$ satisfies the assumptions of Theorem C, which produces an $\varepsilon_1 = \varepsilon_1(n,p,c) > 0$ such that $h \in L^q([0,1])$ for any $q \in [p, p+\varepsilon_1)$. Thus $\phi \in L^q([0,1]^n)$ for any such $q$. That is, we can find an explicit value of $\varepsilon_1$. This is stated as Corollary 3.1.

As a matter of fact we prove Theorem 1 in a much more general setting of a non-atomic probability space $(X,\mu)$ equipped with a tree $\mathcal{T}_k$, which is a $k$-homogeneous tree for a fixed integer $k > 1$, and plays the role of dyadic sets as in $[0,1]^n$ (see the definition in Section 2).

As we shall see later, Theorem 1 is independent of the shape of the dyadic sets and depends only on the homogeneity of the tree $\mathcal{T}_k$. Additionally we mention that the inequality (1.8) need not be satisfied, under the assumptions of Theorem 1, if one replaces the intervals $(0,t]$ by $(t,1]$. That is, $\phi^*$ is not necessarily a weight on $(0,1]$ satisfying a reverse Hölder inequality on all subintervals of $[0,1]$, and one can easily construct a relevant counterexample.

Additionally we mention that a study of dyadic $A_1$-weights appears in [6], where one can find for any $c > 1$ the best possible range $[1,p)$ for which the following holds: $\phi \in A_1^d(c) \Rightarrow \phi \in L^q$ for any $q \in [1,p)$. Finally, results connected with $A_1$ dyadic weights $\phi$ and the behavior of $\phi^*$ as an $A_1$-weight on $\mathbb{R}$ can be found in [9].

**2. Preliminaries.** Let $(X,\mu)$ be a non-atomic probability space. We introduce the notion of a $k$-homogeneous tree on $X$.

DEFINITION 2.1. Let $k > 1$ be an integer. A set $\mathcal{T}_k$ will be called a *k-homogeneous tree* on $X$ if:

(i) $X \in \mathcal{T}_k$.

(ii) For every $I \in \mathcal{T}_k$, there is a subset $C(I) \subseteq \mathcal{T}_k$ consisting of $k$ subsets of $I$ such that

(a) the elements of $C(I)$ are pairwise disjoint,
(b) $I = \bigcup C(I)$,
(c) $\mu(J) = k^{-1}\mu(I)$ for every $J \in C(I)$.

(iii) $\mathcal{T}_k$ differentiates $L^1(X,\mu)$, that is, for every $\phi \in L^1(X,\mu)$,

$$\lim_{\substack{x\in I\in\mathcal{T}_k\\ \mu(I)\to 0}} \frac{1}{\mu(I)}\int_I \phi\,d\mu = \phi(x)$$

$\mu$-almost everywhere on $X$.

For example one can consider $X = [0,1]^n$, the unit cube of $\mathbb{R}^n$. Let $\mu$ be the Lebesgue measure on this cube. Then the set $\mathcal{T}_k$ of all dyadic subcubes of $X$ is a tree of homogeneity $k = 2^n$, with $C(Q)$ being the set of $2^n$ subcubes of $Q$, obtained by bisecting the sides of every $Q \in \mathcal{T}_k$, starting from $Q = X$.

Let now $(X,\mu)$ be as above and let $\mathcal{T}_k$ be a tree on $X$ as in Definition 2.1. From now on, we fix $k$ and write $\mathcal{T} = \mathcal{T}_k$. For any $I \in \mathcal{T}$, $I \neq X$, we denote by $I^*$ the smallest element of $\mathcal{T}$ such that $I^* \supsetneq I$. That is, $I^*$ is the unique element of $\mathcal{T}$ such that $I \in C(I^*)$. Then $\mu(I^*) = k\mu(I)$.

DEFINITION 2.2. For any $(X,\mu)$ and $\mathcal{T}$ as above we define the *dyadic maximal operator* on $X$ with respect to $\mathcal{T}$, denoted $\mathcal{M}_{\mathcal{T}}$, by

$$\mathcal{M}_{\mathcal{T}}\phi(x) = \sup\left\{\frac{1}{\mu(I)}\int_I |\phi|\,d\mu : x \in I \in \mathcal{T}\right\} \tag{2.1}$$

for any $\phi \in L^1(X,\mu)$ and $x \in X$.

