

Grothendieck's inequality in the noncommutative Schwartz space

by

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Dedicated to the memory of Prof. Paweł Domański

Abstract. In the spirit of Grothendieck's famous inequality from the theory of Banach spaces, we study a sequence of inequalities for the noncommutative Schwartz space, a Fréchet algebra of smooth operators. These hold in nonoptimal form by a simple nuclearity argument. We obtain optimal versions and reformulate the inequalities in several different ways.

1. Introduction. The noncommutative Schwartz space \mathcal{S} is a weakly amenable m -convex Fréchet algebra whose properties have been investigated in several recent papers (see e.g. [2, 3, 13, 14]). It is not difficult to see that as a Fréchet space, \mathcal{S} is nuclear. From this, we can easily deduce the following analogue of Grothendieck's inequality, which we call *Grothendieck's inequality in \mathcal{S}* : there exists a constant $K > 0$ such that for any continuous bilinear form $u: \mathcal{S} \times \mathcal{S} \rightarrow \mathbb{C}$ and any $n \in \mathbb{N}$, there exists $k \in \mathbb{N}$ such that for every $m \in \mathbb{N}$ and any $x_1, \dots, x_m, y_1, \dots, y_m \in \mathcal{S}$, we have

$$(1) \quad \left| \sum_{j=1}^m u(x_j, y_j) \right| \leq K \|u\|_n^* \|(x_j)\|_k^{\text{RC}} \|(y_j)\|_k^{\text{RC}}$$

The norms on the right hand side arise naturally from the definition of \mathcal{S} , as explained in Section 2 below. Our goal in this note is to show that in fact $k = 2n + 1$ always suffices, and that this is best possible.

This appears to be the first result concerning Grothendieck's inequality in the category of Fréchet algebras; to the best of our knowledge, all previous re-

2010 *Mathematics Subject Classification*: Primary 47A30, 47A07, 47L10; Secondary 47A63.

Key words and phrases: noncommutative Grothendieck inequality, noncommutative Schwartz space, m -convex Fréchet algebra, bilinear form, state.

Received 26 February 2016; revised 3 June 2016.

Published online 23 August 2016.

sults along these lines concern Banach spaces (including C^* -algebras, general Banach algebras and operator spaces). For Fréchet algebras, Grothendieck's inequality seems to have a specific flavour. Many natural examples of Fréchet algebras, e.g. spaces of holomorphic or smooth functions with pointwise multiplication, the space of all sequences or the power series spaces with coordinatewise multiplication (see [9, Ch. 29]) are nuclear in the sense of Grothendieck, meaning that all tensor product topologies are equal. Since Grothendieck's inequality can be understood as the equivalence of two tensor products, it seems that we can take inequality (1) for granted if the Fréchet algebra in question is nuclear (in the above sense). The interesting question that remains is then optimality.

In the remainder of this section, we recall a C^* -algebraic version of Grothendieck's inequality due to Haagerup, and then review the definition and the basic properties of \mathcal{S} which we require. In Section 2 we explain how nuclearity gives Grothendieck's inequality in \mathcal{S} , and we estimate the constants K and k . Section 3 then settles the optimality question for k via a matricial construction. We conclude with a short section containing several reformulations of the inequality.

1.1. Grothendieck's inequality. Pisier's survey article [12] is a comprehensive reference for Grothendieck's inequality. This presents many equivalent formulations and applications of this famous result, and recounts its evolution from 'commutative' [5] to 'noncommutative'. Of these reformulations and extensions, Haagerup's noncommutative version most closely resembles (1), and we state it here for the convenience of the reader.

THEOREM 1 ([6], [12, Theorem 7.1]). *Let A and B be C^* -algebras. For any bounded bilinear form $u: A \times B \rightarrow \mathbb{C}$ and any finite sequence (x_j, y_j) in $A \times B$, we have*

$$\left| \sum u(x_j, y_j) \right| \leq 2 \|u\| \| (x_j) \|^{RC} \| (y_j) \|^{RC}$$

where $\| (x_j) \|^{RC} := \max\{ \| \sum x_j^* x_j \|^{1/2}, \| \sum x_j x_j^* \|^{1/2} \}$.

