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NEW BRAIDED MONOIDAL CATEGORIES OVER MONOIDAL HOM-HOPF ALGEBRAS

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Abstract. Let (H, α) and (B, β) be two monoidal Hom-Hopf algebras. We introduce the notion of a generalized Hom-Long dimodule and show that the category ${}^{B}_{H}\mathcal{L}$ of generalized Hom-Long dimodules is an autonomous category. We prove that ${}^{B}_{H}\mathcal{L}$ is a braided monoidal category if (H, α) is quasitriangular and (B, β) is coquasitriangular, and we show that ${}^{B}_{H}\mathcal{L}$ is a subcategory of the Hom-Yetter–Drinfeld category ${}^{H\otimes B}_{H\otimes B}\mathcal{HYD}$. Moreover, we prove that the category of Hom-modules (resp., Hom-comodules) over a triangular (resp., cotriangular) Hom-Hopf algebra contains a symmetric generalized Hom-Long dimodule category.

1. Introduction. The definition of braided monoidal categories (or braided tensor categories) was introduced by Joyal and Street [14] to formalize the characteristic properties of the tensor categories of modules over braided bialgebras as well as the ideas of crossing in link and tangle diagrams. One of the main properties of a braided monoidal category is that its braiding may be considered as the categorical version of the Yang–Baxter equation. The class of braided monoidal categories includes categories of (co)modules over (co)quasitriangular Hopf algebras, Yetter–Drinfeld module categories, Long dimodule categories, etc.

Hom-algebras and Hom-coalgebras were introduced by Makhlouf and Silvestrov [21] as a generalization of ordinary algebras and coalgebras in the following sense: the associativity of multiplication is replaced by Hom-associativity, and similarly for Hom-coassociativity. They also defined the structures of Hom-bialgebras and Hom-Hopf algebras, and described some of their properties extending properties of ordinary bialgebras and Hopf algebras [22, 23]. Later, Yau [32, 33] proposed the definition of quasitriangular Hom-Hopf algebras and showed that each quasitriangular Hom-Hopf algebra yields a solution of Hom-Yang–Baxter equations.

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Caenepeel and Goyvaerts [1] studied Hom-bialgebras and Hom-Hopf algebras from the categorical view point, and called them monoidal Hom-bialgebras and monoidal Hom-Hopf algebras respectively; these are slightly different from the above Hom-bialgebras and Hom-Hopf algebras. Chen et al. [4] studied the quasitriangular structures of monoidal Hom-Hopf algebras and gave an equivalent description via a braided monoidal category of Hom-modules. Many more properties and structures of Hom-Hopf algebras have been developed: see [5], [7], [9], [17], [30], [31] and the references cited therein.

Recently, many scholars in mathematics and physics studied braided monoidal categories over monoidal Hom-Hopf algebras from different perspectives. Makhlouf and Panaite [20] defined Yetter–Drinfeld modules over Hom-bialgebras and showed that Yetter–Drinfeld modules over a Hom-bialgebra with bijective structure map provide solutions of the Hom-Yang–Baxter equation. Also Chen and Zhang [6], Liu and Shen [17] and Li and Ma [16] studied Hom-Yetter–Drinfeld modules in a slightly different way to [20]. Later, You and Wang [34] extended the notion of Hom-Yetter–Drinfeld modules to generalized Hom-Yetter–Drinfeld modules. Recently, Guo et al. [10–13] studied separable functors and total integrals of Hom-Yetter–Drinfeld modules and Doi Hom-Hopf modules.

Chen et al. [3] introduced the concept of Long dimodules over a monoidal Hom-bialgebra and discussed its relation to Long equations. Long dimodules are the building blocks of the Brauer–Long group. When H is commutative, cocommutative and faithfully projective, the Yetter–Drinfeld category ${}_H^H\mathcal{YD}$ is precisely the Long dimodule category ${}_H^H\mathcal{L}$. Of course, for an arbitrary H, the categories ${}_H^H\mathcal{YD}$ and ${}_H^H\mathcal{L}$ are basically different. Militaru [24] proved that Long dimodules are also connected to a non-linear equation (called the Long equation). The reader is referred to [2], [19], [29], [35] and references therein for more details on Long dimodules.

One of the main purposes of this paper is to get a general analogue to the Long dimodule category and study its relation to the quantum Yang–Baxter equation. We introduce the notion of a generalized Hom-Long dimodule and we prove that generalized Hom-Long dimodule categories are braided monoidal subcategories of Hom-Yetter–Drinfeld categories. We also study the problem of when a generalized Hom-Long dimodule category is symmetric.

This paper is organized as follows. In Section 2, we recall some basic definitions and facts related to Hom-(co)modules and (co)quasitriangular Hom-Hopf algebras.

In Section 3, we introduce the notion of a generalized Hom-Long dimodule and we show in Theorem 3.8 that the generalized Hom-Long dimodule category admits an autonomous category structure.

In Section 4, we prove that for a quasitriangular Hom-Hopf algebra (H, R, α) and a coquasitriangular Hom-Hopf algebra $(B, \langle | \rangle, \beta)$, the generalized Hom-Long dimodule category ${}^{B}_{H}\mathcal{L}$ is a subcategory of the Hom-Yetter–Drinfeld category ${}^{H\otimes B}_{H\otimes B}\mathcal{HYD}$ (see Theorem 4.8).

In Section 5, we prove that the category $\widetilde{\mathcal{H}}({}_{H}\mathcal{M})$ over a triangular Hom-Hopf algebra (resp., $\widetilde{\mathcal{H}}({}^{H}\mathcal{M})$ over a cotriangular Hom-Hopf algebra) is a generalized Hom-Long dimodule subcategory of ${}^{B}_{H}\mathcal{L}$ (see Propositions 5.1 and 5.2). We also show in Theorem 5.3 that the generalized Hom-Long dimodule category ${}^{B}_{H}\mathcal{L}$ is symmetric whenever (H, R, α) is triangular and $(B, \langle | \rangle, \beta)$ is cotriangular.

Throughout this paper we freely use the Hopf algebras and coalgebras terminology introduced in [8] and [25–28].

2. Preliminaries. Throughout this paper, k is a fixed field. Unless otherwise stated, all vector spaces, algebras, coalgebras, maps and unadorned tensor products are over k. Any unexplained definitions and notation can be found in [15] and [28].

Let $\mathcal{M}_k = (\mathcal{M}_k, \otimes, k, a, l, r)$ be the category of k-modules. There is a new monoidal category $\mathcal{H}(\mathcal{M}_k)$, defined as follows. The objects of $\mathcal{H}(\mathcal{M}_k)$ are pairs (M, μ), where $M \in \mathcal{M}_k$ and $\mu \in \mathrm{Aut}_k(M)$. The morphisms of $\mathcal{H}(\mathcal{M}_k)$ are morphisms $f : (M, \mu) \to (N, \nu)$ in $\mathcal{M}_k$ such that $\nu \circ f = f \circ \mu$. For any objects $(M, \mu), (N, \nu) \in \mathcal{H}(\mathcal{M}_k)$, the monoidal structure is given by

$$(M, \mu) \otimes (N, \nu) = (M \otimes N, \mu \otimes \nu) \quad \text{and} \quad (k, \mathrm{Id}_k).$$

Briefly speaking, all Hom-structures are objects in the monoidal category $\widetilde{\mathcal{H}}(\mathcal{M}_k) = (\mathcal{H}(\mathcal{M}_k), \otimes, (k, \mathrm{Id}), \tilde{a}, \tilde{l}, \tilde{r})$ introduced in [1], where the associator $\tilde{a}$ is given by the formula

$$\tilde{a}_{M,N,L} = a_{M,N,L} \circ ((\mu \otimes \mathrm{Id}) \otimes \varsigma^{-1}) = (\mu \otimes (\mathrm{Id} \otimes \varsigma^{-1})) \circ a_{M,N,L},$$

for any objects $(M, \mu), (N, \nu), (L, \varsigma) \in \widetilde{H}(\mathcal{M}_k)$, and the unitors $\tilde{l}$ and $\tilde{r}$ are

$$\tilde{l}_M = \mu \circ l_M = l_M \circ (\mathrm{Id} \otimes \mu), \quad \tilde{r}_M = \mu \circ r_M = r_M \circ (\mu \otimes \mathrm{Id}).$$

The category $\widetilde{\mathcal{H}}(\mathcal{M}_k)$ is called the *Hom-category* associated to the monoidal category $\mathcal{M}_k$.

Remark 2.1. We recall from [7] that there is an exact functorial isomorphism

$$\Phi : \widetilde{\mathcal{H}}(\mathcal{M}_k) \to \mathrm{Mod}(k[t, t^{-1}])$$

between the monoidal category $\widetilde{\mathcal{H}}(\mathcal{M}_k)$ defined above and the category $\mathrm{Mod}(k[t, t^{-1}])$ of all modules over the K-algebra $k[, t, t^{-1}]$ of all polynomials in one indeterminate t, with coefficients in k, localized at the multiplicative system $\{1, t, t^2, \dots\}$. Therefore the monoidal category $\widetilde{\mathcal{H}}(\mathcal{M}_k)$ is nothing other than the module category $\mathrm{Mod}(k[t,t^{-1}])$. Consequently, the monoidal category $\widetilde{\mathcal{H}}(\mathcal{M}_k)$ can be viewed as a full exact subcategory of the category $\mathrm{Rep}_k\, Q$ of all k-linear representations of the quiver Q with one vertex and one loop (see [26, Sections 14.1–14.4]).

Now we recall from [1] some facts about Hom-structures.

DEFINITION 2.2. A *monoidal Hom-algebra* is an object (A,α) in the category $\widetilde{\mathcal{H}}(\mathcal{M}_k)$ together with an element $1_A \in A$ and a linear map $m : A \otimes A \to A$, $a \otimes b \mapsto ab$, such that

$$\alpha(a)(bc) = (ab)\alpha(c), \quad \alpha(ab) = \alpha(a)\alpha(b), \tag{2.3}$$

$$a1_A = 1_A a = \alpha(a), \quad \alpha(1_A) = 1_A, \tag{2.4}$$

for all $a, b, c \in A$.

