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Jordan product and local spectrum preservers

by

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Abstract. Let X and Y be two infinite-dimensional complex Banach spaces, and fix two nonzero vectors $x_0 \in X$ and $y_0 \in Y$. Let $\mathscr{B}(X)$ (resp. $\mathscr{B}(Y)$) denote the algebra of all bounded linear operators on X (resp. on Y). We show that a map φ from $\mathscr{B}(X)$ onto $\mathscr{B}(Y)$ satisfies

$$\sigma_{\varphi(T)\varphi(S)+\varphi(S)\varphi(T)}(y_0) = \sigma_{TS+ST}(x_0) \quad (T, S \in \mathscr{B}(X))$$

if and only if there exists a bijective bounded linear mapping A from X into Y such that $Ax_0 = y_0$ and either $\varphi(T) = ATA^{-1}$ for all $T \in \mathscr{B}(X)$ or $\varphi(T) = -ATA^{-1}$ for all $T \in \mathscr{B}(X)$.

1. Introduction. In recent years, there has been considerable interest in studying *nonlinear preserver problems*, which involve maps between algebras that leave invariant certain properties or subsets or relations without assuming any algebraic condition like linearity or additivity or multiplicativity; see for instance [3, 4, 37]. In [4], Bhatia, Šemrl and Sourour described the form of all surjective maps defined on the algebra $M_n(\mathbb{C})$ of all complex $n \times n$-matrices and preserving the spectral radius of the difference of matrices, and thus they provided an extension of Marcus and Moyls' result [35] in the absence of linearity. In [37], Molnár studied maps preserving the spectrum of operator or matrix products, and his result has been extended in several directions for uniform algebras and semisimple commutative Banach algebras; see for instance [14, 21–27, 31, 33, 34, 36, 38–40]. Instead of preservers of the usual operator or matrix products, several authors studied maps preserving certain spectral quantities of a Jordan product of operators or matrices; see for instance [18, 20, 21, 30, 29]. In [18], Cui and Li characterized maps preserving the peripheral spectrum of a Jordan product of operators on standard operator algebras. In [21], Gau and Li described

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maps preserving the numerical range of Jordan products of Hilbert space operators. Norm preserver maps for Jordan products on $M_n(\mathbb{C})$ with respect to various norms were investigated in [29].

Besides spectrum preservers, the study of linear and nonlinear *local spectra preserver problems* has attracted the attention of a number of authors. Mainly, maps on matrices or operators were described that preserve local spectrum, local spectral radius, and local inner spectral radius; see for instance [5, 7–13, 15–17] and the references therein. In [8, 9], nonlinear maps on Banach space operators were investigated preserving the local spectrum of the product and the triple product of operators. This paper is a continuation of such recent work, and examines the form of maps preserving the local spectrum of a Jordan product of operators on a complex Banach space.

2. Main result. Throughout this paper, X and Y denote infinite-dimensional complex Banach spaces, and $\mathscr{B}(X,Y)$ denotes the space of all bounded linear maps from X into Y. When $X=Y$, we simply write $\mathscr{B}(X)$ instead of $\mathscr{B}(X,X)$ and denote its identity operator by $\mathbf{1}$. The *local resolvent set*, $\rho_T(x)$, of an operator $T\in\mathscr{B}(X)$ at a point $x\in X$ is the union of all open subsets U of $\mathbb{C}$ for which there is an analytic function $\phi: U\to X$ such that $(T-\lambda)\phi(\lambda)=x$ $(\lambda\in U)$. The *local spectrum* of T at x is defined by

$$\sigma_T(x) := \mathbb{C}\setminus\rho_T(x),$$

and is obviously a closed subset (possibly empty) of $\sigma(T)$, the spectrum of T. In fact, $\sigma_T(x)\neq\emptyset$ for all nonzero vectors x in X precisely when T has the *single-valued extension property* (SVEP). Recall that T is said to have SVEP provided that for every open subset U of $\mathbb{C}$, the equation $(T-\lambda)\phi(\lambda)=0$ $(\lambda\in U)$ has no nontrivial analytic solution ϕ. Every operator $T\in\mathscr{B}(X)$ for which the interior of its point spectrum, $\sigma_p(T)$, is empty enjoys this property.

The following theorem is the main result of this paper. Its proof is presented in Section 6, and uses a new local spectral characterization of rank one nilpotent operators in terms of the local spectrum of a Jordan product of operators given in Section 5. It also relies on a local spectral identity principle which is presented in Section 4 and which characterizes in terms of the local spectrum when two operators are the same.

THEOREM 2.1. *Let $x_0\in X\setminus\{0\}$ and $y_0\in Y\setminus\{0\}$. A map φ from $\mathscr{B}(X)$ onto $\mathscr{B}(Y)$ satisfies*

$$\sigma_{\varphi(T)\varphi(S)+\varphi(S)\varphi(T)}(y_0)=\sigma_{TS+ST}(x_0)\qquad (T,S\in\mathscr{B}(X)) \tag{2.1}$$

if and only if there exists a bijective mapping $A\in\mathscr{B}(X,Y)$ such that $Ax_0=y_0$ and either $\varphi(T)=ATA^{-1}$ for all $T\in\mathscr{B}(X)$ or $\varphi(T)=-ATA^{-1}$ for all $T\in\mathscr{B}(X)$.

Note that the only restriction on the map φ is *surjectivity*; no linearity or additivity or continuity is assumed. We would also like to point out that if X and Y are isomorphic Banach spaces, then the statements of our main result can be reduced to the case when $X = Y$ and $x_0 = y_0$. But the fact that "X and Y are isomorphic" is part of the conclusion of the main result rather than part of its hypothesis. Finally, we point out that the restriction to infinite-dimensional Banach spaces in the statement of our main results is just for simplicity: our results and their proofs remain valid in the finite-dimensional case.

3. Preliminaries. In this section, we recall some usual notation and collect some elementary results that will be used. The first result summarizes some known basic properties of the local spectrum.

LEMMA 3.1. *Let $x, y \in X$ and $\alpha \in \mathbb{C} \setminus \{0\}$. For every $T \in \mathscr{B}(X)$, the following statements hold:*

(1) $\sigma_T(\alpha x) = \sigma_T(x)$ *and* $\sigma_{\alpha T}(x) = \alpha\sigma_T(x)$.
(2) $\sigma_T(x + y) \subset \sigma_T(x) \cup \sigma_T(y)$. *Equality holds if* $\sigma_T(x) \cap \sigma_T(y) = \emptyset$.
(3) *If T has SVEP, $x \neq 0$ and $Tx = \lambda x$ for some $\lambda \in \mathbb{C}$, then $\sigma_T(x) = \{\lambda\}$.*
(4) *If T has SVEP and $Tx = \alpha y$, then $\sigma_T(y) \subset \sigma_T(x) \subset \sigma_T(y) \cup \{0\}$.*
(5) *If $R \in \mathscr{B}(X)$ commutes with T, then $\sigma_T(Rx) \subset \sigma_T(x)$.*

Proof. See for instance [1] or [32]. ∎

In what follows, for any $T \in \mathscr{B}(X)$, let T^* denote as usual its adjoint on the dual space X^* of X. For a vector $x \in X$ and a linear functional $f \in X^*$, let $x \otimes f$ stand for the operator of rank at most one defined by

$$(x \otimes f)y := f(y)x \quad (y \in X).$$

Note that every finite rank operator $T \in \mathscr{B}(X)$ can be written as a finite sum of rank one operators, i.e., $T = \sum_{k=1}^{n} x_k \otimes f_k$ for some $x_k \in X$ and $f_k \in X^*$, $k = 1, \ldots, n$. Denote by $\mathscr{N}_1(X)$ the set of all rank one nilpotent operators on X, and observe that $x \otimes f \in \mathscr{N}_1(X)$ if and only if $f(x) = 0$.

Our second lemma is a known elementary observation which will be needed later. We present its proof here for the sake of completeness.

LEMMA 3.2. *Let $A, B \in \mathscr{B}(X)$ and $x_0 \in X \setminus \{0\}$.*

(1) *If $f(Ax) = f(Bx)$ for all $x \in X$ and $f \in X^*$ such that $f(x) = 0$, then $A = B + \gamma\mathbf{1}$ for some $\gamma \in \mathbb{C}$.*
(2) *If Ax lies in the linear span of x_0 and x for all $x \in X$, then $A = d\mathbf{1} + x_0 \otimes f$ for some $d \in \mathbb{C}$ and $f \in X^*$.*

Proof. Suppose there exists $x \in X$ such that x and $(A-B)x$ are linearly independent, and take $f \in X^*$ such that $f(x) = 0$ and $f((A-B)x) = 1$. Then $f(Ax) \neq f(Bx)$, contradicting our assumption. Hence, x and $(A-B)x$

are linearly dependent for all $x \in X$ and thus there exists $\gamma \in \mathbb{C}$ such that $A - B = \gamma \mathbf{1}$ as desired.

(2) Assume that for every $x \in X$ there exist $c(x), d(x) \in \mathbb{C}$ such that $Ax = c(x)x_0 + d(x)x$. Since A is linear, $d(x) = d$ is a constant for all $x \notin \mathbb{C}x_0$, and consequently $(A - d\mathbf{1})x$ and x_0 are linearly dependent for all $x \in X$. Hence, either $A - d\mathbf{1} = 0$, or $A - d\mathbf{1}$ is a rank one operator and its range is $\mathbb{C}x_0$. Consequently, $A - d\mathbf{1} = x_0 \otimes f$ for some $f \in X^*$, as claimed. ■

The following result, proved in [19, Lemma 2.2], describes bijective maps from $\mathscr{N}_1(X)$ into $\mathscr{N}_1(Y)$ that preserve the rank one nilpotency of sums of operators.

LEMMA 3.3. *If φ is a bijective map from $\mathscr{N}_1(X)$ onto $\mathscr{N}_1(Y)$ for which*

$$N_1 + N_2 \in \mathscr{N}_1(X) \;\Leftrightarrow\; \varphi(N_1) + \varphi(N_2) \in \mathscr{N}_1(Y)$$

for all $N_1, N_2 \in \mathscr{N}_1(X)$, then one of the following statements holds:

(1) *There exists a bijective bounded linear or conjugate linear transformation $A : X \to Y$ such that*

$$\varphi(N) = \tau_N A N A^{-1} \tag{3.2}$$

for all $N \in \mathscr{N}_1(X)$, where τ_N is a scalar depending on N.

(2) *There exists a bijective bounded linear or conjugate linear transformation $A : X^* \to Y$ such that*

$$\varphi(N) = \tau_N A N^* A^{-1} \tag{3.3}$$

for all $N \in \mathscr{N}_1(X)$, where τ_N is a scalar depending on N.