REMARK 2.1. It is not difficult to see that the maximal operator defined by (2.1) satisfies a weak-type $(1,1)$ inequality

$$\mu(\{\mathcal{M}_{\mathcal{T}}\phi > \lambda\}) \le \frac{1}{\lambda}\int_{\{\mathcal{M}_{\mathcal{T}}\phi>\lambda\}} \phi\,d\mu, \qquad \lambda > 0.$$

The above inequality is best possible for every $\lambda > 0$. Also some results in [4] connect such inequalities with differentiation properties of the tree $\mathcal{T}$.

We will also need the following lemma which can again be found in [4].

LEMMA 2.1. *Let $\phi$ be a non-negative function defined on $E \cup \widehat{E} \subseteq X$ such that*

$$\frac{1}{\mu(E)} \int_E \phi \, d\mu = \frac{1}{\mu(\widehat{E})} \int_{\widehat{E}} \phi \, d\mu \equiv A. \tag{2.2}$$

*Additionally suppose that*

$$\phi(x) \le A \quad \text{for every } x \notin E \cap \widehat{E}, \tag{2.3}$$

*and*

$$\phi(x) \le \phi(y) \quad \text{for all } x \in \widehat{E} \setminus E \text{ and } y \in E. \tag{2.4}$$

*Then, for every $p > 1$,*

$$\frac{1}{\mu(E)} \int_E \phi^p \, d\mu \le \frac{1}{\mu(\widehat{E})} \int_{\widehat{E}} \phi^p \, d\mu. \tag{2.5}$$

## 3. Weights on $(X, \mu, \mathcal{T})$

*Proof of Theorem 1.* We suppose that $\phi$ is non-negative defined on $(X, \mu)$ and satisfies a reverse Hölder inequality of the form

$$\frac{1}{\mu(I)} \int_I \phi^p \, d\mu \le c \cdot \left( \frac{1}{\mu(I)} \int_I \phi \, d\mu \right)^p \tag{3.1}$$

for every $I \in \mathcal{T}$, where $c, p$ are fixed such that $p > 1$ and $c \ge 1$. We will prove that for any $t \in (0, 1]$ we have

$$\frac{1}{t} \int_0^t [\phi^*(u)]^p \, du \le (kc - k + 1) \left( \frac{1}{t} \int_0^t \phi^*(u) \, du \right)^p, \tag{3.2}$$

where $\phi^*$ is the non-increasing rearrangement of $\phi$, defined on $(0, 1]$, and $k$ is the homogeneity of $\mathcal{T}$. Fix a $t \in (0, 1]$ and set

$$A_t = \frac{1}{t} \int_0^t \phi^*(u) \, du.$$

Define

$$E_t = \{x \in X : \mathcal{M}_{\mathcal{T}} \phi(x) > A_t\}. \tag{3.3}$$

Then for any $x \in E_t$, there exists $I_x \in \mathcal{T}$ such that

$$x \in I_x \quad \text{and} \quad \frac{1}{\mu(I_x)} \int_{I_x} \phi \, d\mu > A_t. \tag{3.4}$$

Obviously, $I_x \subseteq E_t$. We set $S_{\phi,t} = \{I_x : x \in E_t\}$. This is a family of elements of $\mathcal{T}$ such that $\bigcup \{I : I \in S_{\phi,t}\} = E_t$. Consider now those $I \in S_{\phi,t}$ that are maximal with respect to $\subseteq$. We write this subfamily of $S_{\phi,t}$ as $S'_{\phi,t} = \{I_j : j = 1, 2, \ldots\}$; it may be finite. Then $S'_{\phi,t}$ is a disjoint family of elements of $\mathcal{T}$, because of the maximality of every $I_j$ and the tree structure of $\mathcal{T}$ (see Definition 2.1).

Then by construction, this family still covers $E_t$, that is, $E_t = \bigcup_{j=1}^{\infty} I_j$. For any $I_j \in S'_{\phi,t}$ we have $I_j \neq X$, because if $I_j = X$ for some $j$, then (3.4) would yield

$$\int_0^1 \phi^*(u)\,du = \int_X \phi\,d\mu = \frac{1}{\mu(I_j)} \int_{I_j} \phi\,d\mu > A_t = \frac{1}{t}\int_0^t \phi^*(u)\,du,$$

which is impossible, since $\phi^*$ is non-increasing on $(0,1]$. Thus, for every $I_j \in S'_{\phi,t}$ we see that $I_j^*$ is well defined, but may be common for two or more elements of $S'_{\phi,t}$. We may also have $I_j^* \subseteq I_i^*$ for some $I_j, I_i \in S'_{\phi,t}$.