REMARK 2.5. As noted in [1], the definition of a monoidal Hom-algebra is different from the definition of a Hom-associative algebra given in [23]. Note that the same twisted associativity condition (2.3) holds in both cases. However, the unitality condition in [23] is the usual untwisted one: $a1_A = 1_A a = a$ for any $a \in A$, and the condition (2.4) is not required there.

DEFINITION 2.6. (a) Let (A,α) be a monoidal Hom-algebra. A *left* (A,α)*-Hom-module* consists of $(M,\mu) \in \widetilde{\mathcal{H}}(\mathcal{M}_k)$ together with a morphism $\psi : A \otimes M \to M$, $\psi(a \otimes m) = a \cdot m$, such that

$$\alpha(a)\cdot(b\cdot m) = (ab)\cdot\mu(m), \quad 1_A\cdot m = \mu(m), \quad \mu(a\cdot m) = \alpha(a)\cdot\mu(m), \tag{2.7}$$

for all $a, b \in A$ and $m \in M$.

A monoidal Hom-algebra (A,α) can be viewed as a left (A,α)-Hom-module via Hom-multiplication.

(b) Let (M,μ) and (N,ν) be two left (A,α)-Hom-modules. A morphism $f : M \to N$ is called *left A-linear* if $f(am) = af(m)$ for any $a \in A$ and $m \in M$ and $f \circ \mu = \nu \circ f$. We denote the category of left (A,α)-Hom-modules by $\widetilde{\mathcal{H}}({}_A\mathcal{M})$.

DEFINITION 2.8. A *monoidal Hom-coalgebra* is an object (C,γ) in the category $\widetilde{\mathcal{H}}(\mathcal{M}_k)$ together with three linear maps $\Delta : C \to C \otimes C$, $\Delta(c) = c_1 \otimes c_2$, and $\varepsilon : C \to k$ such that

$$g^{-1}(c_1) \otimes \Delta(c_2) = \Delta(c_1) \otimes \gamma^{-1}(c_2), \quad \Delta(\gamma(c)) = \gamma(c_1) \otimes \gamma(c_2), \tag{2.9}$$

$$c_1\varepsilon(c_2) = \varepsilon(c_1)c_2 = \gamma^{-1}(c), \quad \varepsilon(\gamma(c)) = \varepsilon(c), \tag{2.10}$$

for all $c \in C$.

Note that the first part of (2.9) is equivalent to $c_1 \otimes c_{21} \otimes \gamma(c_{22}) = \gamma(c_{11}) \otimes c_{12} \otimes c_2$, which we use frequently in this paper.

DEFINITION 2.11. (a) Let (C,γ) be a monoidal Hom-coalgebra. A *left (C,γ)-Hom-comodule* consists of $(M,\mu)\in\widetilde{\mathcal{H}}(\mathcal{M}_k)$ together with a morphism $\rho_M: M\to C\otimes M$, $\rho_M(m)=m_{(-1)}\otimes m_0$, such that

$$\Delta_C(m_{(-1)})\otimes\mu^{-1}(m_0)=\gamma^{-1}(m_{(-1)})\otimes(m_{0(-1)}\otimes m_{00}), \tag{2.12}$$

$$\rho_M(\mu(m))=\gamma(m_{(-1)})\otimes\mu(m_0),\qquad \varepsilon(m_{(-1)})m_0=\mu^{-1}(m), \tag{2.13}$$

for all $m\in M$.

Note that (C,γ) is a Hom-comodule on itself via Hom-comultiplication.

(b) Let (M,μ) and (N,ν) be two left (C,γ)-Hom-comodules. A morphism $g: M\to N$ is called *left C-colinear* if $g\circ\mu=\nu\circ g$ and $m_{(-1)}\otimes g(m_0)=g(m)_{(-1)}\otimes g(m)_0$ for any $m\in M$. The category of left (C,γ)-Hom-comodules is denoted by $\widetilde{\mathcal{H}}({}^C\mathcal{M})$.

DEFINITION 2.14. (a) A *monoidal Hom-bialgebra* $H=(H,\alpha,m,1_H,\Delta,\varepsilon)$ is a bialgebra in the category $\widetilde{\mathcal{H}}(\mathcal{M}_k)$, i.e., $(H,\alpha,m,1_H)$ is a monoidal Hom-algebra and $(H,\alpha,\Delta,\varepsilon)$ is a monoidal Hom-coalgebra such that Δ and ε are Hom-algebra maps, that is, for any $h,g\in H$,

$$\Delta(hg)=\Delta(h)\Delta(g),\qquad \Delta(1_H)=1_H\otimes 1_H,$$
$$\varepsilon(hg)=\varepsilon(h)\varepsilon(g),\qquad \varepsilon(1_H)=1_k.$$

(b) A monoidal Hom-bialgebra (H,α) is called a *monoidal Hom-Hopf algebra* if there exists a morphism (called the *antipode*) $S: H\to H$ in $\widetilde{\mathcal{H}}(\mathcal{M}_k)$ (i.e., $S\circ\alpha=\alpha\circ S$), which is the convolution inverse of the identity morphism Id_H (i.e., $S*\mathrm{Id}_H=\eta_H\circ\varepsilon_H=\mathrm{Id}_H*S$); this means that for any $h\in H$,

$$S(h_1)h_2=\varepsilon(h)1_H=h_1S(h_2). \tag{2.15}$$

DEFINITION 2.16. (a) Let (H,α) be a monoidal Hom-Hopf algebra. A *left-left (H,α)-Hom-Yetter–Drinfeld module* is an object $(M,\beta)\in\widetilde{\mathcal{H}}(\mathcal{M}_k)$ such that (M,β) is both a left (H,α)-Hom-module and a left (H,α)-Hom-comodule with the following compatibility condition:

$$h_1m_{(-1)}\otimes h_2\cdot m_0=(h_1\cdot\beta^{-1}(m))_{(-1)}h_2\otimes\beta((h_1\cdot\beta^{-1}(m))_0) \tag{2.17}$$

for all $h\in H$ and $m\in M$.

(b) A *Hom-Yetter–Drinfeld category* ${}^H_H\mathcal{HYD}$ is a braided monoidal category whose objects are left-left (H,α)-Hom-Yetter–Drinfeld modules, morphisms are both left (H,α)-linear and (H,α)-colinear maps, and the braiding $C_{-,-}$ is given by

$$C_{M,N}(m\otimes n)=m_{(-1)}\cdot\nu^{-1}(n)\otimes\mu(m_{(0)}) \tag{2.18}$$

for all $m\in(M,\mu)\in{}^H_H\mathcal{HYD}$ and $n\in(N,\nu)\in{}^H_H\mathcal{HYD}$.

Now, we recall from Chen et al. [4] the notion of (co)quasitriangular Hom-Hopf algebra.

DEFINITION 2.19. (a) A *quasitriangular Hom-Hopf algebra* is a monoidal Hom-Hopf algebra (H,α) together with an invertible element $R = R^{(1)}\otimes R^{(2)} \in H\otimes H$ such that R is α-invariant (i.e., $\alpha(R^{(1)})\otimes\alpha(R^{(2)}) = R^{(1)}\otimes R^{(2)}$) and satisfies the following four conditions:

$$\Delta(R^{(1)})\otimes R^{(2)} = R^{(1)}\otimes r^{(1)}\otimes R^{(2)}r^{(2)}, \tag{QT1}$$

$$R^{(1)}\otimes\Delta(R^{(2)}) = R^{(1)}r^{(1)}\otimes r^{(2)}\otimes R^{(2)}, \tag{QT2}$$

$$\varepsilon(R^{(1)})R^{(2)} = 1_H, \qquad R^{(1)}\varepsilon(R^{(2)}) = 1_H, \tag{QT3}$$

$$\Delta^{\mathrm{cop}}(h)R = R\Delta(h), \tag{QT4}$$

where $\Delta^{\mathrm{cop}}(h) = h_2\otimes h_1$ for all $h\in H$.

(b) A quasitriangular Hom-Hopf algebra (H,R,α) is called *triangular* if $R^{-1} = R^{(2)}\otimes R^{(1)}$.

DEFINITION 2.20. (a) A *coquasitriangular Hom-Hopf algebra* is a monoidal Hom-Hopf algebra (B,β) together with a bilinear form $\langle|\rangle$ on (B,β) (i.e. $\langle|\rangle\in\mathrm{Hom}(B\otimes B,k)$) such that $\langle|\rangle$ is β-*invariant* (i.e. $\langle\beta(h)|\beta(g)\rangle = \langle h|g\rangle$ for all $h,g\in B$) and satisfies the following four axioms:

$$\langle hg|l\rangle = \langle h|l_2\rangle\langle g|l_1\rangle, \tag{BR1}$$

$$\langle h|gl\rangle = \langle h_1|g\rangle\langle h_2|l\rangle, \tag{BR2}$$

$$\langle h_1|g_1\rangle g_2h_2 = h_1g_1\langle h_2|g_2\rangle, \tag{BR3}$$

$$\langle 1|h\rangle = \langle h|1\rangle = \varepsilon(h), \tag{BR4}$$

for all $h,g,l\in B$.

(b) A coquasitriangular Hom-Hopf algebra $(B,\langle|\rangle,\beta)$ is called *cotriangular* if $\langle|\rangle$ is convolution invertible in the sense that

$$\langle h_1|g_1\rangle\langle g_2|h_2\rangle = \varepsilon(h)\varepsilon(g) \tag{BR5}$$

for all $h,g\in B$.

3. Generalized Hom-Long dimodules. In this section, we illustrate generalized Hom-Long dimodules and prove that the category of generalized Hom-Long dimodules forms an autonomous category. We always assume that (H,α) and (B,β) are monoidal Hom-Hopf algebras.