1.2. The noncommutative Schwartz space. Let

$$s = \left\{ \xi = (\xi_j)_{j \in \mathbb{N}} \in \mathbb{C}^{\mathbb{N}} : |\xi|_n := \left(\sum_{j=1}^{\infty} |\xi_j|^2 j^{2n} \right)^{1/2} < \infty \text{ for all } n \in \mathbb{N} \right\}$$

denote the space of *rapidly decreasing sequences*. This space becomes Fréchet when endowed with the above-defined sequence $(|\cdot|_n)_{n \in \mathbb{N}}$ of norms. The basis $(U_n)_{n \in \mathbb{N}}$ of zero neighbourhoods of s is defined by $U_n := \{ \xi \in s : |\xi|_n \leq 1 \}$.

The topological dual of s is the space of *slowly increasing sequences*

$$s' = \left\{ \eta = (\eta_j)_{j \in \mathbb{N}} \in \mathbb{C}^{\mathbb{N}} : |\eta|'_n := \left(\sum_{j=1}^{\infty} |\eta_j|^2 j^{-2n} \right)^{1/2} < \infty \text{ for some } n \in \mathbb{N} \right\}$$

where the duality pairing is given by $\langle \xi, \eta \rangle := \sum_{j \in \mathbb{N}} \xi_j \bar{\eta}_j$ for $\xi \in s, \eta \in s'$.

The *noncommutative Schwartz space* \mathcal{S} is the Fréchet space $L(s', s)$ of all continuous linear operators from s' into s , endowed with the topology of uniform convergence on bounded sets. The formal identity map $\iota : s \hookrightarrow s'$ is a continuous embedding and defines a product on \mathcal{S} by $xy := x \circ \iota \circ y$ for $x, y \in \mathcal{S}$. It is straightforward to see [3] that a natural involution on \mathcal{S} is given by $\langle x^* \xi, \eta \rangle := \langle \xi, x \eta \rangle$ for $x \in \mathcal{S}, \xi, \eta \in s'$. With these operations, \mathcal{S} becomes an m -convex Fréchet $*$ -algebra. The inclusion map $\mathcal{S} \hookrightarrow \mathcal{K}(\ell_2)$ is continuous, and in fact it is a spectrum-preserving $*$ -homomorphism [3]. Moreover, by [13, Proposition 3], an element $x \in \mathcal{S}$ is positive (i.e., $x = y^* y$ for some $y \in \mathcal{S}$) if and only if the spectrum of x is contained in $[0, \infty)$, or equivalently $\langle x \xi, \xi \rangle \geq 0$ for all $\xi \in s'$.

On the other hand, by [3, Cor. 2.4] and [4, Theorems 8.2, 8.3], the topology of \mathcal{S} cannot be given by a sequence of C^* -norms. This causes some technical inconvenience (e.g. there is no bounded approximate identity in \mathcal{S}) meaning we cannot apply C^* -algebraic techniques directly.

2. The inequality. Let $(\|\cdot\|_n)_{n \in \mathbb{N}}$ be a nondecreasing sequence of norms which gives the topology of \mathcal{S} . For $u : \mathcal{S} \times \mathcal{S} \rightarrow \mathbb{C}$ a continuous bilinear form, we write

$$\|u\|_n^* := \sup\{|u(x, y)| : x, y \in \mathcal{U}_n\}$$

where $\mathcal{U}_n = \{x \in \mathcal{S} : \|x\|_n \leq 1\}$; similarly, for a functional $\phi \in \mathcal{S}'$, we write

$$\|\phi\|_n^* := \sup\{|\phi(x)| : x \in \mathcal{U}_n\}.$$

Following Pisier [11, p. 316], for $k \in \mathbb{N}$ and $x_1, \dots, x_m \in \mathcal{S}$, we write

$$\|(x_j)\|_k^{\text{RC}} = \max \left\{ \left\| \sum_{j=1}^m x_j^* x_j \right\|_k^{1/2}, \left\| \sum_{j=1}^m x_j x_j^* \right\|_k^{1/2} \right\}.$$

Relative to our choice of norms $\|\cdot\|_n$, we have thus defined each term in our hoped-for inequality (1). We will now reformulate it using tensor products.

