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# Inverses of disjointness preserving operators

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**Abstract.** A linear operator between (possibly vector-valued) function spaces is disjointness preserving if it maps disjoint functions to disjoint functions. Here, two functions are said to be disjoint if at each point at least one of them vanishes. In this paper, we study linear disjointness preserving operators between various types of function spaces, including spaces of (little) Lipschitz functions, uniformly continuous functions and differentiable functions. It is shown that a disjointness preserving linear isomorphism whose domain is one of these types of spaces (scalar-valued) has a disjointness preserving inverse, subject to some topological conditions on the range space. A representation for a general linear disjointness preserving operator on a space of vector-valued $C^p$ functions is also given.

**1. Introduction.** Let $X, Y$ be Hausdorff topological spaces and let $E, F$ be Banach spaces. Throughout, we consider real vector spaces, although some of our results can readily be extended to the case of complex scalars. Suppose that $A(X,E)$ and $A(Y,F)$ are vector subspaces of the spaces of continuous functions $C(X,E)$ and $C(Y,F)$ respectively. Two functions $f$ and $g$ in, say, $A(X,E)$ are *disjoint* if for each $x \in X$, either $f(x) = 0$ or $g(x) = 0$. An operator $T : A(X,E) \to A(Y,F)$ is said to be *disjointness preserving* if $T$ maps disjoint functions to disjoint functions. In case $E = F = \mathbb{R}$, we simply write $A(X,E) = A(X)$ and $A(Y,F) = A(Y)$.

In the study of linear disjointness preserving operators, two questions naturally arise: representation of such operators and whether the inverse of a disjointness preserving linear isomorphism must also be disjointness preserving. The two questions are related. In fact, a linear isomorphism $T$ such that $T$ and $T^{-1}$ are both disjointness preserving is said to be *biseparating*. Biseparating operators have been well investigated; representation of bisep-

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arating operators (as weighted composition operators) and their automatic continuity are known for many (vector-valued) function spaces (see, e.g., [A1, A2, DL, FH, HBN]). Thus, for a linear disjointness preserving isomorphism $T$, knowing that $T^{-1}$ is also disjointness preserving leads immediately to a concrete representation (and hence a thorough understanding) of $T$ in many instances.

In [J], Jarosz gave a complete analysis of linear disjointness preserving operators $T : C(X) \to C(Y)$, where $X$ and $Y$ are compact Hausdorff spaces. In particular, he proved that a disjointness preserving linear isomorphsim $T : C(X) \to C(Y)$ must have a disjointness preserving inverse.

In the general context of vector lattices $E$ and $F$, Abramovich asked whether for every disjointness linear bijection $T : E \to F$, $T^{-1}$ must preserve disjointness as well (see [HL]). Huijsmans and de Pagter [HD2] and Koldunov [K] independently answered the question in the affirmative when $E$ is a uniformly complete vector lattice and $F$ is a normed vector lattice. The latter also gave examples to show that the result fails if $E$ is not uniformly complete or $F$ is not a normed lattice. For a thorough treatment of disjointness preserving operators in the vector lattice setting, refer to [AK1, AK2].

In [JV], Jiménez-Vargas studied linear disjointness preserving operators $T : \mathrm{lip}_\alpha(X) \to \mathrm{lip}_\alpha(Y)$, where $X, Y$ are compact metric spaces and $\alpha \in (0, 1)$. His methods and results were analogous to Jarosz's. Similar results for linear disjointness preserving operators between regular Banach function algebras which satisfy Ditkin's condition and BSE Ditkin algebras respectively were obtained by Font [F1, F2].

When both $X$ and $Y$ are completely regular Hausdorff spaces, Araujo, Beckenstein and Narici [ABN] showed that a disjointness preserving linear isomorphism $T : C(X) \to C(Y)$ is biseparating if $Y$ is connected. As far as we know, this is the only instance where the connectedness of the underlying space $Y$ has been brought into play in connection with disjointness preserving linear isomorphisms.