DEFINITION 3.1. (a) Let (H,α) and (B,β) be two monoidal Hom-Hopf algebras. A *left-left generalized Hom-Long dimodule* is a quadruple $(M,\cdot,\rho,\mu)$, where $(M,\cdot,\mu)$ is a left (H,α)-Hom-module and (M,ρ,μ) is a left (B,β)-Hom-comodule such that

$$\rho(h\cdot m) = \beta(m_{(-1)})\otimes\alpha^{-1}(h)\cdot m_0 \tag{3.2}$$

for all $h\in H$ and $m\in M$.

(b) We denote by ${}^{B}_{H}\mathcal{L}$ the category of left-left generalized Hom-Long dimodules, morphisms being H-linear B-colinear maps.

REMARK 3.3. Let $(H,\alpha)=(B,\beta)$ be as in Definition 3.1. Then the left-left generalized Hom-Long dimodule is nothing but the Hom-Long dimodule defined in [3].

EXAMPLE 3.4. Let (H,α) and (B,β) be two monoidal Hom-Hopf algebras. Then $(H\otimes B,\alpha\otimes\beta)$ is a left-left generalized Hom-Long dimodule with left (H,α)-action $h\cdot(g\otimes x)=hg\otimes\beta(x)$ and left (B,β)-coaction $\rho(g\otimes x)=x_1\otimes(\alpha^{-1}(h)\otimes x_2)$, where $h,g\in H, x\in B$.

PROPOSITION 3.5. *If $(M,\mu),(N,\nu)$ are two left-left generalized Hom-Long dimodules, then $(M\otimes N,\mu\otimes\nu)$ is a left-left generalized Hom-Long dimodule with the structure actions*

$$h\cdot(m\otimes n)=h_1\cdot m\otimes h_2\cdot n,\qquad \rho(m\cdot n)=m_{(-1)}n_{(-1)}\otimes m_0\otimes n_0,$$

for all $m\in M$, $n\in N$, and $h\in H$.

Proof. By [1, Propositions 2.6 and 2.8], $(M\otimes N,\mu\otimes\nu)$ is both a left (H,α)-Hom-module and a left (B,β)-Hom-comodule. It remains to check that the compatibility condition (3.2). For any $m\in M$, $n\in N$ and $h\in H$, we have

$$\begin{aligned}\rho(h\cdot(m\otimes n)) &= \rho(h_1\cdot m\otimes h_2\cdot n)\\ &= (h_1\cdot m)_{(-1)}(h_2\cdot n)_{(-1)}\otimes(h_1\cdot m)_0\otimes(h_2\cdot n)_0\\ &= \beta(m_{(-1)})\beta(n_{(-1)})\otimes\alpha^{-1}(h_1)\cdot m_0\otimes\alpha^{-1}(h_2)\cdot n_0\\ &\overset{(2.3)}{=} \beta(m_{(-1)}n_{(-1)})\otimes\alpha^{-1}(h)\cdot(m_0\otimes n_0)\\ &= \beta(m\otimes n)_{(-1)})\otimes\alpha^{-1}(h)\cdot(m\otimes n)_0,\end{aligned}$$

as desired. ■

PROPOSITION 3.6. *The category of left-left generalized Hom-Long dimodules ${}^{B}_{H}\mathcal{L}$ is a monoidal category. The tensor product is given in Proposition* 3.5*, the unit $I=(k,\mathrm{Id})$, the associativity and the constraints are given as follows:*

$$a_{U,V,W}:(U\otimes V)\otimes W\to U\otimes(V\otimes W),\ (u\otimes v)\otimes w\mapsto\mu(u)\otimes(v\otimes\omega^{-1}(w)),$$
$$l_V:k\otimes V\to V,\quad k\otimes v\mapsto k\nu(v),\qquad r_V:V\otimes k\to V,\quad v\otimes k\mapsto k\nu(v),$$

where $u\in(U,\mu)\in{}^{B}_{H}\mathcal{L}$, $v\in(V,\nu)\in{}^{B}_{H}\mathcal{L}$, $w\in(W,\omega)\in{}^{B}_{H}\mathcal{L}$.

Proof. Apply the arguments used in the proof of [17, Proposition 4.5]. ■

PROPOSITION 3.7. *Given a left-left generalized Hom-Long dimodule (M,μ) in ${}^{B}_{H}\mathcal{L}$, set $M^*=\mathrm{Hom}_k(M,k)$, with the Hom-module and Hom-comodule structures*

$$\theta_{M^*}: H\otimes M^*\to M^*,\qquad (h\cdot f)(m)=f(S_H(h)\cdot\mu^{-2}(m)),$$
$$\rho_{M^*}: M^*\to B\otimes M^*,\qquad f_{(-1)}\otimes f_{(0)}(m)=S_B^{-1}(m_{(-1)})\otimes f(\mu^2(m_{(0)})),$$

and the Hom-structure map μ^* of M^* is $\mu^*(f)(m)=f(\mu^{-1}(m))$. Then

(a) *M^* is an object in ${}^B_H\mathcal{L}$.*
(b) *${}^B_H\mathcal{L}$ is a left autonomous category.*

Proof. It is not difficult to check that (M^*,μ^*,θ_{M^*}) is an (H,α)-Hom-module and (M^*,μ^*,ρ_{M^*}) is a (B,β)-Hom-comodule. Further, for any $f\in M^*$, $m\in M$ and $h\in H$, we have

$$\begin{aligned}(h\cdot f)_{(-1)}\otimes(h\cdot f)_{(0)}(m)&=S_B^{-1}(m_{(-1)})\otimes(h\cdot f)(\mu^2(m_{(0)}))\\&=S_B^{-1}(m_{(-1)})\otimes f(S_H(h)\cdot m_{(0)}).\end{aligned}$$

Also we get

$$\begin{aligned}&\beta(f_{(-1)})\otimes(\alpha^{-1}(h)\cdot f_{(0)})(m)\\&=\beta(f_{(-1)})\otimes f_{(0)}(S_H(\alpha^{-1}(h))\cdot\mu^{-2}(m))\\&=\beta\big(S_B^{-1}((S_H(\alpha^{-1}(h))\cdot\mu^{-2}(m))_{(-1)})\big)\otimes f\big(\mu^2(S_H(\alpha^{-1}(h))\cdot\mu^{-2}(m))_{(-1)}\big)\\&=\beta(S_B^{-1}(\beta^{-1}(m_{(-1)})))\otimes f(\mu^2\big(S_H(\alpha^{-2}(h))\cdot\mu^{-2}(m_{(0)}))\big)\\&=S_B^{-1}(m_{(-1)})\otimes f(S_H(h)\cdot m_{(0)}).\end{aligned}$$

Thus $M^*\in{}^B_H\mathcal{L}$.

Moreover, for any $f\in M^*$ and $m\in M$, one can define the left evaluation map and the left coevaluation map by

$$\mathrm{ev}_M: f\otimes m\mapsto f(\mu(m)),\qquad \mathrm{coev}_M: 1_k\mapsto\sum\mu^{-1}(e_i)\otimes e^i,$$

where e_i and e^i are dual bases in M and M^* respectively. Next, we claim $(M^*,\mathrm{ev}_M,\mathrm{coev}_M)$ is the left dual of M.

It is easy to see that ev_M and coev_M are morphisms in ${}^B_H\mathcal{L}$. To prove the claim, we need the following computation

$$\begin{aligned}&\big(r_M\circ(\mathrm{Id}_M\otimes\mathrm{ev}_M)\circ a_{M,M^*,M}\circ(\mathrm{coev}_M\otimes\mathrm{Id}_M)\circ l_M^{-1}\big)(m)\\&\qquad=(r_M\circ(\mathrm{Id}_M\otimes\mathrm{ev}_M)\circ a_{M,M^*,M})\Big(\sum_i(\mu^{-1}(e_i)\otimes e^i)\otimes\mu^{-1}(m)\Big)\\&\qquad=(r_M\circ(\mathrm{Id}_M\otimes\mathrm{ev}_M))\Big(\sum_i e_i\otimes(e^i\otimes\mu^{-2}(m))\Big)\\&\qquad=r_M\Big(\sum_i e_i\otimes e^i(\mu^{-1}(m))\Big)=r_M(\mu^{-1}(m)\otimes 1_k)=m.\end{aligned}$$

Similarly, we get

$$\begin{aligned}&\big(l_{M^*}\circ(\mathrm{ev}_M\otimes \mathrm{Id}_{M^*})\circ a^{-1}_{M^*,M,M^*}\circ(\mathrm{Id}_{M^*}\otimes \mathrm{coev}_M)\circ r^{-1}_{M^*}\big)(f)\\ &\qquad=\big(l_{M^*}\circ(\mathrm{ev}_M\otimes \mathrm{Id}_{M^*})\circ a^{-1}_{M^*,M,M^*}\big)\Big(\sum_i \mu^*(f)\otimes(\mu^{-1}(e_i)\otimes e^i)\Big)\\ &\qquad=(l_{M^*}\circ(\mathrm{ev}_M\otimes \mathrm{Id}_{M^*}))\Big(\sum_i(\mu^2)^*(f)\otimes\mu^{-1}(e_i))\otimes(\mu^{-1})^*(e^i)\Big)\\ &\qquad=l_{M^*}\Big(\sum_i(\mu^2)^*(f)(e_i)\otimes(\mu^{-1})^*(e^i)\Big)=l_{M^*}(1_k\otimes\mu^*(f))=f.\end{aligned}$$

It follows that ${}^B_H\mathcal{L}$ admits the left duality and the proof is complete. ∎

THEOREM 3.8. *The generalized Hom-Long dimodule category ${}^B_H\mathcal{L}$ is an autonomous category.*

Proof. By Proposition 3.7, it is sufficient to show that ${}^B_H\mathcal{L}$ is also a right autonomous category. In fact, for any $(M,\mu)\in {}^B_H\mathcal{L}$, its right dual $({}^*M,\widetilde{\mathrm{coev}}_M,\widetilde{\mathrm{ev}}_M)$ is defined as follows:

- ${}^*M=\mathrm{Hom}_k(M,k)$ as k-modules, with the Hom-module and Hom-comodule structures

$$\begin{aligned}&(h\cdot f)(m)=f(S_H^{-1}(h)\cdot\mu^{-2}(m)),\\ &f_{(-1)}\otimes f_{(0)}(m)=S_B(m_{(-1)})\otimes f(\mu^2(m_{(0)})),\end{aligned}$$

where $f\in {}^*M$, $m\in M$, and the Hom-structure map μ^* of M^* is $\mu^*(f)(m)=f(\mu^{-1}(m))$.