For $x_0 \in X \setminus \{0\}$ and $T \in \mathscr{B}(X)$, we use the notation

$$\sigma_T^*(x_0) := \begin{cases} \{0\} & \text{if } \sigma_T(x_0) = \{0\}, \\ \sigma_T(x_0) \setminus \{0\} & \text{if } \sigma_T(x_0) \neq \{0\}. \end{cases}$$

The next lemma gives a complete description of the local spectrum at a fixed vector of the Jordan product of a rank one operator and an arbitrary operator in $\mathscr{B}(X)$. It will be repeatedly used throughout this paper. For a fixed nonzero $x_0 \in X$, $T \in \mathscr{B}(X)$, $f \in X^*$ and $x \in X$, we will use the quantities

$$\begin{aligned} \Gamma_1(T, f, x) &:= \frac{f(x_0)}{2f(x)} - \frac{f(Tx_0)}{2\sqrt{f(T^2x)f(x)}}, \\ \Gamma_2(T, f, x) &:= \frac{f(x_0)}{2f(x)} + \frac{f(Tx_0)}{2\sqrt{f(T^2x)f(x)}}. \end{aligned} \tag{3.4}$$

LEMMA 3.4. *Let $x, x_0 \in X \setminus \{0\}$, $f \in X^*$ and $T \in \mathscr{B}(X)$.*

(1) *If $f(x_0) = f(Tx_0) = 0$, then*

$$\sigma^*_{T(x\otimes f)+(x\otimes f)T}(x_0) = \{0\}.$$

(2) *If $f(x_0) \neq 0$ or $f(Tx_0) \neq 0$, and $f(x) = 0$ or $f(T^2x) = 0$, then*

$$\sigma^*_{T(x\otimes f)+(x\otimes f)T}(x_0) = \{f(Tx)\}.$$

(3) *If $f(x_0) \neq 0$ or $f(Tx_0) \neq 0$, and $f(x) \neq 0$ and $f(T^2x) \neq 0$, then*

$$\sigma^*_{T(x\otimes f)+(x\otimes f)T}(x_0) = \begin{cases} \{f(Tx) \pm \sqrt{f(T^2x)f(x)}\} \setminus \{0\} & \text{if } \mathit{\Gamma}_1(T,f,x) \neq 0 \text{ and } \mathit{\Gamma}_2(T,f,x) \neq 0, \\ \{f(Tx) - \sqrt{f(T^2x)f(x)}\} & \text{if } \mathit{\Gamma}_1(T,f,x) \neq 0 \text{ and } \mathit{\Gamma}_2(T,f,x) = 0, \\ \{f(Tx) + \sqrt{f(T^2x)f(x)}\} & \text{if } \mathit{\Gamma}_1(T,f,x) = 0 \text{ and } \mathit{\Gamma}_2(T,f,x) \neq 0. \end{cases}$$

Proof. (1) If $f(x_0) = f(Tx_0) = 0$, then $(T(x\otimes f) + (x\otimes f)T)x_0 = 0$ and thus

$$\sigma_{T(x\otimes f)+(x\otimes f)T}(x_0) = \{0\}.$$

(2) Assume that $f(x_0) \neq 0$ or $f(Tx_0) \neq 0$, and $f(x) = 0$ or $f(T^2x) = 0$, and let us discuss three cases.

Case 1. If $f(x) = f(Tx) = 0$ or $f(T^2x) = f(Tx) = 0$, then there is nothing to prove since $T(x\otimes f) + (x\otimes f)T$ is nilpotent in this case, and

$$\sigma_{T(x\otimes f)+(x\otimes f)T}(x_0) = \{0\}.$$

Case 2. If $f(x) = 0$ and $f(Tx) \neq 0$, then

$$\sigma_{T(x\otimes f)+(x\otimes f)T}(x_0) \subset \sigma(T(x\otimes f) + (x\otimes f)T) = \{f(Tx), 0\}$$

(see [18, proof of Lemma 2.2]). So, we only need to show that $\{f(Tx)\} \subset \sigma_{T(x\otimes f)+(x\otimes f)T}(x_0)$. Since $(T(x\otimes f) + (x\otimes f)T)x = f(Tx)x$, we have

$$\sigma_{T(x\otimes f)+(x\otimes f)T}(x) = \{f(Tx)\}. \tag{3.5}$$

If $f(x_0) \neq 0$, then since $x\otimes f$ and $T(x\otimes f) + (x\otimes f)T$ commute, we have

$$\{f(Tx)\} = \sigma_{T(x\otimes f)+(x\otimes f)T}(f(x_0)x) \subset \sigma_{T(x\otimes f)+(x\otimes f)T}(x_0), \tag{3.6}$$

as desired. If $f(x_0) = 0$, then $f(Tx_0) \neq 0$ by assumption, and since $(T(x\otimes f) + (x\otimes f)T)(x_0) = f(Tx_0)x$, we have

$$\{f(Tx)\} = \sigma_{T(x\otimes f)+(x\otimes f)T}(f(Tx_0)x) \subset \sigma_{T(x\otimes f)+(x\otimes f)T}(x_0), \tag{3.7}$$

as desired.

Case 3. If $f(T^2x) = 0$ and $f(Tx) \neq 0$, then $\sigma(T(x\otimes f) + (x\otimes f)T) = \{f(Tx), 0\}$ in this case too (see again [18]). Since $(T(x\otimes f)+(x\otimes f)T)(Tx) = f(Tx)Tx$, we have $\sigma_{T(x\otimes f)+(x\otimes f)T}(Tx) = \{f(Tx)\}$. If $f(Tx_0) \neq 0$, then since $Tx\otimes fT$ commutes with $T(x\otimes f) + (x\otimes f)T$, we obtain

$$\begin{aligned}\{f(Tx)\} = \sigma_{T(x\otimes f)+(x\otimes f)T}(f(Tx_0)Tx) &= \sigma_{T(x\otimes f)+(x\otimes f)T}((Tx\otimes fT)x_0) \\ &\subset \sigma_{T(x\otimes f)+(x\otimes f)T}(x_0),\end{aligned}$$

and $\sigma^*_{T(x\otimes f)+(x\otimes f)T}(x_0) = \{f(Tx)\}$. If $f(Tx_0) = 0$ then $f(x_0) \neq 0$ by assumption and $(T(x \otimes f) + (x \otimes f)T)(x_0) = f(x_0)Tx$. Thus by a similar reasoning,

$$\{f(Tx)\} \subset \sigma_{T(x\otimes f)+(x\otimes f)T}(x_0) \subset \{f(Tx), 0\},$$

and $\sigma^*_{T(x\otimes f)+(x\otimes f)T}(x_0) = \{f(Tx)\}$ in this case too.

(3) Assume that $f(x_0) \neq 0$ or $f(Tx_0) \neq 0$, and $f(x) \neq 0$ and $f(T^2x) \neq 0$. Let

$$\alpha_1 := f(Tx) - \sqrt{f(T^2x)f(x)} \quad \text{and} \quad \alpha_2 := f(Tx) + \sqrt{f(T^2x)f(x)},$$

and

$$z_1 := f(x)Tx - \sqrt{f(T^2x)f(x)}x \quad \text{and} \quad z_2 := f(x)Tx + \sqrt{f(T^2x)f(x)}x.$$

Since $f(x) \neq 0$ and $f(T^2x) \neq 0$, either $\alpha_1 \neq 0$ or $\alpha_2 \neq 0$. On the other hand, it is easy to see that $(T(x \otimes f) + (x \otimes f)T)z_i = \alpha_i z_i$ for $i = 1, 2$. We also have

$$(T(x\otimes f)+(x\otimes f)T)x_0 = f(Tx_0)x+f(x_0)Tx = \Gamma_1(T,f,x)z_1+\Gamma_2(T,f,x)z_2.$$

Since $f(x_0) \neq 0$ or $f(Tx_0) \neq 0$, either $\Gamma_1(T,f,x) \neq 0$ or $\Gamma_2(T,f,x) \neq 0$. Hence, Lemma 3.1(2) implies that

$$\sigma_{T(x\otimes f)+(x\otimes f)T}((Tx\otimes f + x\otimes fT)x_0) = \begin{cases} \{\alpha_1, \alpha_2\} & \text{if } \Gamma_1(T,f,x) \neq 0 \text{ and } \Gamma_2(T,f,x) \neq 0, \\ \{\alpha_1\} & \text{if } \Gamma_1(T,f,x) \neq 0 \text{ and } \Gamma_2(T,f,x) = 0, \\ \{\alpha_2\} & \text{if } \Gamma_2(T,f,x) \neq 0 \text{ and } \Gamma_1(T,f,x) = 0. \end{cases}$$

From this and Lemma 3.1(4), we infer that

$$\sigma^*_{Tx\otimes f+x\otimes fT}(x_0) = \begin{cases} \{\alpha_1, \alpha_2\} \setminus \{0\} & \text{if } \Gamma_1(T,f,x) \neq 0 \text{ and } \Gamma_2(T,f,x) \neq 0, \\ \{\alpha_1\} & \text{if } \Gamma_1(T,f,x) \neq 0 \text{ and } \Gamma_2(T,f,x) = 0, \\ \{\alpha_2\} & \text{if } \Gamma_2(T,f,x) \neq 0 \text{ and } \Gamma_1(T,f,x) = 0. \end{cases}$$

The proof of Lemma 3.4 is thus complete. ∎

The following lemma is a useful observation and, together with the promised local spectral identity principle and local spectral characterization of rank one nilpotent operators, allows us to show that if a surjective map φ from $\mathscr{B}(X)$ into $\mathscr{B}(Y)$ preserves the local spectrum at a fixed nonzero vector of a triple product of operators, then it is automatically a bijective linear map from $\mathscr{N}_1(X)$ into $\mathscr{N}_1(Y)$.

Lemma 3.5. *Let $x_0 \in X \setminus \{0\}$. For every $N \in \mathscr{N}_1(X)$,*

$$\sigma^*_{(T+S)N+N(T+S)}(x_0) = \sigma^*_{TN+NT}(x_0) + \sigma^*_{SN+NS}(x_0)$$

for all $T, S \in \mathscr{B}(X)$.

Proof. Let $N = x\otimes f \in \mathscr{N}_1(X)$, where $x \in X$ and $f \in X^*$ with $f(x) = 0$. If $f(x_0) = f(Tx_0) = 0$, Lemma 3.4(1) implies that

$$\sigma^*_{(T+S)N+N(T+S)}(x_0) = \sigma^*_{TN+NT}(x_0) = \sigma^*_{SN+NS}(x_0) = \{0\}$$

for all $T, S \in \mathscr{B}(X)$, and thus the desired identity holds trivially.