For C^* -algebras, such a reformulation is standard. Indeed, by [6, Theorem 1.1] (formulated along the lines of [8, Theorem 2.1]), Haagerup's noncommutative Grothendieck inequality entails the existence of a $K > 0$ such that for any C^* -algebras A, B and z in the algebraic tensor product $A \otimes B$, we have $\|z\|_{\pi} \leq K \|z\|_{ah}$ where $\|\cdot\|_{\pi}$ is the projective tensor norm and $\|\cdot\|_{ah}$

is the absolute Haagerup tensor norm [8, p. 164] on $A \otimes B$, given by

$$\|z\|_{ah} = \inf \left\| \left\| \sum_{j=1}^m |x_j|^2 \right\|^{1/2} \left\| \sum_{j=1}^m |y_j|^2 \right\|^{1/2} \right\|.$$

Here $|x| = \left(\frac{1}{2}(x^*x + xx^*)\right)^{1/2}$ for x an element of a C^* -algebra, and the infimum is taken over all representations $z = \sum_{j=1}^m x_j \otimes y_j$ where $(x_j, y_j) \in A \times B$.

We proceed similarly for \mathcal{S} . For $x \in \mathcal{S}$, let $|x|^2 = \frac{1}{2}(x^*x + xx^*) \in \mathcal{S}$ and consider the sequence of *absolute Haagerup tensor norms* $(\|\cdot\|_{ah,n})_{n \in \mathbb{N}}$ on the algebraic tensor product $\mathcal{S} \otimes \mathcal{S}$ given by

$$\|z\|_{ah,n} := \inf \left\| \left\| \sum_{j=1}^m |x_j|^2 \right\|_n^{1/2} \left\| \sum_{j=1}^m |y_j|^2 \right\|_n^{1/2} \right\|$$

where the infimum runs over all ways to represent $z = \sum_{j=1}^m x_j \otimes y_j$ in $\mathcal{S} \otimes \mathcal{S}$. As usual, we write $(\|\cdot\|_{\pi,n})_{n \in \mathbb{N}}$ for the sequence of projective tensor norms on $\mathcal{S} \otimes \mathcal{S}$.

Just as in the C^* -algebra case, inequality (1) will follow once we show that the sequences of projective and absolute Haagerup tensor norms are equivalent on $\mathcal{S} \otimes \mathcal{S}$. In fact, the equivalence of these norms follows immediately from the nuclearity of \mathcal{S} (see [9, Theorem 28.15] and [7, Ch. 21, §2, Theorem 1] for details). On the other hand, the optimal values of k and K (depending on n and our choice of norms $(\|\cdot\|_n)_{n \in \mathbb{N}}$) for which (1) holds are not given by such general considerations. These optimal parameters will be denoted by $\kappa(n) := k_{\text{best}}$ and $K_n := K_{\text{best}}$.

Henceforth, we focus only on the sequence of norms $(\|\cdot\|_n)_{n \in \mathbb{N}}$ where

$$\|x\|_n := \sup\{|x\xi|_n : \xi \in U_n^\circ\}, \quad n \in \mathbb{N}, \quad x \in \mathcal{S},$$

and $U_n^\circ = \{\xi \in s' : |\xi|'_n \leq 1\}$. In other words, $\|x\|_n$ is the norm of $x \in \mathcal{S}$, considered as a Hilbert space operator from $H'_n := \ell_2((j^{-n})_j)$ to $H_n := \ell_2((j^n)_j)$. This sequence does indeed induce the topology of \mathcal{S} . In this context, we will estimate K_n and compute the exact values of $\kappa(n)$.

We start with the following result, which can be compared with [13, Lemma 8]. To fix some useful notation, for $n \in \mathbb{N}$ we define an infinite diagonal matrix $d_n := \text{diag}(1^n, 2^n, 3^n, 4^n, \dots)$, which we consider as an isometry $d_n : \ell_2 \rightarrow H'_n$ and simultaneously as an isometry $d_n : H_n \rightarrow \ell_2$.