- The right evaluation map and the right coevaluation map are given by

$$\widetilde{\mathrm{ev}}_M: m\otimes f\mapsto f(\mu(m)),\qquad \widetilde{\mathrm{coev}}_M:1_k\mapsto\sum a^i\otimes\mu^{-1}(a_i).$$

where a_i and a^i are dual bases of M and ${}^*M$ respectively. By similar verification as in Proposition 3.7, one can check that ${}^B_H\mathcal{L}$ is a right autonomous category, as required. ∎

4. New braided categories over generalized Hom-Long dimodule categories. In this section, we prove that the category ${}^B_H\mathcal{L}$ over a quasitriangular Hom-Hopf algebra (H,R,α) and a coquasitriangular Hom-Hopf algebra $(B,\langle|\rangle,\beta)$ is a braided monoidal subcategory of the Hom-Yetter–Drinfeld category ${}^{H\otimes B}_{H\otimes B}\mathcal{HYD}$.

THEOREM 4.1. *Assume that (H,R,α) is a quasitriangular Hom-Hopf algebra and $(B,\langle|\rangle,\beta)$ is a coquasitriangular Hom-Hopf algebra. Then the category ${}^B_H\mathcal{L}$ is a braided monoidal category with the braiding*

$$\begin{aligned}&C_{M,N}:M\otimes N\to N\otimes M,\\ &C_{M,N}(m\otimes n)=\langle m_{(-1)}|n_{(-1)}\rangle R^{(2)}\cdot n_0\otimes R^{(1)}\cdot m_0,\end{aligned}\tag{4.2}$$

for all $m\in(M,\mu)\in {}^B_H\mathcal{L}$ and $n\in(N,\nu)\in {}^B_H\mathcal{L}$.

Proof. First we show that the braiding $C_{M,N}$ is a morphism in ${}^B_H\mathcal{L}$. In fact, for any $m \in M$, $n \in N$ and $h \in H$, we have

$$
\begin{aligned}
&C_{M,N}(h_1 \cdot m \otimes h_2 \cdot n)\\
&\qquad = \langle (h_1 \cdot m)_{(-1)} | (h_2 \cdot n)_{(-1)} \rangle R^{(2)} \cdot (h_2 \cdot n)_0 \otimes R^{(1)} \cdot (h_1 \cdot m)_0\\
&\qquad \overset{(3.2)}{=} \langle \beta(m)_{(-1)} | \beta(n)_{(-1)} \rangle R^{(2)} \cdot \alpha^{-1}(h_2) \cdot n_0 \otimes R^{(1)} \cdot \alpha^{-1}(h_1) \cdot m_0\\
&\qquad \overset{(2.7)}{=} \langle m_{(-1)} | n_{(-1)} \rangle \alpha^{-1}(R^{(2)} h_2) \cdot \nu(n_0) \otimes \alpha^{-1}(R^{(1)} h_1) \cdot \mu(m_0)\\
&\qquad \overset{(QT4)}{=} \langle m_{(-1)} | n_{(-1)} \rangle \alpha^{-1}(h_1 R^{(2)}) \cdot \nu(n_0) \otimes \alpha^{-1}(h_2 R^{(1)}) \cdot \mu(m_0)\\
&\qquad \overset{(3.2)}{=} \langle m_{(-1)} | n_{(-1)} \rangle h_1 \cdot (\alpha^{-1}(R^{(2)}) \cdot n_0) \otimes h_2 \cdot (\alpha^{-1}(R^{(1)}) \cdot m_0),
\end{aligned}
$$

$$
\begin{aligned}
h \cdot C_{M,N}(m \otimes n) &= \langle m_{(-1)} | n_{(-1)} \rangle h \cdot (R^{(2)} \cdot n_0 \otimes R^{(1)} \cdot m_0)\\
&= \langle m_{(-1)} | n_{(-1)} \rangle h_1 \cdot (R^{(2)} \cdot n_0) \otimes h_2 \cdot (R^{(1)} \cdot m_0)\\
&= \langle m_{(-1)} | n_{(-1)} \rangle h_1 \cdot (\alpha^{-1}(R^{(2)}) \cdot n_0) \otimes h_2 \cdot (\alpha^{-1}(R^{(1)}) \cdot m_0).
\end{aligned}
$$

The third equality holds since $\langle | \rangle$ is β-invariant and the last equality holds since R is α-invariant. So $C_{M,N}$ is left (H, α)-linear. Similarly, one can check that $C_{M,N}$ is left (B, β)-colinear.

Now we prove that the braiding $C_{M,N}$ is natural. For any $(M, \mu), (M', \mu')$, $(N, \nu), (N', \nu') \in {}^B_H\mathcal{L}$, let $f : M \to M'$ and $g : N \to N'$ be two morphisms in ${}^B_H\mathcal{L}$. It is sufficient to verify the identity $(g \otimes f) \circ C_{M,N} = C_{M',N'} \circ (f \otimes g)$. For this purpose, we take $m \in M$, $n \in N$ and do the following calculation:

$$
\begin{aligned}
(g \otimes f) \circ C_{M,N}(m \otimes n) &= \langle m_{(-1)} | n_{(-1)} \rangle (g \otimes f)(R^{(2)} \cdot n_0 \otimes R^{(1)} \cdot m_0)\\
&= \langle m_{(-1)} | n_{(-1)} \rangle g(R^{(2)} \cdot n_0) \otimes f(R^{(1)} \cdot m_0)\\
&= \langle m_{(-1)} | n_{(-1)} \rangle (R^{(2)} \cdot g(n_0)) \otimes (R^{(1)} \cdot f(m_0)),
\end{aligned}
$$

$$
\begin{aligned}
&C_{M',N'} \circ (f \otimes g)(m \otimes n) = C_{M',N'}(f(m) \otimes g(n))\\
&\qquad = \langle (f(m))_{(-1)} | (g(n))_{(-1)} \rangle (R^{(2)} \cdot (g(n))_0) \otimes (R^{(1)} \cdot (f(m))_0)\\
&\qquad = \langle m_{(-1)} | n_{(-1)} \rangle (R^{(2)} \cdot g(n_0)) \otimes (R^{(1)} \cdot f(m_0)).
\end{aligned}
$$

The last equality holds since f, g are left (B, β)-colinear. So the braiding $C_{M,N}$ is natural, as needed.

Next, we show that the braiding $C_{M,N}$ is an isomorphism with inverse

$$
\begin{aligned}
&C^{-1}_{M,N} : N \otimes M \to M \otimes N,\\
&\qquad n \otimes m \mapsto \langle S^{-1}(m_{(-1)}) | n_{(-1)} \rangle S(R^{(1)}) \cdot m_0 \otimes R^{(2)} \cdot n_0.
\end{aligned}
$$

For all $m \in M$ and $n \in N$, we have

$$
\begin{aligned}
&C^{-1}_{M,N} \circ C_{M,N}(m \otimes n)\\
&\quad= \langle m_{(-1)} | n_{(-1)} \rangle C^{-1}_{M,N}(R^{(2)} \cdot n_0 \otimes R^{(1)} \cdot m_0)\\
&\quad= \langle m_{(-1)} | n_{(-1)} \rangle \langle S^{-1}(\beta(m_{0(-1)})) | \beta(n_{0(-1)}) \rangle\\
&\qquad\quad S(r^{(1)}) \cdot (\alpha^{-1}(R^{(1)}) \cdot m_{00}) \otimes r^{(2)} \cdot (\alpha^{-1}(R^{(2)}) \cdot n_{00})\\
&\quad= \langle \beta(m_{(-1)1}) | \beta(n_{(-1)1}) \rangle \langle S^{-1}(\beta(m_{(-1)2})) | \beta(n_{(-1)2}) \rangle\\
&\qquad\quad S(r^{(1)}) \cdot (\alpha^{-1}(R^{(1)}) \cdot \mu^{-1}(m_0)) \otimes r^{(2)} \cdot (\alpha^{-1}(R^{(2)}) \cdot \nu^{-1}(n_0))\\
&\quad\overset{(\mathrm{BR1})}{=} \langle S^{-1}(\beta(m_{(-1)2}))\beta(m_{(-1)1}) | \beta(n_{(-1)}) \rangle\\
&\qquad\quad S(\alpha^{-1}(r^{(1)}))\alpha^{-1}(R^{(1)})) \cdot m_0 \otimes \alpha^{-1}(r^{(2)}R^{(2)}) \cdot n_0\\
&\quad= \langle \beta(S^{-1}(m_{(-1)2})m_{(-1)1}) | \beta(n_{(-1)}) \rangle\\
&\qquad\quad \alpha^{-1}(S(r^{(1)})R^{(1)}) \cdot m_0 \otimes \alpha^{-1}(r^{(2)}R^{(2)}) \cdot n_0\\
&\quad= \langle \beta(S^{-1}(m_{(-1)2})m_{(-1)1}) | \beta(n_{(-1)}) \rangle 1_H \cdot m_0 \otimes 1_H \cdot n_0\\
&\quad= \langle \varepsilon(m_{(-1)})1_H | \beta(n_{(-1)}) \rangle 1_H \cdot m_0 \otimes 1_H \cdot n_0\\
&\quad= \varepsilon(m_{(-1)})\varepsilon(n_{(-1)})\mu(m_0) \otimes \nu(n_0) = (m \otimes n).
\end{aligned}
$$

The second equality holds since $\rho(R^{(2)} \cdot n_0) = \beta(n_{0(-1)}) \otimes \alpha^{-1}(R^{(2)}) \cdot n_{00}$ and the sixth equality holds since $R^{-1} = S(r^{(1)}) \otimes r^{(2)}$.