If $f(x_0) \neq 0$ or $f(Tx_0) \neq 0$, Lemma 3.4(2) entails that

$$\begin{aligned}\sigma^*_{(T+S)N+N(T+S)}(x_0) &= \{f(T+S)x\} = \{f(T)x\} + \{f(S)x\}\\ &= \sigma^*_{TN+NT}(x_0) + \sigma^*_{SN+NS}(x_0),\end{aligned}$$

as desired. ∎

4. Jordan product and local spectral identity principles. In this section, we establish some *local spectral identity principles* that will be exploited in the proof of Theorem 2.1. We believe that these principles are of interest in their own right. The first principle provides necessary and sufficient conditions for two operators to be the same modulo a scalar operator.

THEOREM 4.1. *Let $x_0 \in X \setminus \{0\}$. For $A, B \in \mathscr{B}(X)$, the following statements are equivalent:*

(1) $A = B + \gamma\mathbf{1}$ *for some* $\gamma \in \mathbb{C}$.
(2) $\sigma^*_{AN+NA}(x_0) = \sigma^*_{BN+NB}(x_0)$ *for all* $N \in \mathscr{N}_1(X)$.

Proof. Assume that (1) holds, and let $N := x\otimes f$ be a rank one nilpotent operator where $x \in X$ and $f \in X^*$ with $f(x) = 0$. If $f(x_0) = f(Ax_0) = 0$ then $f(x_0) = f(Bx_0) = 0$ and Lemma 3.4 implies that $\sigma^*_{AN+NA}(x_0) = \{0\} = \sigma^*_{BN+NB}(x_0)$, as desired. If $f(x_0) \neq 0$ or $f(Ax_0) \neq 0$ then $f(x_0) \neq 0$ or $f(Bx_0) \neq 0$ and again Lemma 3.4 implies $\sigma^*_{AN+NA}(x_0) = \{f(Ax)\} = \{f(Bx)\} = \sigma^*_{BN+NB}(x_0)$. This proves (1)⇒(2).

Conversely, assume that (2) holds, and let us show that $f(Ax) = f(Bx)$ for all $x \in X$ and $f \in X^*$ such that $f(x) = 0$. Indeed, fix $f \in X^*$, and first assume that $f(x_0) \neq 0$. Lemma 3.4 and our assumption imply that

$$\{f(Ax)\} = \sigma^*_{A(x\otimes f)+(x\otimes f)A}(x_0) = \sigma^*_{B(x\otimes f)+(x\otimes f)B}(x_0) = \{f(Bx)\}$$

for all x in X for which $f(x) = 0$. Second, assume that $f(x_0) = 0$ and note that our assumption with $N := x_0 \otimes f$ and Lemma 3.4 imply that $f(Ax_0) = f(Bx_0)$. Now, take $x \in X$ such that $f(x) = 0$ and x and x_0 are linearly independent, and pick $g \in X^*$ such that $g(x) = 0$ and $g(x_0) = 1$. Applying the above to g and $f+g$, we have $g(Ax) = g(Bx)$ and $(f+g)(Ax) = (f+g)(Bx)$. Clearly, $f(Ax) = f(Bx)$ for all $x \in X$ such that $f(x) = 0$, and thus $A = B+\gamma\mathbf{1}$ for some $\gamma \in \mathbb{C}$ (see Lemma 3.2). This establishes (2)⇒(1), and completes the proof of Theorem 4.1. ∎

As an immediate consequence, we obtain the following corollary that characterizes scalar operators in terms of the local spectrum of a Jordan product of operators.

Corollary 4.2. *For $A \in \mathscr{B}(X)$ and a fixed nonzero $x_0 \in X$, the following assertions are equivalent:*

(1) *A is a scalar operator, i.e., $A = \gamma\mathbf{1}$ for some $\gamma \in \mathbb{C}$.*
(2) *$\sigma^*_{NA+AN}(x_0) = \{0\}$ for all $N \in \mathscr{N}_1(X)$.*

Proof. Apply Theorem 4.1 with $B = 0$. ∎

The second principle gives necessary and sufficient conditions for two operators to be the same.

Theorem 4.3. *Let $x_0 \in X \setminus \{0\}$. For $A, B \in \mathscr{B}(X)$, the following statements are equivalent:*

(1) *$A = B$,*
(2) *$\sigma^*_{AT+AT}(x_0) = \sigma^*_{BT+TB}(x_0)$ for all $T \in \mathscr{B}(X)$.*
(3) *$\sigma^*_{AR+RA}(x_0) = \sigma^*_{BR+RB}(x_0)$ for all rank one $R \in \mathscr{B}(X)$.*

Proof. We only need to show (3)⇒(1). So, assume that $\sigma^*_{AR+RT}(x_0) = \sigma^*_{BR+RB}(x_0)$ for all rank one $R \in \mathscr{B}(X)$, and note that Theorem 4.1 entails that $A = B + \gamma\mathbf{1}$ for some $\gamma \in \mathbb{C}$. We only need to show that $\gamma = 0$. To do so, first we show that Ax_0 and Bx_0 are linearly dependent. Assume they are linearly independent, and note that, since $Ax_0 = Bx_0 + \gamma x_0$, we must have $\gamma \neq 0$. Pick $f \in X^*$ such that $f(Ax_0) = 0$ and $f(Bx_0) = -\gamma$. We claim that $f(A^2x_0) = 0$. Indeed, suppose $f(A^2x_0) \neq 0$, and note that $\varGamma_1(A, f, x_0) = \varGamma_2(A, f, x_0) = 1/2$ (see (3.4)). Since $Ax_0 = Bx_0 + \gamma x_0$, we have $f(x_0) = 1$ and thus Lemma 3.4 yields

$$\{\pm\sqrt{f(A^2x_0)}\} = \sigma^*_{A(x_0\otimes f)+(x_0\otimes f)A}(x_0) = \sigma^*_{B(x_0\otimes f)+(x_0\otimes f)B}(x_0). \tag{4.8}$$

On the other hand, since $A^2x_0 = (B+\gamma)^2x_0$, we have $0 \neq f(A^2x_0) = f(B^2x_0) - \gamma^2$; it follows from this and (3.4) that

$$\varGamma_1(B, f, x) = \frac{1}{2}\biggl(1 + \frac{\gamma}{\sqrt{f(B^2x_0)}}\biggr) \neq 0,$$
$$\varGamma_2(B, f, x) = \frac{1}{2}\biggl(1 - \frac{\gamma}{\sqrt{f(B^2x_0)}}\biggr) \neq 0.$$

Again applying Lemma 3.4, we see that

$$\begin{aligned}\sigma^*_{B(x_0\otimes f)+(x_0\otimes f)B}(x_0) &= \{f(Bx_0) \pm \sqrt{f(B^2x_0)f(x_0)}\}\\ &= \{-\gamma \pm \sqrt{f(B^2x_0)}\}.\end{aligned} \tag{4.9}$$

This, (4.8) and (4.9) with the fact that $f(A^2x_0) = f(B^2x_0) - \gamma^2$ show that

$$\{\pm\sqrt{f(B^2x_0)-\gamma^2}\} = \{-\gamma \pm \sqrt{f(B^2x_0)}\},$$

and thus $f(B^2x_0) = \gamma^2$ and $f(A^2x_0) = f(B^2x_0) - \gamma^2 = 0$. This contradiction shows that $f(A^2x_0) = 0$, as claimed.

Second, let us show that $\alpha = 1$ and thus $\gamma = 0$. Suppose that $\alpha \neq 1$ (and hence $\gamma \neq 0$). Then $(\alpha-1)Bx_0 = Ax_0 - Bx_0 = \gamma x_0$, and

$$Bx_0 = ax_0 \quad \text{and} \quad Ax_0 = \alpha a x_0, \quad \text{where} \quad a = \frac{\gamma}{\alpha - 1}.$$

For any $f \in X^*$ such that $f(x_0) = 1$, we have $(A(x_0 \otimes f) + (x_0 \otimes f)A)x_0 = 2\alpha a x_0$ and $(B(x_0 \otimes f) + (x_0 \otimes f)B)(x_0) = 2ax_0$. Consequently,

$$\{2\alpha a\} = \sigma^*_{A(x_0\otimes f)+(x_0\otimes f)A}(x_0) = \sigma^*_{B(x_0\otimes f)+(x_0\otimes f)B}(x_0) = \{2a\}.$$

This implies that $2a = 2a\alpha$ and gives a contradiction. Thus $\alpha = 1$ and $\gamma = 0$, as desired. ∎

The following corollary is an immediate consequence of Theorem 4.3 and characterizes the zero operator in terms of the local spectrum of the Jordan product of operators.

Corollary 4.4. *For $A \in \mathscr{B}(X)$, the following are equivalent:*

(1) $A = 0$.
(2) $\sigma^*_{TA+AT}(x_0) = \{0\}$ *for all* $T \in \mathscr{B}(X)$.
(3) $\sigma^*_{RA+AR}(x_0) = \{0\}$ *for all rank one* $R \in \mathscr{B}(X)$.

Proof. Apply Theorem 4.3 with $B = 0$. ∎

5. Local spectral characterization of rank one nilpotent operators. In this section, we give a local spectral characterization of rank one nilpotent operators in terms of the local spectrum of a Jordan product of operators. From Lemma 3.4, one sees that if x_0 is a nonzero vector in X and N is a rank one nilpotent operator, then $\sigma^*_{TN+NT}(x_0)$ is a singleton for all $T \in \mathscr{B}(X)$. One may expect that the converse is true but we shall see that there are $N \in \mathscr{B}(X)$ that are not rank one nilpotent operators and for which $\sigma^*_{TN+NT}(x_0)$ is a singleton for all $T \in \mathscr{B}(X)$. We prove, in fact, that $N \in \mathscr{B}(X) \setminus \{0\}$ is a rank one nilpotent operator if and and only if either $\sigma_{TN+NT}(x_0)$ is a singleton for all $T \in \mathscr{B}(X)$ and $\sigma_N(x_0) = \{0\}$, or $\sigma^*_{TN+NT}(x_0)$ is a singleton for all $T \in \mathscr{B}(X)$ and there exists $T_0 \in \mathscr{B}(X)$ such that $\sigma_{T_0N+NT_0}(x_0) = \{0, a\}$ for some nonzero $a \in \mathbb{C}$. Such a characterization allows us to show that if a map φ from $\mathscr{B}(X)$ onto $\mathscr{B}(Y)$ satisfies (2.1), then φ preserves rank one nilpotent operators in both directions. The proof is long and requires some auxiliary lemmas.