PROPOSITION 2. *Let $n \in \mathbb{N}$. Then*

- (i) $\|x\|_n = \sup\{x\xi, \xi\} : \xi \in U_n^\circ\}$ for every positive $x \in \mathcal{S}$;
- (ii) $\|x\|_n^2 \leq \|x^2\|_{2n}$ for every self-adjoint $x \in \mathcal{S}$; and
- (iii) $\|x\|_n^2 \leq \|x^*x\|_{2n}^{1/2} \|xx^*\|_{2n}^{1/2}$ for every $x \in \mathcal{S}$.

Moreover, inequalities (ii) and (iii) are sharp.

Proof. (i) Observe that $\|x\|_n = \|d_n x d_n\|_{\mathcal{B}(\ell_2)}$. Furthermore, since x is positive, $d_n x d_n$ is positive and we have

$$\|x\|_n = \|d_n x d_n\|_{\mathcal{B}(\ell_2)} = \sup\{\langle x d_n \xi, d_n \xi \rangle : \|\xi\|_{\ell_2} \leq 1\} = \sup\{\langle x \xi, \xi \rangle : \|\xi\|'_n \leq 1\}.$$

(ii) For x self-adjoint, we have $\|x^2\|_{2n} = \|d_{2n} x^2 d_{2n}\|_{\mathcal{B}(\ell_2)} = \|d_{2n} x\|_{\mathcal{B}(\ell_2)}^2$, and by [1, Proposition II.1.4.2],

$$\|x\|_n = \|d_n x d_n\|_{\mathcal{B}(\ell_2)} = \nu(d_n x d_n) = \nu(d_{2n} x) \leq \|d_{2n} x\|_{\mathcal{B}(\ell_2)},$$

where $\nu(\cdot)$ denotes the spectral radius. This gives the desired inequality.

(iii) Since $\mathcal{S} \hookrightarrow \mathcal{B}(\ell_2)$, any $x \in \mathcal{S}$ is also a Hilbert space operator, and the block-matrix operator $\begin{bmatrix} (xx^*)^{1/2} & x \\ x^* & (x^*x)^{1/2} \end{bmatrix}$ is positive in $\mathcal{B}(\ell_2 \oplus \ell_2)$ (see e.g. [10, p. 117]). Equivalently,

$$(2) \quad |\langle x \xi, \eta \rangle|^2 \leq \langle (xx^*)^{1/2} \eta, \eta \rangle \langle (x^*x)^{1/2} \xi, \xi \rangle \quad \forall \xi, \eta \in \ell_2.$$

For $m \in \mathbb{N}$, let us write $p_m := \begin{bmatrix} I_m & 0 \\ 0 & 0 \end{bmatrix} \in \mathcal{S}$ where $I_m \in \mathcal{M}_m$ is the identity matrix. Now fix $n \in \mathbb{N}$ and choose $\xi, \eta \in H'_n$. Then $p_m \xi, p_m \eta \in \ell_2$ for all $m \in \mathbb{N}$ and (2) gives

$$|\langle p_m x p_m \xi, \eta \rangle|^2 \leq \langle p_m (xx^*)^{1/2} p_m \eta, \eta \rangle \langle p_m (x^*x)^{1/2} p_m \xi, \xi \rangle.$$

Since $(p_m)_{m \in \mathbb{N}}$ is an approximate identity in \mathcal{S} (see [13, Proposition 2]), we obtain

$$|\langle x \xi, \eta \rangle|^2 \leq \langle (xx^*)^{1/2} \eta, \eta \rangle \langle (x^*x)^{1/2} \xi, \xi \rangle.$$

Taking the supremum over all ξ, η in the unit ball of H'_n we get

$$\|x\|_n^2 \leq \|(xx^*)^{1/2}\|_n \|(x^*x)^{1/2}\|_n.$$

Applying (ii) to the positive operators $(xx^*)^{1/2}$ and $(x^*x)^{1/2}$ we conclude that $\|x\|_n^2 \leq \|x^*x\|_{2n}^{1/2} \|xx^*\|_{2n}^{1/2}$.