We now consider the family

$$L_{\phi,t} = \{I_j^* : j = 1,2,\ldots\} \subseteq \mathcal{T}.$$

As mentioned above, this is not necessarily a pairwise disjoint family. We choose a pairwise disjoint subcollection, by considering those $I_j^*$ that are maximal with respect to $\subseteq$. We denote this family as

$$L'_{\phi,t} = \{I_{j_s}^* : s = 1,2,\ldots\}.$$

Then of course

$$\bigcup\{J : J \in L_{\phi,t}\} = \bigcup\{J : J \in L'_{\phi,t}\}.$$

Since each $I_j \in S'_{\phi,t}$ is maximal under the above mentioned integral inequality, we have

$$\frac{1}{\mu(I_{j_s}^*)} \int_{I_{j_s}^*} \phi\,d\mu \le A_t. \tag{3.5}$$

Now note that every $I_{j_s}^*$ contains at least one element $I$ of $S'_{\phi,t}$ such that $I \in C(I_{j_s}^*)$ (one such is $I_{j_s}$). Consider for any $s$ the family of all those $I$ such that $I^* \subseteq I_{j_s}^*$. We write it as

$$S'_{\phi,t,s} = \{I \in S'_{\phi,t} : I^* \subseteq I_{j_s}^*\}.$$

For any $I \in S'_{\phi,t,s}$ we have of course

$$\frac{1}{\mu(I)}\int_I \phi\,d\mu > A_t, \quad \text{so if we set} \quad K_s = \bigcup\{I : I \in S'_{\phi,t,s}\},$$

we must have, because of the disjointness of the elements of $S'_{\phi,t}$,

$$\frac{1}{\mu(K_s)} \int_{K_s} \phi\,d\mu > A_t. \tag{3.6}$$

Additionally, $K_s \subseteq I_{j_s}^*$ and by the comments above we easily see that

$$\frac{1}{k}\mu(I_{j_s}^*) \le \mu(K_s) < \mu(I_{j_s}^*). \tag{3.7}$$

By (3.5) and (3.6) we can now choose (because $\mu$ is non-atomic), for any $s$, a measurable set $B_s \subseteq I^*_{j_s} \setminus K_s$ such that if we define $\Gamma_s = K_s \cup B_s$, then

$$\frac{1}{\mu(\Gamma_s)} \int_{\Gamma_s} \phi\, d\mu = A_t.$$

We now set

$$E^*_t = \bigcup_s I^*_{j_s}, \qquad \Gamma = \bigcup_s \Gamma_s, \qquad \Delta = \bigcup_s \Delta_s,$$

where $\Delta_s = I^*_{j_s} \setminus \Gamma_s$ for any $s = 1, 2, \ldots$. Then by the above,

$$\Gamma \cup \Delta = E^*_t \quad \text{and} \quad \frac{1}{\mu(\Gamma)} \int_{\Gamma} \phi\, d\mu = A_t,$$

which is true in view of the pairwise disjointness of $(I^*_{j_s})^\infty_{s=1}$.

Define

$$h := (\phi/\Gamma)^* : (0, \mu(\Gamma)] \to \mathbb{R}^+.$$

Then obviously

$$\frac{1}{\mu(\Gamma)} \int_0^{\mu(\Gamma)} h(u)\, du = \frac{1}{\mu(\Gamma)} \int_{\Gamma} \phi\, d\mu = A_t.$$

By the definition of $h$ we have $h(u) \le \phi^*(u)$ for any $u \in (0, \mu(\Gamma)]$. Thus

$$\text{(3.8)} \qquad \frac{1}{\mu(\Gamma)} \int_0^{\mu(\Gamma)} \phi^*(u)\, du \ge \frac{1}{\mu(\Gamma)} \int_0^{\mu(\Gamma)} h(u)\, du = A_t = \frac{1}{t} \int_0^t \phi^*(u)\, du.$$

From (3.8), we see that $\mu(\Gamma) \le t$, since $\phi^*$ is non-increasing.

We now consider a set $E \subseteq X$ such that $(\phi/E)^* = \phi^*/(0, t]$ with $\mu(E) = t$ and for which $\{\phi > \phi^*(t)\} \subseteq E \subseteq \{\phi \ge \phi^*(t)\}$. Its existence is guaranteed by the equimeasurability of $\phi$ and $\phi^*$, and the fact that $(X, \mu)$ is non-atomic. Then we see immediately that

$$\frac{1}{\mu(E)} \int_E \phi\, d\mu = \frac{1}{t} \int_0^t \phi^*(u)\, du = A_t.$$

We are now going to construct a second set $\widehat{E} \subseteq X$. We first set $\widehat{E}_1 = \Gamma$.