The first two lemmas summarize some properties of operators $N \in \mathscr{B}(X)$ for which $\sigma^*_{TN+NT}(x_0)$ is a singleton for all $T \in \mathscr{B}(X)$.

LEMMA 5.1. *Let $N \in \mathscr{B}(X)$ be a nonzero operator for which $\sigma^*_{NT+TN}(x_0)$ is a singleton for all $T \in \mathscr{B}(X)$ with rank at most 2. Then:*

(1) *N is not a scalar operator.*
(2) *If $N^2 = \beta N$ for some $\beta \in \mathbb{C}$, then $N^2 = 0$.*
(3) *If $N = \gamma\mathbf{1} + x_0 \otimes f$ for some $\gamma \in \mathbb{C}$ and $f \in X^*$, then $2\gamma + f(x_0) = 0$.*
(4) *If $N^2 = \beta N + \lambda\mathbf{1}$ for some $\beta, \lambda \in \mathbb{C}$, then either $N^2 = 0$ or $N = \gamma\mathbf{1} + x_0 \otimes f$ for some $\lambda \in \mathbb{C}$ and $f \in X^*$ for which $2\gamma + f(x_0) = 0$.*

Proof. (1) Suppose that $N = \alpha\mathbf{1}$ for some $\alpha \in \mathbb{C}$; since $N \neq 0$, we have $\alpha \neq 0$. Take two linearly independent vectors x_1 and x_2 in X such that $x_0 = x_1 + x_2$ and pick f_1 and f_2 in X^* such that $f_i(x_j) = \delta_{ij}$ for $i, j = 1, 2$, where δ_{ij} is the Kronecker delta. Let $T := (x_1 \otimes f_1) + 2(x_2 \otimes f_2)$; then $Tx_1 = x_1$ and $Tx_2 = 2x_2$. By Lemma 3.1, we have $\sigma_T(x_0) = \sigma_T(x_1) \cup \sigma_T(x_2) = \{1, 2\}$. On the other hand, since $N = \alpha\mathbf{1}$, we have

$$\sigma_{NT+TN}(x_0) = \sigma_{\alpha T+\alpha T}(x_0) = 2\alpha\sigma_T(x_0) = \{\alpha, 2\alpha\}.$$

This contradiction shows that N is not a scalar operator.

(2) Assume that $N^2 = \beta N$ for some $\beta \in \mathbb{C}$, and suppose $\beta \neq 0$. If $\{x_0, Nx_0\}$ is a linearly independent set, let f_0 in X^* be such that $f_0(x_0) = 4/\beta$ and $f_0(Nx_0) = 1$, and note that $\Gamma_1(N, f_0, x_0) = 1/4$ and $\Gamma_2(N, f_0, x_0) = 3/4$. By Lemma 3.4,

$$\sigma^*_{N(x_0\otimes f_0)+(x_0\otimes f_0)N}(x_0) = \{f_0(Nx_0) \pm \sqrt{f_0(N^2x_0)f_0(x_0)}\} \setminus \{0\} = \{-1, 3\}.$$

This contradiction shows that $Nx_0 = \alpha x_0$ for some $\alpha \in \mathbb{C}$. Note that, by (1), there is x_1 in X such that x_1 and Nx_1 are linearly independent. Moreover, since $N^2x_0 = \alpha Nx_0$, either $Nx_0 = 0$ or $\alpha = \beta$. If $Nx_0 = 0$, then pick f_1 in X^* such that $f_1(x_0) \neq 0$, $f_1(x_1) = 4/\beta$ and $f_1(Nx_1) = 1$, and note that $\Gamma_1(N, f_1, x_1) = \Gamma_2(N, f_1, x_1) = \beta f(x_0)/8 \neq 0$. By Lemma 3.4, we have

$$\sigma^*_{N(x_1\otimes f_1)+(x_1\otimes f_1)N}(x_0) = \{f_1(Nx_1) \pm \sqrt{f_1(N^2x_1)f_1(x_1)}\} \setminus \{0\} = \{-1, 3\}.$$

This contradiction shows that $\alpha = \beta$.

If x_0, x_1 and Nx_1 are linearly independent, then take f_2 in X^* such that $f_2(x_0) = 1$, $f_2(x_1) = 4/\beta$ and $f_2(Nx_1) = 1$, and note that

$$\Gamma_1(N, f_2, x_1) = -\beta/8 \neq 0 \quad \text{and} \quad \Gamma_2(N, f_2, x_1) = 3\beta/8 \neq 0.$$

Thus Lemma 3.4 shows that $\sigma^*_{N(x_1\otimes f_2)+(x_1\otimes f_2)N}(x_0) = \{-1, 3\}$ contains two different elements. This contradiction shows that $x_0 = ax_1 + bNx_1$ for some $a, b \in \mathbb{C}$. Now, let t be a large enough positive scalar so that $(a/\beta)t + b \neq 0$, and take f_3 in X^* such that $f_3(x_1) = t/\beta$ and $f_3(Nx_1) = 1$. Observe that

$f_3(x_0) = (a/\beta)t + b \neq 0$, and note that

$$\Gamma_1(N, f_3, x_1) = \frac{\beta f_3(x_0)}{2}\left(\frac{1}{t} - \frac{1}{\sqrt{t}}\right) \neq 0,$$
$$\Gamma_2(N, f_3, x_1) = \frac{\beta f_3(x_0)}{2}\left(\frac{1}{t} + \frac{1}{\sqrt{t}}\right) \neq 0.$$

Thus Lemma 3.4 shows that $\sigma^*_{N(x_1\otimes f_3)+(x_1\otimes f_3)N}(x_0) = \{1-\sqrt{t}, 1+\sqrt{t}\}$ contains two different elements. This contradiction shows that $\beta = 0$.

(3) Assume that $N = \gamma\mathbf{1} + x_0 \otimes f$ for some $\lambda \in \mathbb{C}$ and $f \in X^*$, and let $x \in X$ be such that $f(x) = 1$, and x_0 and x are linearly independent. Note that

$$N^2 = (2\gamma + f(x_0))N - (\gamma^2 + \gamma f(x_0))\mathbf{1}.$$

If $\gamma = 0$, then $N^2 = f(x_0)N$ and $f(x_0) = 0$ by (2), and thus there is nothing to prove. So, assume $\gamma \neq 0$ and note that x and $Nx = \gamma x + x_0$ are linearly independent. If $\beta := 2\gamma + f(x_0) \neq 0$ and $\alpha := -\gamma^2 - \gamma f(x_0) \neq 0$, let s be a positive real such that $1 - \alpha s \neq 0$ and $|(\beta + \alpha s)s| \neq 1, |(\gamma + f(x_0))s|^2$. Pick $f_1 \in X^*$ such that $f_1(Nx) = 1$ and $f_1(x) = s$, and note that $f_1(x_0) = 1 - \alpha s \neq 0$ and $f_1(N^2x) = \beta + \alpha s$. Moreover, since $1 - \alpha s \neq 0$ and $|(\beta + \alpha s)s| \neq |(\gamma + f(x_0))s|^2$, we have

$$\Gamma_1(N, f_1, x) = (1 - \alpha s)\left(\frac{1}{2s} - \frac{\gamma + f(x_0)}{2\sqrt{(\beta + \alpha s)s}}\right) \neq 0,$$
$$\Gamma_2(N, f_1, x) = (1 - \alpha s)\left(\frac{1}{2s} + \frac{\gamma + f(x_0)}{2\sqrt{(\beta + \alpha s)s}}\right) \neq 0.$$

By Lemma 3.4, we have

$$\begin{aligned}\sigma^*_{N(x\otimes f_1)+(x\otimes f_1)N}(x_0) &= \{f_1(Nx) \pm \sqrt{f_1(N^2x)f_1(x)}\} \setminus \{0\}\\ &= \{1 \pm \sqrt{(\beta + \alpha s)s}\}.\end{aligned}$$

This contradicts our assumption, and shows that either $\alpha = 0$ or $\beta = 0$. In either case, we have $\beta = 2\gamma + f(x_0) = 0$ by (2) as desired.

(4) Suppose that $N^2 = \beta N + \lambda\mathbf{1}$ for some scalars $\beta, \lambda \in \mathbb{C}$, and note that, by (3) and Lemma 3.2, we may and shall assume that there is a nonzero vector x in X such that x_0, x and Nx are linearly independent, and then show that $N^2 = 0$. Let us first prove that if $\lambda \neq 0$ then x_0 and Nx_0 are linearly dependent. If not, pick $f_0 \in X^*$ such that $f_0(x_0) = 1$ and $f_0(Nx_0) = 0$. Since $N^2 = \beta N + \lambda\mathbf{1}$, we have $f_0(N^2x_0) = \lambda \neq 0$ and $\Gamma_1(N, f_0, x_0) = \Gamma_2(N, f_0, x_0) = 1/2$. By Lemma 3.4, we see that $\sigma^*_{N(x_0\otimes f_0)+(x_0\otimes f_0)N}(x_0) = \{-\sqrt{\lambda}, \sqrt{\lambda}\}$. This contradiction shows that $Nx_0 = \theta x_0$ for some $\theta \in \mathbb{C}$.

Second, let us show that either $\beta = 0$ or $\lambda = 0$. Indeed, otherwise let s be a positive real such that $|(\beta + \lambda s)s| \neq 1, |\theta s|^2$, and pick $f_1 \in X^*$ such that $f_1(x_0) = 1$, $f_1(Nx) = 1$ and $f_1(x) = s$. Since $|(\beta + \lambda s)s| \neq |\theta s|^2$, we have

$$\Gamma_1(N, f_1, x) = \frac{1}{2s} - \frac{\theta}{2\sqrt{(\beta + \lambda s)s}} \neq 0,$$
$$\Gamma_2(N, f_1, x) = \frac{1}{2s} + \frac{\theta}{2\sqrt{(\beta + \lambda s)s}} \neq 0.$$

Lemma 3.4 yields

$$\begin{aligned}\sigma^*_{N(x\otimes f_1)+(x\otimes f_1)N}(x_0) &= \{f_1(Nx) \pm \sqrt{f_1(N^2x)f_1(x)}\} \setminus \{0\} \\ &= \{1 \pm \sqrt{(\beta + \lambda s)s}\}.\end{aligned}$$

This contradiction shows that either $\beta = 0$ or $\lambda = 0$. Of course, if $\lambda = 0$ then $N^2 = 0$ by (2) and we are done.