For sharpness, observe that if x is a diagonal rank one matrix unit then we have equality in both (ii) and (iii). ■

PROPOSITION 3. For any $n, m \in \mathbb{N}$ and $x_1, \dots, x_m, y_1, \dots, y_m \in \mathcal{S}$,

$$\begin{aligned} & \sum_{j=1}^m \|x_j\|_n \|y_j\|_n \\ & \leq \frac{\pi^2}{6} \left\| \sum_{j=1}^m x_j^* x_j \right\|_{2n+1}^{1/4} \left\| \sum_{j=1}^m x_j x_j^* \right\|_{2n+1}^{1/4} \left\| \sum_{j=1}^m y_j^* y_j \right\|_{2n+1}^{1/4} \left\| \sum_{j=1}^m y_j y_j^* \right\|_{2n+1}^{1/4}. \end{aligned}$$

Proof. Let $C := \pi^2/6$ and let $p \in \mathbb{N}$. We claim that

$$\sum_{k=1}^m \|x_k^* x_k\|_p \leq C \left\| \sum_{k=1}^m x_k^* x_k \right\|_{p+1}.$$

By the Cauchy–Schwarz inequality and Proposition 2(iii) this will then imply the desired inequality. To establish the claim, let $\xi^1, \dots, \xi^m \in U_p^\circ$ and let us write $(e_j)_{j \in \mathbb{N}}$ for the standard basis vectors in ℓ^2 . We have

$$\sum_{k=1}^m \langle x_k^* x_k \xi^k, \xi^k \rangle = \sum_{k=1}^m \sum_{i,j=1}^\infty \langle x_k^* x_k e_j, e_i \rangle (ij)^p \overline{\xi_i^k} i^{-p} \xi_j^k j^{-p}.$$

Applying the Cauchy–Schwarz inequality to summation over $i, j \in \mathbb{N}$ gives

$$\sum_{k=1}^m \langle x_k^* x_k \xi^k, \xi^k \rangle \leq \sum_{k=1}^m \left(\sum_{i,j=1}^\infty |\langle x_k^* x_k e_j, e_i \rangle|^2 (ij)^{2p} \right)^{1/2}.$$

Since x^*x is positive for any $x \in \mathcal{S}$, and for positive operators $y \in \mathcal{S}$ we have $|y_{ij}|^2 \leq y_{ii}y_{jj}$ (where $y_{ij} := \langle ye_j, e_i \rangle$), this implies that

$$\begin{aligned} \sum_{k=1}^m \langle x_k^* x_k \xi^k, \xi^k \rangle &\leq \sum_{j=1}^\infty \left\langle \left(\sum_{k=1}^m x_k^* x_k \right) j^p e_j, j^p e_j \right\rangle \\ &\leq \sum_{j=1}^\infty j^{-2} \sup_{i \in \mathbb{N}} \left\langle \left(\sum_{k=1}^m x_k^* x_k \right) i^{p+1} e_i, i^{p+1} e_i \right\rangle \\ &\leq C \left\| \sum_{k=1}^m x_k^* x_k \right\|_{p+1}. \end{aligned}$$

By Proposition 2(i), for any $\varepsilon > 0$ there are $\xi^1, \dots, \xi^m \in U_p^\circ$ with

$$\sum_{k=1}^m \|x_k^* x_k\|_p < \sum_{k=1}^m \langle x_k^* x_k \xi^k, \xi^k \rangle + \varepsilon < C \left\| \sum_{k=1}^m x_k^* x_k \right\|_{p+1} + \varepsilon.$$

Taking the infimum over $\varepsilon > 0$ yields the claim. ■

As a straightforward consequence of Proposition 3, we obtain:

THEOREM 4 (Grothendieck’s inequality in \mathcal{S}). *There is a constant $K \leq \pi^2/6$ such that $\|z\|_{\pi,n} \leq 2K \|z\|_{ah,2n+1}$ for any $n \in \mathbb{N}$ and $z \in \mathcal{S} \otimes \mathcal{S}$. Moreover, every continuous bilinear form $u: \mathcal{S} \times \mathcal{S} \rightarrow \mathbb{C}$ satisfies inequality (1) with $k = 2n + 1$, for any $n, m \in \mathbb{N}$ and any $x_1, \dots, x_m, y_1, \dots, y_m \in \mathcal{S}$. In particular, taking $u(x, y) := \phi(x)\phi(y)$ where $\phi \in \mathcal{S}'$, we obtain*

$$(3) \quad \sum_{j=1}^m |\phi(x_j)|^2 \leq K (\|\phi\|_n^* \|(x_j)\|_{2n+1}^{\text{RC}})^2$$

REMARK. This shows that $\kappa(n) \leq 2n + 1$. On the other hand, it is easy to show that $\kappa(n) > 2n - 1$. Indeed, if not, then (3) would hold with $2n + 1$ replaced by some $\ell \leq 2n - 1$. Take $m \in \mathbb{N}$, define $\xi_m := \sum_{j=1}^m j^n e_j$ and $\phi_m \in \mathcal{S}'$ by $\phi_m(x) := \langle x \xi_m, \xi_m \rangle$. Then for $x_j := e_{jj}$, $j = 1, \dots, m$, we get $\|(x_j)\|_\ell^{\text{RC}} = m^\ell$ and $\|\phi_m\|_n^* = m$. On the other hand,

$\sum_{j=1}^m |\phi_m(x_j)|^2$ is equivalent (up to a constant) to m^{4n+1} . Therefore (3) takes the form $m^{4n+1} \leq Cm^{2\ell+2}$ for some constant C (independent of m). Letting $m \rightarrow \infty$, we obtain $\ell \geq 2n - 1/2$, a contradiction. Hence $\kappa(n) \in \{2n, 2n + 1\}$.

3. Optimality. We will now show that $\kappa(n) = 2n + 1$. For this, we will use the tensor product formulation, noting that

$$\kappa(n) = \min \left\{ k \in \mathbb{N} : \sup \left\{ \frac{\|z\|_{\pi,n}}{\|z\|_{ah,k}} : z \in \mathcal{S} \otimes \mathcal{S}, z \neq 0 \right\} < \infty \right\}.$$

Recall the diagonal operator d_n defined before Proposition 2. For every $x \in \mathcal{S}$ the operator $\ell_2 \xrightarrow{d_n} H'_n \xrightarrow{x} H_n \xrightarrow{d_n} \ell_2$ is bounded. This leads to the following observation.

PROPOSITION 5. *If $z = \sum_{j=1}^k x_j \otimes y_j \in \mathcal{S} \otimes \mathcal{S}$, then*

$$\|z\|_{\pi,n} = \left\| \sum_{j=1}^k d_n x_j d_n \otimes d_n y_j d_n \right\|_{\pi}.$$

Proof. Write

$$\Delta_n z = \sum_{j=1}^k d_n x_j d_n \otimes d_n y_j d_n \in \mathcal{B}(\ell_2) \otimes \mathcal{B}(\ell_2).$$

If $\Delta_n z = \sum_{l=1}^m a_l \otimes b_l \in \mathcal{B}(\ell_2) \otimes \mathcal{B}(\ell_2)$ and $\sum_{l=1}^m \|a_l\| \|b_l\| < \|\Delta_n z\|_{\pi} + \varepsilon$ for some $\varepsilon > 0$, then $z = \sum_{l=1}^m d_n^{-1} a_l d_n^{-1} \otimes d_n^{-1} b_l d_n^{-1}$ and

$$\|z\|_{\pi,n} \leq \sum_{l=1}^m \|d_n^{-1} a_l d_n^{-1}\|_n \|d_n^{-1} b_l d_n^{-1}\|_n = \sum_{l=1}^m \|a_l\| \|b_l\| < \|\Delta_n z\|_{\pi} + \varepsilon.$$

This gives $\|z\|_{\pi,n} \leq \|\Delta_n z\|_{\pi}$. The reverse inequality is proved similarly. ■

We also need the following well-known fact.