Our results are in a similar spirit in this respect. Specifically, some of our main results show that if $T : A(X) \to A(Y)$ is a disjointness preserving linear isomorphism, where $A(X)$ may be certain spaces of (little) Lipschitz functions or differentiable functions, then $T^{-1}$ is disjointness preserving provided that $Y$ has few connected components (see Theorems 4.7 and 6.6). In the course of the investigation, we also obtain representations of linear disjointness preserving operators defined on certain spaces of vector-valued $C^p$ functions (see Theorems 5.6 and 6.3). In §3, we give a slight generalization of the result of Jarosz mentioned above to noncompact spaces $X$ and $Y$. Our results can be compared with those of [ABN, FH, J, JV].

We now set some notation and terminology. In particular, we recall the notion and the construction of the support map of a linear disjointness preserving operator. Let $X$ be a Hausdorff topological space and let $A(X)$ be a vector subspace of $C(X)$ that separates points from closed sets. Denote by $\mathbb{R}_\infty$ the 1-point compactification of $\mathbb{R}$. The map $i : X \to \mathbb{R}_\infty^{A(X)}$ defined by $i(x) = (f(x))_{f\in A(X)}$ is a homeomorphic embedding. We denote the closure of $i(X)$ in $\mathbb{R}_\infty^{A(X)}$ by $\mathcal{A}X$. Identify $X$ with its homeomorphic copy $i(X)$. Then $\mathcal{A}X$ is a compact Hausdorff space that contains $X$ as a dense subspace. Furthermore, every $f \in A(X)$ has a unique continuous extension $\hat{f}$ onto $\mathcal{A}X$ given by $\hat{f}(x) = x_f$ for every $x = (x_f)_{f\in A(X)} \in \mathcal{A}X$.

We say that $A(X)$ is *closed under $C_*^\infty$ operations* if for any $n \in \mathbb{N}$, any $C^\infty$ function $\varphi : \mathbb{R}^n \to \mathbb{R}$ such that $\sup_{t\in\mathbb{R}^n} |\partial^\xi \varphi(t)| < \infty$ for any multiindex $\xi$, and any set of functions $f_1, \ldots, f_n \in A(X)$, the function

$$x \mapsto \varphi(f_1(x), \ldots, f_n(x)) \text{ belongs to } A(X).$$

Note that if $A(X)$ is closed under $C_*^\infty$ operations, then it contains all real-valued constant functions on $X$. The usefulness of being closed under $C_*^\infty$ operations lies in the following results.

Proposition 1.1 ([LW, Proposition 3]). *Let $X$ be a Hausdorff topological space and let $A(X)$ be a subspace of $C(X)$ that separates points from closed sets and is closed under $C_*^\infty$ operations. If $P$ and $Q$ are subsets of $X$ such that $\overline{P}^{\mathcal{A}X} \cap \overline{Q}^{\mathcal{A}X} = \emptyset$, then there exists $f \in A(X)$ with $f(P) \subseteq \{0\}$ and $f(Q) \subseteq \{1\}$.*

Here $\overline{\cdot}^{\mathcal{A}X}$ denotes the closure operation in $\mathcal{A}X$. Assume that $Y$ is a Hausdorff topological space and let $A(Y)$ be a vector subspace of $C(Y)$. Suppose that $T : A(X) \to A(Y)$ is a linear disjointness preserving mapping. For $f \in A(X)$, let $C(f) = \{x \in X : f(x) \neq 0\}$ and $\overline{C}(f) = \overline{C(f)}^{\mathcal{A}X}$.