So, assume $\lambda \neq 0$ and thus $\beta = 0$ and $N^2 = \lambda \mathbf{1}$. Since $\lambda \neq 0$, there is a $\theta \in \mathbb{C}$ such that $Nx_0 = \theta x_0$. In fact, since $N^2 = \gamma \mathbf{1}$, we must have $\theta^2 = \lambda \neq 0$. After replacing N by $(1/\theta)N$, we may and shall assume that $N^2 = \mathbf{1}$ and $Nx_0 = x_0$. Now, pick f_2 and f_3 in X^* such that $f_2(x_0) = f_3(x) = 1$ and $f_2(x) = f_2(Nx) = f_3(x_0) = f_3(Nx) = 0$. For $T := (x_0 + Nx) \otimes f_2 + (2Nx - x) \otimes f_3$, we have

$$(NT+TN)(x+Nx) = x+Nx, \quad (NT+TN)(x_0+x+Nx) = 2(x_0+x+Nx).$$

Thus, $\sigma_{NT+TN}(x + Nx) = \{1\}$ and $\sigma_{NT+TN}(x_0 + x + Nx) = \{2\}$. Lemma 3.1 yields

$$\sigma_{NT+TN}(x_0) = \sigma_{NT+TN}(x + Nx) \cup \sigma_{NT+TN}(x_0 + x + Nx) = \{1, 2\}.$$

This contradiction shows that $\lambda = 0$, and thus $N^2 = 0$ by (2). ∎

Lemma 5.2. *Let $N \in \mathscr{B}(X) \setminus \{0\}$. If $\sigma^*_{NT+TN}(x_0)$ is a singleton for all $T \in \mathscr{B}(X)$ of rank at most 2, then either $N^2 = 0$, or $N = \gamma \mathbf{1} + x_0 \otimes f$ for some $\gamma \in \mathbb{C}$ and $f \in X^*$ with $2\gamma + f(x_0) = 0$.*

Proof. Assume that $\sigma^*_{NT+TN}(x_0)$ is as in the statement, and let us show that $N^2 = \beta N + \gamma \mathbf{1}$ for some $\beta, \gamma \in \mathbb{C}$. So, assume for contradiction that there is a nonzero $x_1 \in X$ such that x_1, Nx_1 and N^2x_1 are linearly independent. We claim that in this case x_0, x_1, Nx_1 and N^2x_1 are linearly dependent. Indeed, suppose the contrary, and let us first show that neither $\{x_0, Nx_0, x_1, Nx_1, N^2x_1\}$ is linearly independent, nor $Nx_0 = 0$. If not, pick $f_1 \in X^*$ such that $f_1(x_0) = f_1(x_1) = f(N^2x_1) = 1$ and $f_1(Nx_1) = f_1(Nx_0) = 0$. Then $\Gamma_1(N, f_1, x_1) = \Gamma_2(N, f_1, x_1) = 1/2$, and thus Lemma 3.4 implies that

$$\sigma^*_{N(x_1\otimes f_1)+(x_1\otimes f_1)N}(x_0) = \{f_1(Nx_1) \pm \sqrt{f_1(N^2x_1)f_1(x_1)}\} \setminus \{0\} = \{-1, 1\}.$$

This contradiction shows that x_0, Nx_0, x_1, Nx_1 and N^2x_1 are linearly dependent and $Nx_0 \neq 0$. If Nx_0 is a multiple of any of the vectors x_0, x_1,

Nx_1, N^2x_1, choose positive scalars s and t such that

$$(s,t)=\begin{cases}(1,2|\theta|+1) & \text{if } Nx_0=\theta x_0 \text{ for some } \theta\in\mathbb{C},\\ (1,2|\theta|+1) & \text{if } Nx_0=\theta x_1 \text{ for some } \theta\in\mathbb{C},\\ (1,2|\theta|+1) & \text{if } Nx_0=\theta Nx_1 \text{ for some } \theta\in\mathbb{C},\\ (2|\theta|+1,\ 2) & \text{if } Nx_0=\theta N^2x_1 \text{ for some } \theta\in\mathbb{C}.\end{cases}$$

Pick $f_2\in X^*$ for which $f_2(x_0)=s$, $f_2(x_1)=f_2(Nx_1)=1$ and $f_2(N^2x_1)=t^2$, and note that $st>|f_2(Nx_0)|$. We have

$$\Gamma_1(N,f_2,x_1)=\frac{1}{2}\left(s-\frac{f_2(Nx_0)}{t}\right)\neq 0,$$
$$\Gamma_2(N,f_2,x_1)=\frac{1}{2}\left(s+\frac{f_2(Nx_0)}{t}\right)\neq 0,$$

and Lemma 3.4 gives

$$\sigma^*_{N(x_1\otimes f_2)+(x_1\otimes f_2)N}(x_0)=\{f_2(Nx_1)\pm\sqrt{f_2(N^2x_1)f_2(x_1)}\}\setminus\{0\}=\{1\mp t\}.$$

This contradiction shows that $Nx_0=ax_0+bx_1+cNx_1+dN^2x_1$ with at least two of the scalars a,b,c and d nonzero. Let s, t, u and v be nonzero scalars such that $as+bt+cu+dv=0$ and $u^2\neq tv$. Pick $f_3\in X^*$ satisfying $f_3(x_0)=s$, $f_3(x_1)=t$, $f_3(Nx_1)=u$ and $f_3(N^2x_1)=v$. We have $f_3(Nx_0)=0$, and $\Gamma_1(N,f_3,x_1)=\Gamma_2(N,f_3,x_1)=s/(2t)\neq 0$, and thus Lemma 3.4 entails that

$$\sigma^*_{N(x_1\otimes f_3)+(x_1\otimes f_3)N}(x_0)=\{f_3(Nx_1)\pm\sqrt{f_3(N^2x_1)f_3(x_1)}\}\setminus\{0\}=\{u\mp\sqrt{tv}\}$$

contains two different elements as $u^2\neq tv$. This contradiction establishes our claim and shows that x_0, x_1, Nx_1 and N^2x_1 are linearly dependent, and thus $x_0=\alpha_0x_1+\beta_0Nx_1+\gamma_0N^2x_1$ for some $\alpha_0,\beta_0,\gamma_0\in\mathbb{C}$. Here, we shall discuss two cases.

Case 1. Either $Nx_0=0$, or Nx_0, x_1, Nx_1 and N^2x_1 are linearly independent. In this case, choose $f_4\in X^*$ such that $f_4(Nx_0)=0$, $f_4(x_1)=1$, $f_4(Nx_1)=s$ and $f_4(N^2x_1)=s^4$ where s is a scalar such that $s>1$ and $\alpha_0+\beta_0s+\gamma_0s^4\neq 0$. We have $f(x_0)=\alpha_0+\beta_0s+\gamma_0s^4\neq 0$ and $\Gamma_1(N,f_4,x_1)=\Gamma_2(N,f_4,x_1)=(\alpha_0+\beta_0s+\gamma_0s^4)/2\neq 0$, and thus Lemma 3.4 yields

$$\begin{aligned}\sigma^*_{N(x_1\otimes f_4)+(x_1\otimes f_4)N}(x_0)&=\{f_4(Nx_1)\pm\sqrt{f_4(N^2x_1)f_4(x_1)}\}\setminus\{0\}\\&=\{s-s^2,s+s^2\}.\end{aligned}$$

This is a contradiction.

Case 2. $Nx_0\neq 0$ and Nx_0, x_1, Nx_1 and N^2x_1 are linearly dependent. Since x_1, Nx_1 and N^2x_1 are linearly independent, $Nx_0=\alpha_1x_1+\beta_1Nx_1+\gamma_1N^2x_1$ for some $\alpha_1,\beta_1,\gamma_1\in\mathbb{C}$, not all zero. It is easy to see that there is a positive scalar $s \neq 1$ such that

$$s^2|\alpha_1 + \beta_1 s + \gamma_1 s^4| \neq |\alpha_0 + \beta_0 s + \gamma_0 s^4|. \tag{5.10}$$

Choose $f_5 \in X^*$ such that $f_5(x_1) = 1$, $f_5(Nx_1) = s$ and $f_5(N^2x_1) = s^4$, and note that, in view of (5.10), we have

$$\begin{aligned}\Gamma_1(N, f_5, x_1) &= \frac{\alpha_0 + \beta_0 s + \gamma_0 s^4}{2} - \frac{\alpha_1 + \beta_1 s + \gamma_1 s^4}{2s^2} \neq 0,\\ \Gamma_2(N, f_5, x_1) &= \frac{\alpha_0 + \beta_0 s + \gamma_0 s^4}{2} + \frac{\alpha_1 + \beta_1 s + \gamma_1 s^4}{2s^2} \neq 0.\end{aligned}$$

By Lemma 3.4, we have

$$\begin{aligned}\sigma^*_{N(x_1\otimes f_5)+(x_1\otimes f_5)N}(x_0) &= \{f_5(Nx_1) \pm \sqrt{f_5(N^2x_1)f_5(x_1)}\} \setminus \{0\}\\ &= \{s - s^2, s + s^2\}.\end{aligned}$$

This is a contradiction as well.

Finally, we have shown that x, Nx and N^2x are linearly dependent for all $x \in X$, and so Kaplansky's Lemma tells us that there exist $\alpha, \beta, \gamma \in \mathbb{C}$ not all zero such that $\alpha N^2 + \beta N + \gamma \mathbf{1} = 0$ (see for example [2, 28]). Lemma 5.1 yields $\alpha \neq 0$, and thus we may assume that

$$N^2 = \beta N + \gamma \mathbf{1}$$

for some $\beta, \gamma \in \mathbb{C}$. Again by Lemma 5.1, either $N^2 = 0$, or $N = \gamma \mathbf{1} + x_0 \otimes f$ for some $\lambda \in \mathbb{C}$ and $f \in X^*$ for which $2\gamma + f(x_0) = 0$, as desired. ∎

In terms of the local spectrum at a fixed nonzero vector $x_0 \in X$ of an operator Jordan product, the following result characterizes all rank one nilpotent operators $N = x \otimes f$ for which x is linearly independent of x_0.