PROPOSITION 6. *If \mathcal{H} is a Hilbert space and $x_1, \dots, x_m \in \mathcal{B}(\mathcal{H})$, then*

$$\left\| \sum_{j=1}^m x_j \otimes x_j^* \right\|_h = \left\| \sum_{j=1}^m x_j x_j^* \right\|.$$

Proof. By [15, Theorem 4.3], the Haagerup norm on the left hand side is equal to the completely bounded norm of the map on $\mathcal{B}(\mathcal{H})$ given by $a \mapsto \sum_{j=1}^m x_j a x_j^*$, which is completely positive, so attains its completely bounded norm at the identity operator. ■

THEOREM 7. *For every $n \in \mathbb{N}$, we have $\kappa(n) = 2n + 1$.*

Proof. By Theorem 4, it only remains to show that $\kappa(n) > 2n$. Choose $k_n \in \mathbb{N}$ sufficiently large that $k + 1 \leq 2^k (k^{1/(4n)} - 1)$ for all $k \geq k_n$. This

inequality ensures that for every $k \geq k_n$, if we define

$$i_1 = 2^k, \quad i_{k+1} = \lfloor k^{1/(4n)} 2^k \rfloor, \quad i_j = i_{k+1} + j - (k + 1), \quad 2 \leq j \leq k,$$

then $i_1 < i_2 < \dots < i_{k+1}$. Observe that these indices depend on k . Denote by $(e_{ij})_{i,j \in \mathbb{N}}$ the standard matrix units, and for $j = 2, \dots, k + 1$, consider the self-adjoint operators

$$x_j := e_{i_1, i_j} + e_{i_j, i_1} \in \mathcal{M}_{i_{k+1}} \subset \mathcal{S}.$$

Let $z_k := \sum_{j=2}^{k+1} x_j \otimes x_j$. Since $d_n x_j d_n = i_1^n i_j^n (e_{i_1, i_j} + e_{i_j, i_1})$ and $(d_n x_j d_n)^2 = i_1^{2n} i_j^{2n} (e_{i_1, i_1} + e_{i_j, i_j})$, by Propositions 5 and 6 we obtain

$$\begin{aligned} \|z_k\|_{\pi, n} &= \left\| \sum_{j=2}^{k+1} d_n x_j d_n \otimes d_n x_j d_n \right\|_{\pi} \\ &\geq \left\| \sum_{j=2}^{k+1} d_n x_j d_n \otimes d_n x_j d_n \right\|_h = \left\| \sum_{j=2}^{k+1} (d_n x_j d_n)^2 \right\| = i_1^{2n} \sum_{j=2}^{k+1} i_j^{2n}. \end{aligned}$$

On the other hand,

$$|x_j|^2 = x_j^2 = e_{i_1, i_1} + e_{i_j, i_j} \quad \text{and} \quad \sum_{j=2}^{k+1} d_{2n} x_j^2 d_{2n} = i_1^{4n} k e_{i_1, i_1} + \sum_{j=2}^{k+1} i_j^{4n} e_{i_j, i_j}.$$

Therefore

$$\begin{aligned} \|z_k\|_{ah, 2n} &\leq \left\| \sum_{j=2}^{k+1} |x_j|^2 \right\|_{2n} = \left\| \sum_{j=2}^{k+1} d_{2n} x_j^2 d_{2n} \right\| = \max\{i_1^{4n} k, i_{k+1}^{4n}\} \\ &< i_1^{4n} k + i_{k+1}^{4n}. \end{aligned}$$

Hence

$$\frac{\|z_k\|_{\pi, n}}{\|z_k\|_{ah, 2n}} > \frac{i_1^{2n} \sum_{j=2}^{k+1} i_j^{2n}}{i_1^{4n} k + i_{k+1}^{4n}} > \frac{i_1^{-2n} i_2^{2n}}{1 + k^{-1} i_1^{-4n} i_{k+1}^{4n}} \rightarrow \infty \quad \text{as } k \rightarrow \infty,$$

by our choice of i_1, \dots, i_{k+1} . So $\kappa(n) > 2n$ as required. ■

4. Reformulations of the inequality. Here we give several different ways of stating our inequality; in each case, an analogous result for C*-algebras may be found in [12]. The methods here are fairly standard, so full proofs are often omitted. Throughout, we write $K = \sup_{n \in \mathbb{N}} K_n \leq \pi^2/6$.