Now let us verify the hexagon axioms (H_1, H_2) from [15, Section XIII.1.1]. We need to show that the following diagram commutes for any $(U,\mu), (V,\nu)$, (W,ω) in ${}^B_H\mathcal{L}$:

$$
\begin{array}{ccccc}
(U\otimes V)\otimes W & \xrightarrow{a_{U,V,W}} & U\otimes(V\otimes W) & \xrightarrow{c_{U,V\otimes W}} & (V\otimes W)\otimes U\\
\downarrow{\scriptstyle c_{U,V}\otimes \mathrm{Id}_W} & & & & \downarrow{\scriptstyle a_{V,W,U}}\\
(V\otimes U)\otimes W & \xrightarrow{a_{V,U,W}} & V\otimes(U\otimes W) & \xrightarrow{\mathrm{Id}_V\otimes c_{U,W}} & V\otimes(W\otimes U)
\end{array}
\tag{4.3}
$$

For this purpose, we fix $u \in U$, $v \in V$ and $w \in W$. Then we have

$$
\begin{aligned}
&a_{V,U,W} \circ C_{U,V\otimes W} \circ a_{U,V,W}((u \otimes v) \otimes w)\\
&\qquad= a_{V,U,W} \circ C_{U,V\otimes W}(\mu(u) \otimes (v \otimes \omega^{-1}(w)))\\
&\qquad= \langle \beta(u_{(-1)}) | v_{(-1)}\beta^{-1}(w_{(-1)}) \rangle a_{V,U,W}\\
&\qquad\qquad \big(R^{(2)} \cdot (v_0 \otimes \omega^{-1}(w_0)) \otimes R^{(1)} \cdot \mu(u_0)\big)\\
&\qquad= \langle \beta(u_{(-1)}) | v_{(-1)}\beta^{-1}(w_{(-1)}) \rangle a_{V,U,W}\\
&\qquad\qquad \big((R_1^{(2)} \cdot v_0 \otimes R_2^{(2)} \cdot \omega^{-1}(w_0)) \otimes R^{(1)} \cdot \mu(u_0)\big)
\end{aligned}
$$

$$
\begin{aligned}
&= \langle\beta(u_{(-1)})|v_{(-1)}\beta^{-1}(w_{(-1)})\rangle\beta(R_1^{(2)})\cdot\nu(v_0)\\
&\quad\otimes\big(R_2^{(2)}\cdot\omega^{-1}(w_0)\otimes\beta^{-1}(R^{(1)})\cdot u_0\big)\\
&\overset{\text{(QT2)}}{=} \langle\beta(u_{(-1)})|v_{(-1)}\beta^{-1}(w_{(-1)})\rangle\beta(r^{(2)})\cdot\nu(v_0)\\
&\quad\otimes\big(R^{(2)}\cdot\omega^{-1}(w_0)\otimes\beta^{-1}(R^{(1)}r^{(1)})\cdot u_0\big)
\end{aligned}
$$

and

$$
\begin{aligned}
&(\mathrm{Id}_V\otimes C_{U,W})\circ a_{V,U,W}\circ(C_{U,V}\otimes\mathrm{Id}_W)((u\otimes v)\otimes w)\\
&\quad= \langle u_{(-1)}|v_{(-1)}\rangle(\mathrm{Id}_V\otimes C_{U,W})\circ a_{V,U,W}\big((R^{(2)}\cdot v_0\otimes R^{(1)}\cdot u_0)\otimes w\big)\\
&\quad= \langle u_{(-1)}|v_{(-1)}\rangle(\mathrm{Id}_V\otimes C_{U,W})\big(\beta(R^{(2)})\cdot\nu(v_0)\otimes(R^{(1)}\cdot u_0\otimes\omega^{-1}(w))\big)\\
&\quad= \langle u_{(-1)}|v_{(-1)}\rangle\langle\beta(u_{0(-1)})|\beta^{-1}(w_{(-1)})\rangle\\
&\qquad\quad \beta(R^{(2)})\cdot\nu(v_0)\otimes\big(r^{(2)}\cdot\omega^{-1}(w_0)\otimes r^{(1)}\cdot(\beta^{-1}(R^{(1)})\cdot u_{00})\big)\\
&\quad= \langle\beta(u_{(-1)1})|v_{(-1)}\rangle\langle\beta(u_{(-1)2})|\beta^{-1}(w_{(-1)})\rangle\\
&\qquad\quad \beta(R^{(2)})\cdot\nu(v_0)\otimes\big(r^{(2)}\cdot\omega^{-1}(w_0)\otimes(\beta^{-1}(r^{(1)}R^{(1)})\cdot u_0)\big)\\
&\quad\overset{\text{(BR2)}}{=} \langle\beta(u_{(-1)})|v_{(-1)}\beta^{-1}(w_{(-1)})\rangle\beta(R^{(2)})\cdot\nu(v_0)\\
&\qquad\quad\otimes\big(r^{(2)}\cdot\omega^{-1}(w_0)\otimes(\beta^{-1}(r^{(1)}R^{(1)})\cdot u_0)\big).
\end{aligned}
$$

Since $r = R$, it follows that $a_{V,U,W}\circ C_{U,V\otimes W}\circ a_{U,V,W} = (\mathrm{Id}_V\otimes C_{U,W})\circ a_{V,U,W}\circ(C_{U,V}\otimes\mathrm{Id}_W)$, that is, the diagram (4.3) commutes.

Now we check that the following diagram commutes for any $(U,\mu),(V,\nu)$, (W,ω) in ${}^B_H\mathcal{L}$:

$$
\begin{array}{ccccc}
U\otimes(V\otimes W) & \xrightarrow{a^{-1}_{U,V,W}} & (U\otimes V)\otimes W & \xrightarrow{c_{U\otimes V,W}} & W\otimes(U\otimes V)\\
\Big\downarrow{\scriptstyle \mathrm{Id}_U\otimes t_{V,W}} & & & & \Big\downarrow{\scriptstyle a^{-1}_{W,U,V}}\\
U\otimes(W\otimes V) & \xrightarrow{a^{-1}_{U,W,V}} & (U\otimes W)\otimes V & \xrightarrow{c_{U,W}\otimes\mathrm{Id}_V} & (W\otimes U)\otimes V
\end{array}
\tag{4.4}
$$

In fact, for any $u\in U, v\in V, w\in W$, we obtain

$$
\begin{aligned}
&a^{-1}_{W,U,V}\circ C_{U\otimes V,W}\circ a^{-1}_{U,V,W}(u\otimes(v\otimes w))\\
&\quad= a^{-1}_{W,U,V}\circ C_{U\otimes V,W}\big((\mu^{-1}(u)\otimes v)\otimes\omega(w)\big)\\
&\quad= \langle\beta^{-1}(u_{(-1)})v_{(-1)}|\beta(w_{(-1)})\rangle a^{-1}_{W,U,V}\\
&\qquad\quad \big(R^{(2)}\cdot\omega(w_0)\otimes R^{(1)}\cdot(\mu^{-1}(u_0)\otimes v_0)\big)\\
&\quad= \langle\beta^{-1}(u_{(-1)})v_{(-1)}|\beta(w_{(-1)})\rangle a^{-1}_{W,U,V}\\
&\qquad\quad \big(R^{(2)}\cdot\omega(w_0)\otimes(R_1^{(1)}\cdot\mu^{-1}(u_0)\otimes R_2^{(1)}\cdot v_0)\big)\\
&\quad= \langle\beta^{-1}(u_{(-1)})v_{(-1)}|\beta(w_{(-1)})\rangle\\
&\qquad\quad \big(\omega^{-1}(R^{(2)}\cdot\omega(w_0))\otimes R_1^{(1)}\cdot\mu^{-1}(u_0)\big)\otimes\nu(R_2^{(1)}\cdot v_0)
\end{aligned}
$$

$$
\begin{aligned}
&= \langle \beta^{-1}(u_{(-1)})v_{(-1)}|\beta(w_{(-1)})\\
&\qquad \big(\alpha^{-1}(R^{(2)})\cdot w_0 \otimes R_1^{(1)}\cdot \mu^{-1}(u_0)\big)\otimes \alpha(R_2^{(1)})\cdot \nu(v_0)\\
&\overset{(\mathrm{QT1})}{=} \langle \beta^{-1}(u_{(-1)})v_{(-1)}|\beta(w_{(-1)})\\
&\qquad \big(\alpha^{-1}(R^{(2)}r^{(2)})\cdot w_0 \otimes R^{(1)}\cdot \mu^{-1}(u_0)\big)\otimes \alpha(r^{(1)})\cdot \nu(v_0).
\end{aligned}
$$