THEOREM 5.3. *Let $x_0 \in X \setminus \{0\}$, and $N \in \mathscr{B}(X) \setminus \{0\}$ not of the form $\gamma \mathbf{1} + x_0 \otimes f$ where $\gamma \in \mathbb{C}$ and $f \in X^*$. Then the following statements are equivalent:*

(1) *N is a rank one nilpotent operator, i.e., $N \in \mathscr{N}_1(X)$.*
(2) *$\sigma^*_{NT+TN}(x_0)$ is a singleton for all $T \in \mathscr{B}(X)$.*
(3) *$\sigma^*_{NT+TN}(x_0)$ is a singleton for all $T \in \mathscr{B}(X)$ with rank at most 2.*

Proof. The implication (1)⇒(2) follows immediately from Lemma 3.4, and (2)⇒(3) is obvious. So, we only need to establish (3)⇒(1). Assume that $\sigma^*_{NT+TN}(x_0)$ is a singleton for all $T \in \mathscr{B}(X)$ with rank at most 2, and note that, by Lemma 5.2, we have $N^2 = 0$. Assume to the contrary that N has rank at least 2 and let us first show that x_0 is not in the range of N. If it is, there are x and y in X such that $x_0 = Nx$ and Ny are linearly independent. Set $x_1 := (x - y)/2$ and $x_2 := (x + y)/2$, and note that Nx_1 and Nx_2 are linearly independent. Take f_1 and f_2 in X^* such that $f_i(Nx_j) = \delta_{ij}$ for $i, j = 1, 2$. For $T := x_1 \otimes f_1 + 2x_2 \otimes f_2$, we have $(NT + TN)Nx_1 = Nx_1$

and $(NT+TN)Nx_2 = 2Nx_2$. As $x_0 = Nx_1 + Nx_2$, Lemma 3.1 tells us that

$$\begin{aligned}\sigma_{NT+TN}(x_0) &= \sigma_{NT+TN}(Nx_1 + Nx_2)\\ &= \sigma_{NT+TN}(Nx_1) \cup \sigma_{NT+TN}(Nx_2) = \{1,2\}.\end{aligned}$$

This contradiction shows that x_0 is not in the range of N, as desired.

Now, to establish a contradiction and show that N is a rank one operator, we shall discuss two cases.

Case 1. Assume that $Nx_0 \neq 0$. Since the rank of N is supposed to be at least 2, there exists $x \in X$ such that Nx_0 and Nx are linearly independent. Observe that x_0 and x are also linearly independent, and set

$$x_3 := \frac{x_0 - x}{2} \quad \text{and} \quad x_4 := \frac{x_0 + x}{2}.$$

Note that x_3, x_4, Nx_3 and Nx_4 are linearly independent. Indeed, assume that

$$ax_3 + bx_4 + cNx_3 + dNx_4 = 0 \tag{5.11}$$

for some $a, b, c, d \in \mathbb{C}$. Applying N to (5.11) and keeping in mind that $N^2 = 0$, one has $aNx_3 + bNx_4 = 0$, so $a = b = 0$ since Nx_3 and Nx_4 are linearly independent. Thus (5.11) becomes $cNx_3 + dNx_4 = 0$, which in turn gives $c = d = 0$. Hence, $\{x_3, x_4, Nx_3, Nx_4\}$ is a linearly independent set, as claimed. Now, take f_3 and f_4 in X^* such that

$$f_3(x_3) = f_3(x_4) = f_3(Nx_4) = 0 \quad \text{and} \quad f_3(Nx_3) = 1,$$

and

$$f_4(x_3) = f_4(x_4) = f_4(Nx_3) = 0 \quad \text{and} \quad f_4(Nx_4) = 2.$$

Let $T := x_3 \otimes f_3 + x_4 \otimes f_4$. Then $(NT+TN)x_3 = x_3$ and $(NT+TN)x_4 = 2x_4$. As $x_0 = x_3 + x_4$, we have

$$\sigma_{NT+TN}(x_0) = \sigma_{NT+TN}(x_3 + x_4) = \sigma_{NT+TN}(x_3) \cup \sigma_{NT+TN}(x_4) = \{1,2\},$$

by Lemma 3.1. This contradiction shows that N is a rank one operator.

Case 2. Assume that $Nx_0 = 0$. Since the rank of N is at least 2, there exist x and y in X such that Nx and Ny are linearly independent. Since x_0 is not in the range of N, one can easily check that x_0, Nx and Ny are linearly independent. Now, choose f and g in X^* such that

$$f(x_0) = g(x_0) = f(Nx) = g(Ny) = 1 \quad \text{and} \quad g(Nx) = f(Ny) = 0,$$

and let $T := x \otimes f + y \otimes g$. We have $(NT+TN)Nx = Nx$ and $(NT+TN)Ny = 2Ny$, and thus $\sigma_{NT+TN}(Nx + Ny) = \sigma_{NT+TN}(Nx) \cup \sigma_{NT+TN}(Ny) =$

$\{1,2\}$, by Lemma 3.1. As $(NT+TN)x_0 = Nx+Ny$, we get

$$\begin{aligned}\{1,2\} &= \sigma_{NT+TN}(Nx+Ny) \subset \sigma_{NT+TN}(x_0)\\ &\subset \sigma_{NT+TN}(Nx+Ny) \cup \{0\} = \{0,1,2\}\end{aligned}$$

(see Lemma 3.1). Hence, $\sigma^*_{NT+TN}(x_0) = \{1,2\}$, which is a contradiction, and thus N is a rank one operator in this case too. ∎

The following result shows, in particular, that there are operators $N \in \mathscr{B}(X)$ other than rank one nilpotent operators for which $\sigma^*_{NT+TN}(x_0)$ is a singleton for all $T \in \mathscr{B}(X)$.

Theorem 5.4. *Let $x_0 \in X\backslash\{0\}$ and $N \in \mathscr{B}(X)\backslash\{0\}$. Then the following statements are equivalent:*

(1) *$N = \gamma\mathbf{1}+x_0\otimes f$ for some $\gamma \in \mathbb{C}$ and $f \in X^*$ for which $2\gamma+f(x_0) = 0$.*
(2) *$\sigma_{NT+TN}(x_0)$ is a singleton for all $T \in \mathscr{B}(X)$.*
(3) *$\sigma_{NT+TN}(x_0)$ is a singleton for all $T \in \mathscr{B}(X)$ with rank at most 2.*

Proof. Assume that $N = \gamma\mathbf{1} + x_0 \otimes f$ for some $\gamma \in \mathbb{C}$ and $f \in X^*$ such that $2\gamma + f(x_0) = 0$. Then, for every $T \in \mathscr{B}(X)$, we have

$$\begin{aligned}(TN+NT)x_0 &= (T(\gamma\mathbf{1}+x_0\otimes f) + (\gamma\mathbf{1}+x_0\otimes f)T)x_0\\ &= \gamma Tx_0 + f(x_0)Tx_0 + \gamma Tx_0 + f(Tx_0)x_0\\ &= (2\gamma+f(x_0))Tx_0 + f(Tx_0)x_0 = f(Tx_0)x_0.\end{aligned}$$

Hence,

$$\sigma_{TN+NT}(x_0) \subset \{f(Tx_0)\}$$

for all operators T, and (1)⇒(2) follows.

Since the implication (2)⇒(3) holds trivially, we only need to establish (3)⇒(1) in order to finish the proof. So, assume that $\sigma_{NT+TN}(x_0)$ is a singleton for all $T \in \mathscr{B}(X)$ with rank at most 2, and suppose for contradiction that $N \in \mathscr{B}(X)$ is not of the form $\gamma\mathbf{1} + x_0 \otimes f$ where $\gamma \in \mathbb{C}$ and $f \in X^*$. By Theorem 5.3, $N = x \otimes f$ for some $x \in X \setminus \{0\}$ and $f \in X^* \setminus \{0\}$ such that $f(x) = 0$ and x_0 and x are linearly independent. If $f(x_0) = 0$, take $x_1 \in X$ such that $f(x_1) = 1$ and f_1 in X^* such that $f_1(x_0) = f_1(x) = 1$. Set $T_1 := x_1 \otimes f_1$ and note that

$$(T_1N+NT_1)x = x \quad\text{and}\quad (T_1N+NT_1)(x_0-x) = 0.$$

Hence, $\sigma_{T_1N+NT_1}(x_0) = \sigma_{T_1N+NT_1}(x_0-x) \cup \sigma_{T_1N+NT_1}(x) = \{0,1\}$, by Lemma 3.1. This contradiction shows that $f(x_0) \neq 0$. Choose z in X such that $f(z) = 1$ and x_0, x and z are linearly independent. Pick $f_2, f_3 \in X^*$ such that $f_2(x) = f_3(z) = 1$ and $f_2(z) = f_3(x) = 0$, and note that, for $T_2 := z \otimes f_2 + x \otimes f_3$, we have

$$(NT_2+T_2N)x = x \quad\text{and}\quad (NT_2+T_2N)z = z.$$

It follows that $\sigma_{NT_2+T_2N}(x) = \sigma_{NT_2+T_2N}(z) = \{1\}$, and thus $\sigma_{NT_2+T_2N}(w) = \{1\}$ for all nonzero $w \in \bigvee\{x, z\}$ (see Lemma 3.1(2)). Also $(NT_2 + T_2N)w = 0$ for all $w \in \ker(f) \cap \ker(fT_2)$ and $\sigma_{NT_2+T_2N}(w) = \{1\}$ for all nonzero $w \in \ker(f) \cap \ker(fT_2)$. Moreover,

$$X = \ker(f) \cap \ker(fT_2) \oplus \bigvee\{x, z\}.$$

Let p_1 and p_2 be the projections on $\ker(f) \cap \ker(fT_2)$ and $\bigvee\{x, z\}$, respectively. Since $x_0 \notin \ker(f) \cap \ker(fT)$, we have $p_2(x_0) \neq 0$. Also, since $x_0 \notin \bigvee\{x, z\}$, we have $p_1(x_0) \neq 0$. It follows that

$$\sigma_{NT_2+T_2N}(x_0) = \sigma_{NT_2+T_2N}(p_1(x_0)) \cup \sigma_{NT_2+T_2N}(p_2(x_0)) = \{0, 1\},$$

again by Lemma 3.1(2). This contradiction shows that N has the desired form. ∎

As a consequence of Theorem 5.4, we obtain the following corollary that describes, in terms of the local spectrum at a fixed nonzero vector $x_0 \in X$ of an operator Jordan product, all rank one nilpotent operators of the form $x_0 \otimes f$ where $f \in X^*$ for which $f(x_0) = 0$.