4.1. Grothendieck’s inequality with states. For $\xi \in U_n^\circ$, let $\phi_\xi \in \mathcal{S}'$ be given by $\phi_\xi(x) = \langle x\xi, \xi \rangle$, $x \in \mathcal{S}$. We call an element of the closed convex hull of $\{\phi_\xi : \xi \in U_n^\circ\}$ an n -state on \mathcal{S} . Note that by Proposition 2(i), for any positive element $x \in \mathcal{S}$ we have $\|x\|_n = \sup\{\phi(x) : \phi \in V_n\}$, where $V_n \subseteq \mathcal{S}'$

is the set of all n -states on \mathcal{S} . The next result may be deduced from Theorem 4 by closely following the Hahn–Banach Separation argument of [12, §23].

THEOREM 8. *For any continuous bilinear form $u: \mathcal{S} \times \mathcal{S} \rightarrow \mathbb{C}$ and $n \in \mathbb{N}$, there are $(2n + 1)$ -states $\phi_1, \phi_2, \psi_1, \psi_2$ on \mathcal{S} with*

$$|u(x, y)| \leq K \|u\|_n^* (\phi_1(x^*x) + \phi_2(xx^*))^{1/2} (\psi_1(y^*y) + \psi_2(yy^*))^{1/2}$$

for all $x, y \in \mathcal{S}$.

4.2. ‘Little’ Grothendieck inequality. As a consequence we obtain the following ‘little’ Grothendieck inequality in \mathcal{S} . Recall that if $T: X \rightarrow Y$ is a linear map between Fréchet spaces, then $\|T\|_{n,k} := \sup\{\|Tx\|_k: \|x\|_n \leq 1\}$.

THEOREM 9. *For any Fréchet–Hilbert space H , if $u_1, u_2: \mathcal{S} \rightarrow H$ are continuous linear maps, $k, m, n \in \mathbb{N}$ and $x_1, \dots, x_m, y_1, \dots, y_m \in \mathcal{S}$, then*

$$\left| \sum_{j=1}^m \langle u_1(x_j), u_2(y_j) \rangle_k \right| \leq K \|u_1\|_{n,k} \|u_2\|_{n,k} \|(x_j)\|_{2n+1}^{\text{RC}} \|(y_j)\|_{2n+1}^{\text{RC}}.$$

Equivalently, for any $k, n \in \mathbb{N}$ there are $(2n + 1)$ -states $\phi_1, \phi_2, \psi_1, \psi_2$ such that for all $x, y \in \mathcal{S}$ we have

$$\begin{aligned} |\langle u_1(x), u_2(y) \rangle_k| &\leq K \|u_1\|_{n,k} \|u_2\|_{n,k} \\ &\quad \times (\phi_1(x^*x) + \phi_2(xx^*))^{1/2} (\psi_1(y^*y) + \psi_2(yy^*))^{1/2}. \end{aligned}$$

Proof. Apply Theorems 4 and 8 to $u_k(x, y) := \langle u_1(x), u_2(y) \rangle_k$ for $k \in \mathbb{N}$. ■

Using the same argument as in the proof of Theorem 8 we can obtain an equivalent version of the ‘little’ Grothendieck inequality.

THEOREM 10. *For any Fréchet–Hilbert space H , if $u: \mathcal{S} \rightarrow H$ is a continuous linear map and $k, n \in \mathbb{N}$, then there exist $(2n + 1)$ -states ϕ_1, ϕ_2 on \mathcal{S} such that $\|ux\|_k \leq \sqrt{K} \|u\|_{n,k} (\phi_1(x^*x) + \phi_2(xx^*))^{1/2}$ for all $x \in \mathcal{S}$.*

Acknowledgements. The research of the second-named author has been supported by the National Center of Science, Poland, grant no. UMO-2013/10/A/ST1/00091. The authors are grateful to the anonymous referee, whose comments greatly improved the presentation of this paper.

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