Let $x \notin \widehat{E}_1$. Since $\Gamma \supseteq \{\mathcal{M}_{\mathcal{T}}\phi > A_t\}$, we must have $\mathcal{M}_{\mathcal{T}}\phi(x) \le A_t$. But since $\mathcal{T}$ differentiates $L^1(X, \mu)$, we obviously have $\phi(y) \le \mathcal{M}_{\mathcal{T}}\phi(y)$ for $\mu$-almost every $y \in X$. Thus the set $\Omega = \{x \notin \widehat{E}_1 : \phi(x) > \mathcal{M}_{\mathcal{T}}\phi(x)\}$ has $\mu$-measure zero.

Finally, we set $\widehat{E} = \widehat{E}_1 \cup \Omega = \Gamma \cup \Omega$. Then $\mu(\widehat{E}) = \mu(\Gamma)$ and $\phi(x) \le \mathcal{M}_{\mathcal{T}}\phi(x) \le A_t$ for every $x \notin \widehat{E}$.

Let now $x \notin E$. By the construction of $E$ we immediately see that $\phi(x) \le \phi^*(t) \le (1/t)\int_0^t \phi^*(u)\, du = A_t$. Thus, if $x \notin E$ or $x \notin \widehat{E}$, we must have $\phi(x) \le A_t$, that is, (2.3) of Lemma 2.1 is satisfied for these choices of $E$ and $\widehat{E}$. Let now $x \in \widehat{E} \setminus E$ and $y \in E$. Then obviously by the above discussion, $\phi(x) \le \phi^*(t) \le \phi(y)$. Thus (2.4) is also satisfied. Also since $\widehat{E} = \Gamma \cup \Omega$, we obviously have $\mu(\widehat{E})^{-1} \int_{\widehat{E}} \phi\, d\mu = A_t$, and as a consequence (2.2) is also satisfied.

Applying Lemma 2.1, we conclude that

$$\frac{1}{\mu(E)} \int_E \phi^p\, d\mu \le \frac{1}{\mu(\widehat{E})} \int_{\widehat{E}} \phi^p\, d\mu,$$

or by the definitions of $E$ and $\widehat{E}$,

$$\frac{1}{t} \int_0^t [\phi^*(u)]^p\, du \le \frac{1}{\mu(\Gamma)} \int_\Gamma \phi^p\, d\mu. \tag{3.9}$$

Our aim now is to show that the right integral average in (3.9) is less than or equal to $(kc - k + 1)(A_t)^p$. We proceed as follows:

We set $\ell_\Gamma = \mu(\Gamma)^{-1} \int_\Gamma \phi^p\, d\mu$. Then with the notation above, we have

$$\begin{aligned} \ell_\Gamma &= \frac{1}{\mu(\Gamma)} \Big( \int_{E_t^*} \phi^p\, d\mu - \int_\Delta \phi^p\, d\mu \Big) \\ &= \frac{1}{\mu(\Gamma)} \Big( \sum_{s=1}^\infty \int_{I_{j_s}^*} \phi^p\, d\mu - \sum_{s=1}^\infty \int_{\Delta_s} \phi^p\, d\mu \Big) = \frac{1}{\mu(\Gamma)} \sum_{s=1}^\infty p_s, \end{aligned} \tag{3.10}$$

where the $p_s$ are given by

$$p_s = \int_{I_{j_s}^*} \phi^p\, d\mu - \int_{\Delta_s} \phi^p\, d\mu \quad \text{for any } s = 1, 2, \ldots.$$

We now find an effective lower bound for $\int_{\Delta_s} \phi^p\, d\mu$. By Hölder's inequality,

$$\int_{\Delta_s} \phi^p\, d\mu \ge \frac{1}{\mu(\Delta_s)^{p-1}} \Big( \int_{\Delta_s} \phi\, d\mu \Big)^p, \tag{3.11}$$

Since $\Delta_s = I_{j_s}^* \setminus \Gamma_s$, (3.11) can be written as

$$\int_{\Delta_s} \phi^p\, d\mu \ge \frac{\big(\int_{I_{j_s}^*} \phi\, d\mu - \int_{\Gamma_s} \phi\, du\big)^p}{(\mu(I_{j_s}^*) - \mu(\Gamma_s))^{p-1}}. \tag{3.12}$$