Corollary 5.5. *Let $x_0 \in X \setminus \{0\}$ and $N \in \mathscr{B}(X) \setminus \{0\}$. Then the following statements are equivalent:*

(1) *$N = x_0 \otimes f$ for some $f \in X^*$ for which $f(x_0) = 0$.*
(2) *$\sigma_N(x_0) = \{0\}$, and $\sigma_{NT+TN}(x_0)$ is a singleton for all $T \in \mathscr{B}(X)$.*
(3) *$\sigma_N(x_0) = \{0\}$, and $\sigma_{NT+TN}(x_0)$ is a singleton for all $T \in \mathscr{B}(X)$ with rank at most 2.*

Proof. Obviously, the implications (1)⇒(2)⇒(3) always hold. So, assume that $\sigma_N(x_0) = \{0\}$ and $\sigma_{NT+TN}(x_0)$ is a singleton for all $T \in \mathscr{B}(X)$ with rank at most 2, and note that, by Theorem 5.4, $N = \gamma\mathbf{1} + x_0 \otimes f$ for some $\gamma \in \mathbb{C}$ and $f \in X^*$ for which $2\gamma + f(x_0) = 0$. We have $Nx_0 = (\gamma + f(x_0))x_0$ and

$$\{0\} = \sigma_N(x_0) = \{\gamma + f(x_0)\},$$

and $\gamma + f(x_0) = 0$. As $2\gamma + f(x_0) = 0$, we see that $\gamma = f(x_0) = 0$ and $N = x_0 \otimes f$ is a rank one nilpotent operator. This establishes (3)⇒(1). ∎

6. Proof of the main result, Theorem 2.1. Checking the "if" part is straightforward. For the "only if" part assume that φ satisfies (2.1), and let us show that φ has the desired form.

Step 1. *φ is injective and $\varphi(0) = 0$.*

If $\varphi(A) = \varphi(B)$ for some $A, B \in \mathscr{B}(X)$, then by (2.1) we obtain

$$\begin{aligned}\sigma_{AT+TA}(x_0) &= \sigma_{\varphi(A)\varphi(T)+\varphi(T)\varphi(A)}(y_0)\\ &= \sigma_{\varphi(B)\varphi(T)+\varphi(T)\varphi(B)}(y_0) = \sigma_{BT+TB}(x_0)\end{aligned}$$

for all $T \in \mathscr{B}(X)$. By Theorem 4.3, we see that $A = B$. This together with the assumed surjectivity implies that φ is a bijection.

In a similar way, we show that $\varphi(0) = 0$. Indeed,

$$\{0\} = \sigma_{T0+0T}(x_0) = \sigma_{\varphi(T)\varphi(0)+\varphi(0)\varphi(T)}(y_0)$$

for all $T \in \mathscr{B}(X)$. By the bijectivity of φ and by Corollary 4.4, we have $\varphi(0) = 0$, as claimed.

Step 2. *φ preserves rank one nilpotent operators in both directions.*

Let $N = x \otimes f \in \mathscr{B}(X)$ be a nonzero rank one nilpotent operator. If $x = \alpha x_0$ for a nonzero $\alpha \in \mathbb{C}$, then $N = x_0 \otimes (\alpha f)$ and thus, by Corollary 5.5, we see that $\sigma_N(x_0) = \{0\}$, and $\sigma_{NT+TN}(x_0)$ is a singleton for all $T \in \mathscr{B}(X)$. Thus

$$\begin{aligned}\{\sigma_{\varphi(N)}(y_0)\}^2 &= \tfrac{1}{2}\sigma_{\varphi(N)\varphi(N)+\varphi(N)\varphi(N)}(y_0) = \tfrac{1}{2}\sigma_{NN+NN}(x_0)\\ &= \{\sigma_N(x_0)\}^2 = \{0\},\end{aligned}$$

and $\sigma_{\varphi(N)\varphi(T)+\varphi(T)\varphi(N)}(y_0)$ is a singleton for all $T \in \mathscr{B}(X)$. By Corollary 5.5, $\varphi(N) = y_0 \otimes g$ for some $g \in Y^*$ for which $g(y_0) = 0$.

If x and x_0 are linearly independent, then obviously N is not of the form $\gamma \mathbf{1} + x_0 \otimes f$ with $2\gamma + f(x_0) = 0$, and thus, by Theorem 5.4, $\varphi(N)$ is not of this form either. Now, $\sigma^*_{NT+TN}(x_0)$ is a singleton for all $T \in \mathscr{B}(X)$, and so is $\sigma^*_{\varphi(N)\varphi(T)+\varphi(T)\varphi(N)}(y_0)$ for all $T \in \mathscr{B}(X)$. Since $\varphi(N) \neq 0$, by Theorem 5.3 and the bijectivity of φ, we see that $\varphi(N)$ is a rank one nilpotent operator. Conversely, since φ is bijective and φ^{-1} satisfies (2.1), we see that if $\varphi(N)$ is a rank one nilpotent operator, then so is N.

Step 3. *φ is homogeneous.*

Indeed, for every $\lambda \in \mathbb{C}$ and $S, T \in \mathscr{B}(X)$, we have

$$\begin{aligned}\sigma_{\varphi(T)(\lambda\varphi(S))+(\lambda\varphi(S))\varphi(T)}(y_0) &= \lambda\sigma_{\varphi(T)\varphi(S)+\varphi(S)\varphi(T)}(y_0) = \lambda\sigma_{TS+ST}(x_0)\\ &= \sigma_{T(\lambda S)+(\lambda S)T}(x_0) = \sigma_{\varphi(T)\varphi(\lambda S)+\varphi(\lambda S)\varphi(T)}(y_0).\end{aligned}$$

Since φ is bijective, Theorem 4.3 shows that $\varphi(\lambda S) = \lambda\varphi(S)$ for all $\lambda \in \mathbb{C}$ and $S \in \mathscr{B}(X)$.

Step 4. *For $N_1, N_2 \in \mathscr{N}_1(X)$ for which $N_1 + N_2 \in \mathscr{N}_1(X)$, we have*

$$\varphi(N_1 + N_2) = \varphi(N_1) + \varphi(N_2). \tag{6.12}$$

Since φ preserves rank one nilpotent operators in both directions, it suffices to show that for any $S, T \in \mathscr{B}(X)$, there is $\delta_{S,T}$ such that

$$\varphi(S + T) = \varphi(S) + \varphi(T) + \delta_{S,T}\mathbf{1}. \tag{6.13}$$

Indeed, let $S, T \in \mathscr{B}(X)$ and $N \in \mathscr{N}_1(X)$. By Lemma 3.5 and (2.1),

$$\begin{aligned}
&\sigma^*_{\varphi(T+S)\varphi(N)+\varphi(N)\varphi(S+T)}(y_0)\\
&\qquad = \sigma^*_{(S+T)N+N(S+T)}(x_0) = \sigma^*_{TN+NT}(x_0) + \sigma^*_{SN+NS}(x_0)\\
&\qquad = \sigma^*_{\varphi(T)\varphi(N)+\varphi(N)\varphi(T)}(y_0) + \sigma^*_{\varphi(S)\varphi(N)+\varphi(N)\varphi(S)}(y_0)\\
&\qquad = \sigma^*_{(\varphi(T)+\varphi(S))\varphi(N)+\varphi(N)(\varphi(T)+\varphi(S))}(y_0).
\end{aligned}$$

Since φ preserves rank one nilpotent operators, Theorem 4.1 ensures that there is $\delta_{S,T}$ such that $\varphi(T+S) = \varphi(T) + \varphi(S) + \delta_{S,T}\mathbf{1}$, as claimed.

STEP 5. *There exists a bijective transformation $A \in \mathscr{B}(X, Y)$ such that $Ax_0 = y_0$ and*

$$\begin{aligned}
&\text{either} \quad \varphi(N) = ANA^{-1} \quad \text{for all } N \in \mathscr{N}_1(X),\\
&\text{or} \quad \varphi(N) = -ANA^{-1} \quad \text{for all } N \in \mathscr{N}_1(X).
\end{aligned} \tag{6.14}$$

We have shown that φ is a bijective map from $\mathscr{N}_1(X)$ into $\mathscr{N}_1(Y)$, and thus (6.12) applied to both φ and φ^{-1} shows that

$$N_1 + N_2 \in \mathscr{N}_1(X) \;\Leftrightarrow\; \varphi(N_1 + N_2) \in \mathscr{N}_1(Y)$$

for all $N_1, N_2 \in \mathscr{N}_1(X)$. By Lemma 3.3, φ restricted to $\mathscr{N}_1(X)$ has either the form (3.2) or (3.3). We claim that φ cannot take the second form. Indeed, assume that there exists a bijective bounded linear or conjugate linear transformation $A : X^* \to Y$ such that

$$\varphi(N) = \tau_N AN^*A^{-1} \tag{6.15}$$

for all $N \in \mathscr{N}_1(X)$, where τ_N is a scalar depending on N. Take nonzero vectors x and y in $\ker(A^{-1}y_0)$ such that x_0, x and y are linearly independent. Now, take f and g in X^* such that

$$f(x) = 0, \quad f(y) = f(x_0) = 1, \quad \text{and} \quad g(y) = 0, \quad g(x) = 1.$$

For $N_1 := x \otimes f$ and $N_2 := y \otimes g$, we have $(N_1^*N_2^* + N_2^*N_1^*)A^{-1}y_0 = 0$, and thus, by Lemma 3.4,

$$\begin{aligned}
\{0\} &= \sigma_{\tau_{N_1}\tau_{N_2}(A(N_1^*N_2^*+N_2^*N_1^*)A^{-1})}(y_0) = \sigma_{\tau_{N_1}\tau_{N_2}(AN_1^*N_2^*A^{-1}+AN_2^*N_1^*A^{-1})}(y_0)\\
&= \sigma_{\varphi(N_1)\varphi(N_2)+\varphi(N_2)\varphi(N_1)}(y_0) = \sigma_{N_1N_2+N_2N_1}(x_0) = \sigma_{(x\otimes f)N_2+N_2(x\otimes f)}(x_0)\\
&= \{f(N_2x)\} = \{1\}.
\end{aligned}$$

This contradiction shows that φ has the form (3.2), i.e., there exists a bijective bounded linear or conjugate linear transformation $A : X \to Y$ such that

$$\varphi(N) = \tau_N ANA^{-1} \tag{6.16}$$

for all $N \in \mathscr{N}_1(X)$, where τ_N is a scalar depending on N. Let $N_1 := x_1 \otimes f_1 \in \mathscr{N}_1(X)$ and $N_2 := x_2 \otimes f_2 \in \mathscr{N}_1(X)$, where $x_1, x_2 \in X$ and $f_1, f_2 \in X^*$ satisfy

$f_i(x_i) = 0$ for $i = 1, 2$. To show that $\tau_{N_1} = \tau_{N_2}$, take $g \in X^* \setminus \{0\}$ such that $g(x_1) = g(x_2) = 0$, and note that since

$$\begin{aligned}\tau_{(x_1+x_2)\otimes g} A((x_1+x_2)\otimes g)A^{-1} &= \varphi((x_1+x_2)\otimes g) = \varphi(x_1\otimes g) + \varphi(x_2\otimes g)\\ &= \tau_{x_1\otimes g} A(x_1\otimes g)A^{-1} + \tau_{x_2\otimes g} A(x_2\otimes g)A^{-1},\end{aligned}$$

we have

$$\tau_{(x_1+x_2)\otimes g} = \tau_{x_1\otimes g} = \tau_{x_2\otimes g}. \tag{6.17}$$

In a similar way, one shows

$$\tau_{x_1\otimes(f_1+g)} = \tau_{x_1\otimes f_1} = \tau_{x_1\otimes g}, \tag{6.18}$$

$$\tau_{x_2\otimes(f_2+g)} = \tau_{x_2\otimes f_2} = \tau_{x_2\otimes g}. \tag{6.19}$$

From (6.17)–(6.19), we see that $\tau_{N_2} = \tau_{x_2\otimes f_2} = \tau_{x_1\otimes f_1} = \tau_{N_1}$. Thus $\tau_N = \epsilon$ is a nonzero constant independent of $N \in \mathscr{N}_1(X)$, and $\varphi(N) = \epsilon ANA^{-1}$ for all $N \in \mathscr{N}_1(X)$, as desired.