Also we get

$$
\begin{aligned}
&(C_{U,W}\otimes \mathrm{Id}_V)\circ a^{-1}_{U,W,V}\circ (\mathrm{Id}_U\otimes C_{V,W})(u\otimes (v\otimes w))\\
&\quad= \langle v_{(-1)})|w_{(-1)}\rangle (C_{U,W}\otimes \mathrm{Id}_V)\circ a^{-1}_{U,W,V}(u\otimes (R^{(2)}\cdot w_0\otimes R^{(1)}\cdot w_0))\\
&\quad= \langle v_{(-1)})|w_{(-1)}\rangle (C_{U,W}\otimes \mathrm{Id}_V)((\mu^{-1}(u)\otimes R^{(2)}\cdot w_0)\otimes \omega(R^{(1)}\cdot w_0)))\\
&\quad= \langle v_{(-1)})|w_{(-1)}\rangle \langle \beta^{-1}(u_{(-1)})|\beta(w_{0(-1)})\rangle\\
&\qquad\quad r^{(2)}\cdot(\alpha^{-1}(R^{(2)})\cdot w_{00})\otimes r^{(1)}\cdot \mu^{-1}(u_0)\otimes \omega(R^{(1)}\cdot w_0)\\
&\quad= \langle v_{(-1)})|\beta(w_{(-1)1})\rangle \langle \beta^{-1}(u_{(-1)})|\beta(w_{(-1)2})\rangle\\
&\qquad\quad \alpha^{-1}(r^{(2)}R^{(2)})\cdot w_0)\otimes r^{(1)}\cdot \mu^{-1}(u_0)\otimes \alpha(R^{(1)})\cdot \omega(w_0)\\
&\quad\overset{(\mathrm{BR1})}{=} \langle \beta^{-1}(u_{(-1)})v_{(-1)}|\beta(w_{(-1)})\rangle \alpha^{-1}(r^{(2)}R^{(2)})\cdot w_0)\\
&\qquad\quad \otimes r^{(1)}\cdot \mu^{-1}(u_0)\otimes \alpha(R^{(1)})\cdot \omega(w_0).
\end{aligned}
$$

So the diagram (4.4) commutes since $r = R$. ∎

COROLLARY 4.5. *Under the hypotheses of Theorem* 4.1, *the braiding* C *is a solution of the quantum Yang–Baxter equation*

$$
\begin{aligned}
&(\mathrm{Id}_W\otimes C_{U,V})\circ a_{W,U,V}\circ (C_{U,W}\otimes \mathrm{Id}_V)\circ a^{-1}_{W,V,U}\circ (\mathrm{Id}_U\otimes C_{V,W})\circ a_{U,V,W}\\
&\quad = a_{W,V,U}\circ (C_{W,V}\otimes \mathrm{Id}_U)\circ a^{-1}_{W,V,U}\circ (\mathrm{Id}_V\otimes C_{U,W})\circ a_{V,U,W}\circ (C_{U,V}\otimes \mathrm{Id}_W).
\end{aligned}
$$

Proof. Straightforward. ∎

LEMMA 4.6. *Let* (H, R, α) *be a quasitriangular Hom-Hopf algebra and* $(B, \langle|\rangle, \beta)$ *a coquasitriangular Hom-Hopf algebra. Define a linear map*

$$(H\otimes B)\otimes M\to M,\qquad (h\otimes x)\cdot m = \langle x|m_{(-1)}\rangle h\cdot \mu(m_0),$$

for any $h\in H$, $x\in B$ *and* $m\in (M,\mu)\in {}^{B}_{H}\mathcal{L}$. *Then* (M,μ) *becomes a left* $(H\otimes B, \alpha\circ\beta)$*-Hom-module.*

Proof. It is sufficient to show that the Hom-module action defined above satisfies (2.7). For any $h, g\in H$, $x, y\in B$ and $m\in M$, we have

$$(1_H\otimes 1_B)\cdot m = \langle 1_B|m_{(-1)}\rangle 1_H\cdot \mu(m_0) = \varepsilon(m_{(-1)})\mu^2(m_0) = \mu(m_0).$$

That is, $(1_H\otimes 1_B)\cdot m = \mu(m_0)$. For the equality

$$\mu((h\otimes x)\cdot m) = ((\alpha\otimes\beta)(h\otimes x))\cdot \mu(m),$$

we have

$$
\begin{aligned}
((\alpha\otimes\beta)(h\otimes x))\cdot\mu(m) &= (\alpha(h)\otimes\beta(x))\cdot\mu(m)\\
&= \langle\beta(x)|\beta(m_{(-1)})\rangle\alpha(h)\cdot\mu^2(m_0) = \langle x|m_{(-1)}\rangle\alpha(h)\cdot\mu^2(m_0)\\
&= \langle x|m_{(-1)}\rangle\mu(h\cdot\mu(m_0)) = \mu((h\otimes x)\cdot m),
\end{aligned}
$$

as required. Finally, we check that

$$((h\otimes x)(g\otimes y))\cdot\mu(m) = (\alpha(h)\otimes\beta(x))\cdot((g\otimes y)\cdot m).$$

Indeed, on the one hand, we have

$$((h\otimes x)(g\otimes y))\cdot\mu(m) = (hg\otimes xy)\cdot\mu(m) = \langle xy|\beta(m_{(-1)})\rangle(hg)\cdot\mu^2(m_0).$$

On the other hand,

$$
\begin{aligned}
&(\alpha(h)\otimes\beta(x))\cdot((g\otimes y)\cdot m)\\
&\quad= \langle y|m_{(-1)}\rangle(\alpha(h)\otimes\beta(x))\cdot(g\cdot\mu(m_0))\\
&\quad= \langle y|m_{(-1)}\rangle\langle\beta(x)|\beta^2(m_{0(-1)})\rangle\alpha(h)\cdot\mu(\alpha^{-1}(g)\cdot\mu(m_{00}))\\
&\quad\overset{(2.13)}{=} \langle y|\beta(m_{(-1)1})\rangle\langle x|\beta(m_{(-1)2})\rangle\alpha(h)\cdot\mu(\alpha^{-1}(g)\cdot m_0)\\
&\quad\overset{(BR1)}{=} \langle xy|\beta(m_{(-1)})\rangle\alpha(h)\cdot(g\cdot\mu(m_0)) \overset{(2.7)}{=} \langle xy|\beta(m_{(-1)})\rangle(hg)\cdot\mu^2(m_0).
\end{aligned}
$$

So (M,μ) is a left $(H\otimes B)$-Hom-module. ∎

LEMMA 4.7. *Assume that (H,R,α) is a quasitriangular Hom-Hopf algebra and $(B,\langle|\rangle,\beta)$ is a coquasitriangular Hom-Hopf algebra. Define a linear map*

$$
\begin{aligned}
&\overline{\rho}: M\to(H\otimes B)\otimes M,\\
&\overline{\rho}(m) = m_{[-1]}\otimes m_{[0]} = \alpha(R^{(2)})\otimes\beta^{-1}(m_{(-1)})\otimes R^{(1)}\cdot\mu^{-1}(m_0),
\end{aligned}
$$

for any $m\in(M,\mu)$. Then (M,μ) becomes a left $(H\otimes B)$-Hom-comodule.

Proof. First we show that $\overline{\rho}$ satisfies (2.12). On the one hand,

$$
\begin{aligned}
&\Delta(m_{[-1]})\otimes\mu^{-1}(m_{[0]})\\
&\quad= \Delta(\alpha(R^{(2)})\otimes\beta^{-1}(m_{(-1)}))\otimes\mu^{-1}(R^{(1)}\cdot\mu^{-1}(m_0))\\
&\quad= (\alpha(R^{(2)}_1)\otimes\beta^{-1}(m_{(-1)1}))\\
&\qquad\otimes(\alpha(R^{(2)}_2)\otimes\beta^{-1}(m_{(-1)2}))\otimes\alpha^{-1}(R^{(1)})\cdot\mu^{-2}(m_0)\\
&\quad= (R^{(2)}_1\otimes\beta^{-1}(m_{(-1)1}))\otimes(R^{(2)}_2\otimes\beta^{-1}(m_{(-1)2}))\otimes\alpha^{-2}(R^{(1)})\cdot\mu^{-2}(m_0)\\
&\quad\overset{(QT2)}{=} (r^{(2)}\otimes\beta^{-1}(m_{(-1)1}))\otimes(R^{(2)}\otimes\beta^{-1}(m_{(-1)2}))\\
&\qquad\otimes\alpha^{-2}(R^{(1)}r^{(1)})\cdot\mu^{-2}(m_0).
\end{aligned}
$$

The third equality holds since R is α-invariant. On the other hand,

$$\begin{aligned}
&(\alpha^{-1}\otimes\beta^{-1})(m_{[-1]})\otimes\overline{\rho}(m_{[0]})\\
&\qquad= (\alpha^{-1}\otimes\beta^{-1})(\alpha(R^{(2)})\otimes\beta^{-1}(m_{(-1)}))\otimes\overline{\rho}(R^{(1)}\cdot\mu^{-1}(m_0))\\
&\qquad= (R^{(2)}\otimes\beta^{-2}(m_{(-1)}))\otimes(\alpha(r^{(2)})\otimes\beta^{-1}(m_{0(-1)}))\\
&\qquad\quad\otimes r^{(1)}\cdot\mu^{-1}(\alpha^{-1}(R^{(1)})\cdot\mu^{-1}(m_{00}))\\
&\qquad= (R^{(2)}\otimes\beta^{-2}(m_{(-1)}))\otimes(\alpha(r^{(2)})\otimes\beta^{-1}(m_{0(-1)}))\\
&\qquad\quad\otimes r^{(1)}\cdot(\alpha^{-2}(R^{(1)})\cdot\mu^{-2}(m_{00}))\\
&\qquad\overset{(2.13)}{=} (R^{(2)}\otimes\beta^{-1}(m_{(-1)1}))\otimes(r^{(2)}\otimes\beta^{-1}(m_{(-1)2}))\\
&\qquad\quad\otimes\alpha^{-1}(r^{(1)})\cdot(\alpha^{-2}(R^{(1)})\cdot\mu^{-3}(m_0))\\
&\qquad\overset{(2.7)}{=} (R^{(2)}\otimes\beta^{-1}(m_{(-1)1}))\otimes(r^{(2)}\otimes\beta^{-1}(m_{(-1)2}))\\
&\qquad\quad\otimes\alpha^{-2}(r^{(1)}R^{(1)})\cdot\mu^{-2}(m_0).
\end{aligned}$$