We now use Hölder's inequality in the form

$$\frac{(\lambda_1 + \lambda_2)^p}{(\sigma_1 + \sigma_2)^{p-1}} \le \frac{\lambda_1^p}{\sigma_1^{p-1}} + \frac{\lambda_2^p}{\sigma_2^{p-1}} \quad \text{for } \lambda_i \ge 0 \text{ and } \sigma_i > 0,$$

which holds since $p > 1$. Thus (3.12) gives

$$\int_{\Delta_s} \phi^p \, d\mu \geq \frac{1}{\mu(I_{j_s}^*)^{p-1}} \Big( \int_{I_{j_s}^*} \phi \, d\mu \Big)^p - \frac{1}{\mu(\Gamma_s)^{p-1}} \Big( \int_{\Gamma_s} \phi \, d\mu \Big)^p. \tag{3.13}$$

Since $\mu(\Gamma_s)^{-1} \int_{\Gamma_s} \phi \, d\mu = A_t$, (3.13) gives

$$\int_{\Delta_s} \phi^p \, d\mu \geq \frac{1}{\mu(I_{j_s}^*)^{p-1}} \Big( \int_{I_{j_s}^*} \phi \, d\mu \Big)^p - \mu(\Gamma_s) \cdot (A_t)^p,$$

so we conclude, by the definition of $p_s$, that

$$p_s \leq \int_{I_{j_s}^*} \phi^p \, d\mu - \frac{1}{\mu(I_{j_s}^*)^{p-1}} \Big( \int_{I_{j_s}^*} \phi \, d\mu \Big)^p + \mu(\Gamma_s) \cdot (A_t)^p. \tag{3.14}$$

Using now (3.1) for $I = I_{j_s}^*$, $s = 1, 2, \ldots$, we have

$$p_s \leq (c-1) \frac{1}{\mu(I_{j_s}^*)^{p-1}} \Big( \int_{I_{j_s}^*} \phi \, d\mu \Big)^p + \mu(\Gamma_s) \cdot (A_t)^p. \tag{3.15}$$

Summing (3.15) for $s = 1, 2, \ldots$ we obtain, in view of (3.10),

$$\ell_\Gamma \leq \frac{1}{\mu(\Gamma)} \bigg[ (c-1) \sum_{s=1}^{\infty} \frac{1}{\mu(I_{j_s}^*)^{p-1}} \Big( \int_{I_{j_s}^*} \phi \, d\mu \Big)^p + \Big( \sum_{s=1}^{\infty} \mu(\Gamma_s) \Big) (A_t)^p \bigg]. \tag{3.16}$$

Now from $\mu(I_{j_s}^*)^{-1} \int_{I_{j_s}^*} \phi \, d\mu \leq A_t$, we see that

$$\begin{aligned} \ell_\Gamma &\leq \frac{1}{\mu(\Gamma)} \Big[ (c-1) \sum_{s=1}^{\infty} \mu(I_{j_s}^*) \cdot (A_t)^p + \mu(\Gamma) \cdot (A_t)^p \Big] \\ &= \bigg[ (c-1) \frac{\mu(E_t^*)}{\mu(\Gamma)} + 1 \bigg] \cdot (A_t)^p. \end{aligned} \tag{3.17}$$

Since now $E_t^* \supseteq \Gamma \supseteq E_t$, by (3.7) we have

$$\mu(E_t^*) \leq k\mu(E_t) \leq k\mu(\Gamma).$$

Thus (3.17) gives

$$\frac{1}{\mu(\Gamma)} \int_{\Gamma} \phi^p \, d\mu \leq [k(c-1) + 1] (A_t)^p.$$

Using now (3.9) and the last inequality we obtain the desired result. ∎

Corollary 3.1. *If $\phi$ satisfies (3.1) for every $I \in \mathcal{T}$, then $\phi \in L^q$ for any $q \in [p, p_0)$, where $p_0$ is defined by*

$$\frac{p_0 - p}{p_0} \cdot \Big( \frac{p_0}{p_0 - 1} \Big)^p \cdot (kc - k + 1) = 1.$$

*Proof.* Immediate from Theorems 1 and A. ∎

Remark 3.1. All the above holds if we replace the condition (3.1) by the known Muckenhoupt condition on $\phi$ over the dyadic sets of $X$. Then the same proof shows that the Muckenhoupt condition should hold for $\phi^*$ for intervals of the form $(0,t]$, and for the constant $kc-k+1$. This is true since there exists a lemma analogous to Lemma 2.1 for this case (as can be seen in [4]). Also the inequality that is used in order to produce (3.13) from (3.12) is true even for $p<0$. We omit the details.

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