Moreover, we claim that A must be linear. Indeed, take $x, z \in X$, $f \in X^*$ such that $f(x) = 0$ and $f(z) = 1$. By homogeneity of φ, we have

$$\lambda\varepsilon A(x\otimes f)A^{-1} = \lambda\varphi(x\otimes f) = \varphi(\lambda(x\otimes f)) = \varepsilon A(\lambda(x\otimes f))A^{-1},$$

and $\lambda A(x\otimes f)A^{-1} = A(\lambda(x\otimes f))A^{-1}$. It follows that

$$\lambda Ax = \lambda A(x\otimes f)A^{-1}Az = A(\lambda(x\otimes f))A^{-1}Az = A(\lambda x),$$

and A is linear, as desired.

Now, we show that $Ax_0 = \alpha y_0$ for some nonzero $\alpha \in \mathbb{C}$. Assume that x_0 and $A^{-1}y_0$ are linearly independent and take $x_3, x_4 \in X$ such that x_3, x_4, x_0 and $A^{-1}y_0$ are linearly independent. Pick f_3 and f_4 in X^* such that

$$f_3(x_0) = f_3(x_4) = 1, \quad f_3(x_3) = f_3(A^{-1}y_0) = 0,$$

and

$$f_4(x_3) = 1, \quad f_4(x_4) = f_4(A^{-1}y_0) = 0.$$

By Lemma 3.4,

$$\begin{aligned}\{1\} &= \{f_3((x_4\otimes f_4)x_3)\}\\ &= \sigma_{(x_4\otimes f_4)(x_3\otimes f_3)+(x_3\otimes f_3)(x_4\otimes f_4)}(x_0)\\ &= \sigma_{\varphi(x_4\otimes f_4)\varphi(x_3\otimes f_3)+\varphi(x_3\otimes f_3)\varphi(x_4\otimes f_4)}(y_0)\\ &= \sigma_{\epsilon^2 A(x_4\otimes f_4)(x_3\otimes f_3)A^{-1}+\epsilon^2 A(x_3\otimes f_3)(x_4\otimes f_4)A^{-1}}(y_0) = \{0\}.\end{aligned}$$

This contradiction shows that $A^{-1}y_0 = \alpha x_0$ for some nonzero $\alpha \in \mathbb{C}$. After replacing A by αA, we may assume that $Ax_0 = y_0$, and

$$\varphi(N) = \epsilon ANA^{-1}$$

for all $N \in \mathscr{N}_1(X)$.

Now, let us show that $\epsilon = \pm 1$. Take $x_5, x_6 \in X$ such that x_5, x_6, and x_0 are linearly independent. Pick f_5 and f_6 in X^* such that

$$f_5(x_0) = f_5(x_6) = 1, \quad f_5(x_5) = 0, \quad \text{and} \quad f_6(x_5) = 1, \quad f_6(x_6) = f_6(x_0) = 0.$$

By Lemma 3.4,

$$\begin{aligned}\{1\} = \{f_5((x_6 \otimes f_6)x_5)\} &= \sigma_{(x_6\otimes f_6)(x_5\otimes f_5)+(x_5\otimes f_5)(x_6\otimes f_6)}(x_0)\\ &= \sigma_{\varphi(x_6\otimes f_6)\varphi(x_5\otimes f_5)+\varphi(x_5\otimes f_5)\varphi(x_6\otimes f_6)}(y_0)\\ &= \sigma_{\epsilon^2 A(x_6\otimes f_6)(x_5\otimes f_5)A^{-1}+\epsilon^2 A(x_5\otimes f_5)(x_6\otimes f_6)A^{-1}}(y_0)\\ &= \sigma_{\epsilon^2(x_6\otimes f_6)(x_5\otimes f_5)+\epsilon^2(x_5\otimes f_5)(x_6\otimes f_6)}(x_0) = \{\epsilon^2\}.\end{aligned}$$

Hence, $\epsilon = \pm 1$ and thus we may and shall assume that

$$\varphi(N) = ANA^{-1}$$

for all $N \in \mathscr{N}_1(X)$.

Step 6. *$\varphi(T) = ATA^{-1}$ for all $T \in \mathscr{B}(X)$.*

First, we show that for every $T \in \mathscr{B}(X)$ there is $\alpha_T \in \mathbb{C}$ such that

$$\varphi(T) = ATA^{-1} + \alpha_T \mathbf{1}. \tag{6.20}$$

For every $R \in \mathscr{N}_1(X)$ and $T \in \mathscr{B}(X)$, we have

$$\begin{aligned}\sigma_{\varphi(N)ATA^{-1}+ATA^{-1}\varphi(N)}(y_0) &= \sigma_{ANA^{-1}ATA^{-1}+ATA^{-1}ANA^{-1}}(y_0)\\ &= \sigma_{A(NT+NT)A^{-1}}(y_0)\\ &= \sigma_{NT+TN}(x_0) = \sigma_{\varphi(N)\varphi(T)+\varphi(T)\varphi(N)}(y_0).\end{aligned}$$

By Theorem 4.1, the above identity (6.20) holds. In particular, $\varphi(\mathbf{1}) = (1+\alpha_{\mathbf{1}})\mathbf{1}$ for some $\alpha_{\mathbf{1}} \in \mathbb{C}$. Since

$$\{2\} = \sigma_{\mathbf{1}+\mathbf{1}}(x_0) = \sigma_{\varphi(\mathbf{1})\varphi(\mathbf{1})+\varphi(\mathbf{1})\varphi(\mathbf{1})}(y_0) = \{2(1+\alpha_{\mathbf{1}})^2)\},$$

we see that $\alpha_{\mathbf{1}} = 0$ or $\alpha_{\mathbf{1}} = -2$ and thus $\varphi(\mathbf{1}) = \pm\mathbf{1}$. Now, let us check that $\varphi(\mathbf{1}) = -\mathbf{1}$ cannot occur. If $\varphi(\mathbf{1}) = -\mathbf{1}$, then

$$\begin{aligned}\sigma_T(x_0) &= \sigma_{\varphi(\mathbf{1})\varphi(\frac{1}{2}T)+\varphi(\mathbf{1})\varphi(\frac{1}{2}T)}(y_0)\\ &= \sigma_{-\frac{1}{2}\varphi(T)-\frac{1}{2}\varphi(T)}(y_0) = -\sigma_{\varphi(T)}(y_0)\end{aligned} \tag{6.21}$$

for all $T \in \mathscr{B}(X)$. Now, take $x, y, z \in X$ such that x_0, x, y and z are linearly independent. Since $x_1 := x_0 + x$, $x_2 := y - x - z$ and $x_3 := z - y$ are linearly independent as well, there are $f_1, f_2, f_3 \in X^*$ such that $f_i(x_j) = \delta_{ij}$. For $T := x_1 \otimes f_1 + 2x_2 \otimes f_2 + ix_3 \otimes f_3$, we have $Tx_1 = x_1$, $Tx_2 = 2x_2$ and $Tx_3 = ix_3$, and thus

$$\begin{aligned}\{1, 2, i\} &= \sigma_T(x_1) \cup \sigma_T(x_2) \cup \sigma_T(x_3) = \sigma_T(x_1 + x_2 + x_3)\\ &= \sigma_T(x_0) = -\sigma_{\varphi(T)}(y_0) = -\sigma_{ATA^{-1}+\alpha_T}(y_0)\\ &= -\sigma_{ATA^{-1}}(y_0) - \{\alpha_T\} = -\sigma_T(A^{-1}y_0) - \{\alpha_T\} = -\sigma_T(x_0) - \{\alpha_T\}\\ &= \{-1-\alpha_T, -2-\alpha_T, -i-\alpha_T\}.\end{aligned}$$

As no $\alpha_T \in \mathbb{C}$ satisfies this equality, this contradiction shows that $\varphi(\mathbf{1}) = \mathbf{1}$, as desired.

Next, we claim that $\varphi(R) = ARA^{-1}$ for all rank one operators $R \in \mathscr{B}(X)$. Since $\varphi(\mathbf{1}) = \mathbf{1}$, we see that

$$\begin{aligned}\sigma_R(x_0) &= \sigma_{\varphi(R)}(y_0) = \sigma_{ARA^{-1}+\alpha_R}(y_0)\\ &= \sigma_{ARA^{-1}}(y_0) + \{\alpha_R\} = \sigma_R(A^{-1}y_0) + \{\alpha_R\} = \sigma_R(x_0) + \{\alpha_R\}.\end{aligned}$$

As $\sigma_R(x_0)$ is a nonempty set containing at most one nonzero number, we see that $\alpha_R = 0$, hence the claim.

To finish the proof, note that for every rank one operator $R \in \mathscr{B}(X)$ and every $T \in \mathscr{B}(X)$, we have

$$\begin{aligned}\sigma_{\varphi(R)ATA^{-1}+ATA^{-1}\varphi(R)}(y_0) &= \sigma_{ARA^{-1}ATA^{-1}+ATA^{-1}ARA^{-1}}(y_0)\\ &= \sigma_{A(RT+RT)A^{-1}}(y_0) = \sigma_{RT+TR}(x_0)\\ &= \sigma_{\varphi(R)\varphi(T)+\varphi(T)\varphi(R)}(y_0).\end{aligned}$$

By Theorem 4.3, we see that $\varphi(T) = ATA^{-1}$ for all $T \in \mathscr{B}(X)$. The proof of Theorem 2.1 is now complete.

Note added in proof. After this paper was submitted for publication, the authors showed in [6] that Theorem 2.1 remains valid when $X = Y = \mathbb{C}^n$ is a finite-dimensional space and without the surjectivity of the map φ. The proof given therein is completely different from the one presented in the current paper.

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