The fourth equality holds since R is α-invariant. Thus $\Delta(m_{[-1]})\otimes\mu^{-1}(m_{[0]}) = (\alpha^{-1}\otimes\beta^{-1})(m_{[-1]})\otimes\overline{\rho}(m_{[0]})$, as needed. For (2.13), we have

$$\begin{aligned}
(\varepsilon_H\otimes\varepsilon_B)(m_{[-1]})m_{[0]} &= (\varepsilon_H\otimes\varepsilon_B)(\alpha(R^{(2)})\otimes\beta^{-1}(m_{(-1)}))R^{(1)}\cdot\mu^{-1}(m_0)\\
&= \varepsilon_H(\alpha(R^{(2)}))\varepsilon_B(\beta^{-1}(m_{(-1)}))R^{(1)}\cdot\mu^{-1}(m_0)\\
&= \varepsilon_H(R^{(2)})\varepsilon_B(m_{(-1)})R^{(1)}\cdot\mu^{-1}(m_0)\\
&= 1_H\cdot\mu^{-2}(m) = \mu^{-1}(m),
\end{aligned}$$

$$\begin{aligned}
(\alpha\otimes\beta)(m_{[-1]})\otimes\mu(m_{[0]}) &= (\alpha\otimes\beta)(\alpha(R^{(2)})\otimes\beta^{-1}(m_{(-1)}))\otimes\mu(R^{(1)}\cdot\mu^{-1}(m_0))\\
&= (\alpha^2(R^{(2)})\otimes m_{(-1)})\otimes\alpha(R^{(1)})\cdot m_0,\\
\overline{\rho}(\mu(m)) &= \alpha(R^{(2)})\otimes\beta^{-1}(\beta(m_{(-1)}))\otimes R^{(1)}\cdot\mu^{-1}(\mu(m_0))\\
&= (\alpha^2(R^{(2)})\otimes m_{(-1)})\otimes\alpha(R^{(1)})\cdot m_0. \ \blacksquare
\end{aligned}$$

THEOREM 4.8. *Assume that (H,R,α) is a quasitriangular Hom-Hopf algebra and $(B,\langle|\rangle,\beta)$ is a coquasitriangular Hom-Hopf algebra. Then ${}^B_H\mathcal{L}$ is a monoidal subcategory of the Hom-Yetter–Drinfeld category ${}^{H\otimes B}_{H\otimes B}\mathcal{HYD}$.*

Proof. Let $m\in(M,\mu)\in{}^B_H\mathcal{L}$ and $h\in H$. Here we first note that $\rho(h\cdot\mu(m_0)) = \beta^2(m_{0(-1)})\otimes\alpha^{-1}(h)\cdot\mu(m_{00})$. It is sufficient to show that the left $(H\otimes B)$-Hom-module action in Lemma 4.6 and the left $(H\otimes B)$-Hom-comodule structure in Lemma 4.7 satisfy the compatibility condition (2.17). On the one hand, we have

$$\begin{aligned}
&\overline{\rho}((h\otimes x)\cdot m) = \langle x|m_{(-1)}\rangle\overline{\rho}(h\cdot\mu(m_0))\\
&\qquad= \langle x|m_{(-1)}\rangle\alpha(R^{(2)})\otimes\beta^{-1}(\beta^2(m_{0(-1)}))\otimes R^{(1)}\cdot\mu^{-1}(\alpha^{-1}(h)\cdot\mu(m_{00}))
\end{aligned}$$

$$
\begin{aligned}
&= \langle x|m_{(-1)}\rangle\alpha(R^{(2)})\otimes\beta(m_{0(-1)})\otimes R^{(1)}\cdot(\alpha^{-2}(h)\cdot m_{00})\\
&\overset{(2.7)}{=} \langle x|m_{(-1)}\rangle\alpha(R^{(2)})\otimes\beta(m_{0(-1)})\otimes(\alpha^{-1}(R^{(1)})\alpha^{-2}(h))\cdot\mu(m_{00})\\
&\overset{(2.12)}{=} \langle x|\beta(m_{(-1)1})\rangle\alpha(R^{(2)})\otimes\beta(m_{(-1)2})\otimes(\alpha^{-1}(R^{(1)})\alpha^{-2}(h))\cdot m_0.
\end{aligned}
$$

On the other hand, we get

$$
\begin{aligned}
&[(h\otimes x)_{11}(\alpha^{-1}\otimes\beta^{-1})(m_{[-1]})]((S_H\otimes S_B)(h\otimes x)_2)\\
&\qquad\otimes((\alpha\otimes\beta)(h\otimes x)_{12})\cdot m_{[0]}\\
&= (h_{11}\alpha^{-1}(\alpha(R^{(2)})))S_H(h_2)\otimes(x_{11}\beta^{-2}(m_{(-1)}))S_B(x_2)\\
&\qquad\otimes(\alpha(h_{12})\otimes\beta(x_{12}))\cdot(R^{(1)}\cdot\mu^{-1}(m_0))\\
&= (h_{11}R^{(2)})S_H(h_2)\otimes(x_{11}\beta^{-2}(m_{(-1)}))S_B(x_2)\\
&\qquad\otimes\langle\beta(x_{12})|m_{0(-1)}\rangle\alpha(h_{12})\cdot\mu(\alpha^{-1}(R^{(1)})\cdot\mu^{-1}(m_{00}))\\
&\overset{(2.12)}{=} (h_{11}R^{(2)})S_H(h_2)\otimes(x_{11}\beta^{-1}(m_{(-1)1}))S_B(x_2)\\
&\qquad\otimes\langle\beta(x_{12})|m_{(-1)2}\rangle\alpha(h_{12})\cdot(R^{(1)}\cdot\mu^{-1}(m_0))\\
&\overset{(2.7)}{=} (h_{11}R^{(2)})S_H(h_2)\otimes(x_{11}\beta^{-1}(m_{(-1)1}))S_B(x_2)\\
&\qquad\otimes\langle x_{12}|\beta^{-1}(m_{(-1)2})\rangle(h_{12}R^{(1)})\cdot m_0\\
&\overset{(\mathrm{QT4,BR3})}{=} (R^{(2)}h_{12})S_H(h_2)\otimes(\beta^{-1}(m_{(-1)2})x_{12})S_B(x_2)\\
&\qquad\otimes\langle x_{11}|\beta^{-1}(m_{(-1)1})\rangle(R^{(1)}h_{11})\cdot m_0\\
&\overset{(2.7)}{=} \alpha(R^{(2)})(h_{12}S_H(\alpha^{-1}(h_2)))\otimes m_{(-1)2}(x_{12}S_B(\beta^{-1}(x_2)))\\
&\qquad\otimes\langle x_{11}|\beta^{-1}(m_{(-1)1})\rangle(R^{(1)}h_{11})\cdot m_0\\
&\overset{(2.9)}{=} \alpha(R^{(2)})(h_{21}S_H(h_{22}))\otimes m_{(-1)2}(x_{21}S_B(x_{22}))\\
&\qquad\otimes\langle\beta^{-1}(x_1)|\beta^{-1}(m_{(-1)1})\rangle(R^{(1)}\alpha^{-1}(h_1))\cdot m_0\\
&= \alpha(R^{(2)})(\varepsilon(h_2)1_H)\otimes m_{(-1)2}(\varepsilon(x_2)1_B)\\
&\qquad\otimes\langle x_1|m_{(-1)1}\rangle(R^{(1)}\alpha^{-1}(h_1))\cdot m_0\\
&= \langle\beta^{-1}(x)|m_{(-1)1}\rangle\alpha^2(R^{(2)})\otimes\beta(m_{(-1)2})\otimes(R^{(1)}\alpha^{-2}(h))\cdot m_0\\
&= \langle x|\beta(m_{(-1)1})\rangle\alpha(R^{(2)})\otimes\beta(m_{(-1)2})\otimes(\alpha^{-1}(R^{(1)})\alpha^{-2}(h))\cdot m_0.
\end{aligned}
$$

It follows that $(M,\mu)\in {}^{H\otimes B}_{H\otimes B}\mathcal{HYD}$. ■

Proposition 4.9. *Under the hypotheses of Theorem 4.8, ${}^{B}_{H}\mathcal{L}$ is a braided monoidal subcategory of ${}^{H\otimes B}_{H\otimes B}\mathcal{HYD}$.*

Proof. It is sufficient to show that the braiding in ${}^{B}_{H}\mathcal{L}$ is compatible with the braiding in ${}^{H\otimes B}_{H\otimes B}\mathcal{HYD}$. In fact, for any $m\in(M,\mu)$ and $n\in(N,\nu)$, we have

$$\begin{aligned}C_{M\otimes N}(m\otimes n) &= m_{[-1]}\cdot\nu^{-1}(n)\otimes\mu(m_{[0]})\\ &= (\alpha(R^{(2)})\otimes\beta^{-1}(m_{(-1)}))\cdot\nu^{-1}(n)\otimes\mu(R^{(1)}\cdot\mu^{-1}(m_0))\\ &= \langle\beta^{-1}(m_{(-1)})|\beta^{-1}(n_{(-1)})\rangle\alpha(R^{(2)})\cdot\nu(\nu^{-1}(n_0))\otimes\alpha(R^{(1)})\cdot m_0\\ &= \langle m_{(-1)}|n_{(-1)}\rangle R^{(2)}\cdot n_0\otimes R^{(1)}\cdot m_0,\end{aligned}$$

as desired. ■

5. Symmetries in generalized Hom-Long dimodule categories. In this section, we present a sufficient condition for a generalized Hom-Long dimodule category ${}^{B}_{H}\mathcal{L}$ to be symmetric.

PROPOSITION 5.1. *Assume that (H,R,α) is a triangular Hom-Hopf algebra and (B,β) is a monoidal Hom-Hopf algebra. Then the category $\widetilde{\mathcal{H}}({}_H\mathcal{M})$ of left (H,α)-Hom-modules is a symmetric subcategory of ${}^{B}_{H}\mathcal{L}$ under the left (B,β)-comodule structure $\rho(m)=1_B\otimes\mu^{-1}(m)$, where $m\in(M,\mu)\in\widetilde{\mathcal{H}}({}_H\mathcal{M})$, and the braiding is defined as*

$$C_{M,N}: M\otimes N\to N\otimes M,\qquad m\otimes n\mapsto R^{(2)}\cdot n\otimes R^{(1)}\cdot m,$$

for all $m\in(M,\mu)\in\widetilde{\mathcal{H}}({}_H\mathcal{M})$ and $n\in(N,\nu)\in\widetilde{\mathcal{H}}({}_H\mathcal{M})$.

Proof. It is clear that (M,ρ,μ) is a left (B,β)-Hom-comodule under the left (B,β)-comodule structure given above. Now we check that the left (B,β)-comodule structure satisfies the compatibility condition (3.2). For this purpose, let $h\in H$ and $m\in(M,\mu)\in\widetilde{\mathcal{H}}({}_H\mathcal{M})$. Then

$$\rho(h\cdot m)=1_B\otimes\alpha^{-1}(h)\cdot\mu^{-1}(m)=\beta(m_{(-1)})\otimes\alpha^{-1}(h)\cdot m_0.$$

So, (3.2) holds. That is, (M,ρ,μ) is a generalized Hom-Long dimodule.

Next we verify that any morphism in $\widetilde{\mathcal{H}}({}_H\mathcal{M})$ is also left (B,β)-colinear. Indeed, take any $m\in(M,\mu)\in\widetilde{\mathcal{H}}({}_H\mathcal{M})$ and $n\in(N,\nu)\in\widetilde{\mathcal{H}}({}_H\mathcal{M})$. If $f:(M,\mu)\to(N,\nu)$ is a morphism in $\widetilde{\mathcal{H}}({}_H\mathcal{M})$, then

$$(\mathrm{Id}_H\otimes f)\rho(m)=1_B\otimes f(\mu^{-1}(m))=1_B\otimes\nu^{-1}(f(m))=\rho(f(m)).$$

So f is left (B,β)-colinear, as desired. Therefore, $\widetilde{\mathcal{H}}({}_H\mathcal{M})$ is a subcategory of ${}^{B}_{H}\mathcal{L}$.

Finally, we prove that $\widetilde{\mathcal{H}}({}_H\mathcal{M})$ is a symmetric subcategory of ${}^{B}_{H}\mathcal{L}$. From [30, Proposition 5.2], $C_{M,N}(m\otimes n)=R^{(2)}\cdot n\otimes R^{(1)}\cdot m$ for all $m\in(M,\mu)\in\widetilde{\mathcal{H}}({}_H\mathcal{M})$ and $n\in(N,\nu)\in\widetilde{\mathcal{H}}({}_H\mathcal{M})$. Thus, we have

$$\begin{aligned}C_{N,M}\circ C_{M,N}(m\otimes n) &= C_{N,M}(R^{(2)}\cdot n\otimes R^{(1)}\cdot m)\\ &= r^{(2)}\cdot(R^{(1)}\cdot m)\otimes r^{(1)}\cdot(R^{(2)}\cdot n)\\ &= (\alpha^{-1}(r^{(2)})R^{(1)})\cdot\mu(m)\otimes(\alpha^{-1}(r^{(1)})R^{(2)})\cdot\nu(n)\\ &= (r^{(2)}R^{(1)})\cdot\mu(m)\otimes(r^{(1)}R^{(2)})\cdot\nu(n)=1_H\cdot\mu(m)\otimes 1_H\cdot\nu(n)=m\otimes n,\end{aligned}$$

where the fourth equality holds since R is α-invariant. It follows that the braiding $C_{M,N}$ is symmetric. ■

PROPOSITION 5.2. *Assume that $(B,\langle|\rangle,\beta)$ is a cotriangular Hom-Hopf algebra and (H,α) is a monoidal Hom-Hopf algebra. Then the category $\widetilde{\mathcal{H}}({}^B\mathcal{M})$ of left (B,β)-Hom-comodules is a symmetric subcategory of ${}^B_H\mathcal{L}$ under the left (H,α)-module action $h\cdot m=\varepsilon(h)\mu(m)$, where $h\in H$, $m\in(M,\mu)\in\widetilde{\mathcal{H}}({}^B\mathcal{M})$, and the braiding is given by*

$$C_{M,N}:M\otimes N\to N\otimes M,\qquad m\otimes n\mapsto\langle m_{(-1)}|n_{(-1)}\rangle\nu(n_0)\otimes\mu(m_0),$$

for all $m\in(M,\mu)\in\widetilde{\mathcal{H}}({}^B\mathcal{M})$ and $n\in(N,\nu)\in\widetilde{\mathcal{H}}({}^B\mathcal{M})$.

Proof. We have to show that the left (H,α)-module action defined above forces (M,μ) to be a left (H,α)-module, but this is easy to check. For the compatibility condition (3.2), we take $h\in H$ and $m\in(M,\mu)\in\widetilde{\mathcal{H}}({}^B\mathcal{M})$ and calculate as follows:

$$\begin{aligned}\rho(h\cdot m)&=\varepsilon(h)\beta(m_{(-1)})\otimes\mu(m_0)=\varepsilon(\alpha^{-1}(h))\beta(m_{(-1)})\otimes\mu(m_0)\\&=\beta(m_{(-1)})\otimes\alpha^{-1}(h)\cdot m_0.\end{aligned}$$

So, (3.2) holds, as required. Therefore, (M,ρ,μ) is a generalized Hom-Long dimodule.

Now we verify that any morphism in $\widetilde{\mathcal{H}}({}^B\mathcal{M})$ is left (H,α)-linear. Indeed, pick any $m\in(M,\mu)\in\widetilde{\mathcal{H}}({}^B\mathcal{M})$ and $n\in(N,\nu)\in\widetilde{\mathcal{H}}({}^B\mathcal{M})$. If $f:(M,\mu)\to(N,\nu)$ is a morphism in $\widetilde{\mathcal{H}}({}^B\mathcal{M})$, then

$$f(h\cdot m)=f(\varepsilon(h)\mu(m))=\varepsilon(h)\mu(f(m))=h\cdot f(m).$$

So f is left (H,α)-linear, as desired. Therefore, $\widetilde{\mathcal{H}}({}^B\mathcal{M})$ is a subcategory of ${}^B_H\mathcal{L}$.

Finally, we show that $\widetilde{\mathcal{H}}({}^B\mathcal{M})$ is a symmetric subcategory of ${}^B_H\mathcal{L}$. From [30, Proposition 6.2], $C_{M,N}(m\otimes n)=\langle m_{(-1)}|n_{(-1)}\rangle\nu(n_0)\otimes\mu(m_0)$ for all $m\in(M,\mu)\in\widetilde{\mathcal{H}}({}^B\mathcal{M})$ and $n\in(N,\nu)\in\widetilde{\mathcal{H}}({}^B\mathcal{M})$. Thus, we have

$$\begin{aligned}C_{N,M}\circ C_{M,N}(m\otimes n)&=\langle m_{(-1)}|n_{(-1)}\rangle C_{N,M}(\nu(n_0)\otimes\mu(m_0))\\&=\langle m_{(-1)}|n_{(-1)}\rangle\langle n_{0(-1)}|m_{0(-1)}\rangle(\mu^2(m_{00})\otimes\nu^2(n_{00})\\&=\langle\beta(m_{(-1)1})|\beta(n_{(-1)1})\rangle\langle n_{(-1)2}|m_{(-1)2}\rangle\mu(m_0)\otimes\nu(n_0)\\&=\langle m_{(-1)1}|n_{(-1)1}\rangle\langle n_{(-1)2}|m_{(-1)2}\rangle\mu(m_0)\otimes\nu(n_0)\\&=\varepsilon(m_{(-1)})\varepsilon(n_{(-1)})\mu(m_0)\otimes\nu(n_0)=m\otimes n,\end{aligned}$$

where the fourth equality holds since $\langle|\rangle$ is α-invariant. It follows that the braiding $C_{M,N}$ is symmetric. ■

THEOREM 5.3. *If (H,α) is a triangular Hom-Hopf algebra and $(B,\langle|\rangle,\beta)$ is a cotriangular Hom-Hopf algebra, then the category ${}^B_H\mathcal{L}$ is symmetric.*

Proof. For any $m \in (M,\mu) \in {}^B_H\mathcal{L}$ and $n \in (N,\nu) \in {}^B_H\mathcal{L}$, we have

$$\begin{aligned}
C_{N,M} \circ C_{M,N}(m \otimes n) &= \langle m_{(-1)} | n_{(-1)} \rangle C_{N,M}(R^{(2)} \cdot n_0 \otimes R^{(1)} \cdot m_0) \\
&= \langle m_{(-1)} | n_{(-1)} \rangle \langle \beta(n_{0(-1)}) | \beta(m_{0(-1)}) \rangle \\
&\qquad r^{(2)} \cdot (\alpha^{-1}(R^{(1)}) \cdot m_{00}) \otimes r^{(1)} \cdot (\alpha^{-1}(R^{(2)}) \cdot n_{00}) \\
&= \langle \beta(m_{(-1)1}) | \beta(n_{(-1)1}) \rangle \langle \beta(n_{(-1)2}) | \beta(m_{(-1)2}) \rangle \\
&\qquad \alpha^{-1}(r^{(2)} R^{(1)}) \cdot m_0 \otimes \alpha^{-1}(r^{(1)} R^{(2)}) \cdot n_0 \\
&= \varepsilon(m_{(-1)}) \varepsilon(n_{(-1)}) 1_H \cdot m_0 \otimes 1_H \cdot n_0 \\
&= \varepsilon(m_{(-1)}) \varepsilon(n_{(-1)}) \mu(m_0) \otimes \nu(n_0) \\
&= m \otimes n,
\end{aligned}$$

as desired. ∎

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