Proposition 1.2 ([LW, Proposition 4]). *Let $X$ and $Y$ be Hausdorff topological spaces and let $A(X)$ and $A(Y)$ be vector subspaces of $C(X)$ and $C(Y)$ respectively. Assume that $A(X)$ separates points from closed sets of $X$ and is closed under $C_*^\infty$ operations. Define*

$$Y_s = \{y \in Y : \text{there exists } f \in A(X) \text{ such that } Tf(y) \neq 0\}.$$

*Then there is a continuous function $\beta : Y_s \to \mathcal{A}X$ such that if $f \in A(X)$ and $\beta(y) \notin \overline{C}(f)$, then $Tf(y) = 0$.*

The map $\beta$ is usually referred to as the "support map" of the disjointness preserving operator $T$ (see, e.g., [BNT]). It is of fundamental importance to our subsequent considerations.

## 2. A sufficient condition for $T^{-1}$ to be disjointness preserving.

From here on, assume that $A(X)$ and $A(Y)$ are as in Proposition 1.2 unless otherwise stated. Also assume that $A(Y)$ contains all real-valued constant functions on $Y$.

PROPOSITION 2.1. *Let $T : A(X) \to A(Y)$ be a linear disjointness preserving operator. If $T$ is onto, then $Y_s = Y$. If $T$ is one-to-one, then $\beta(Y_s)$ is dense in $\mathcal{A}X$.*

*Proof.* If $T$ is onto, there is a function $f \in A(X)$ such that $Tf$ is the constant function 1 on $Y$. Clearly, it follows that $Y_s = Y$.

Suppose that $\beta(Y_s)$ is not dense in $\mathcal{A}X$. There exists a nonempty open set $U$ in $\mathcal{A}X$ such that $\beta(Y_s) \cap U = \emptyset$. Choose a nonempty open set $V$ in $\mathcal{A}X$ such that $\overline{V}^{\mathcal{A}X} \subseteq U$. Since $X$ is dense in $\mathcal{A}X$, there exists $x_0 \in V \cap X$. Set $P = X \setminus \overline{V}^{\mathcal{A}X}$ and $Q = \{x_0\}$. Then $\overline{P}^{\mathcal{A}X} \subseteq \mathcal{A}X \setminus V$ and $\overline{Q}^{\mathcal{A}X} = \{x_0\}$. Hence $\overline{P}^{\mathcal{A}X} \cap \overline{Q}^{\mathcal{A}X} = \emptyset$. By Proposition 1.1, there exists $f \in A(X)$ such that $f(P) \subseteq \{0\}$ and $f(x_0) = 1$. Observe that $\mathcal{A}X \setminus \overline{V}^{\mathcal{A}X}$ is an open set in $\mathcal{A}X$ and hence $P = X \cap (\mathcal{A}X \setminus \overline{V}^{\mathcal{A}X})$ is dense in $\mathcal{A}X \setminus \overline{V}^{\mathcal{A}X}$. By continuity, $\hat{f}(\mathcal{A}X \setminus \overline{V}^{\mathcal{A}X}) \subseteq \{0\}$.

Consider any $y \in Y$. If $y \notin Y_s$, then $Tf(y) = 0$. On the other hand, if $y \in Y_s$, then $\beta(y) \notin U$ and so $\beta(y) \in \mathcal{A}X \setminus \overline{V}^{\mathcal{A}X}$. Thus the latter set in an open neighborhood of $\beta(y)$ in $\mathcal{A}X$ on which $\hat{f}$ vanishes. Therefore, $\beta(y) \notin \overline{C}(f)$. By Proposition 1.2, $Tf(y) = 0$. This shows that $Tf$ is the constant function 0. However, $f \neq 0$. Hence $T$ is not one-to-one. ∎

The difficulty in working with the support map is that it is neighborhood determined: in order to conclude that $Tf(y) = 0$, one needs to know that $\hat{f}$ vanishes on a neighborhood of $\beta(y)$ in $\mathcal{A}X$. The set of points where $T$ is point determined exhibits much simpler behavior. Let $T : A(X) \to A(Y)$ be a linear disjointness preserving operator with support map $\beta : Y_s \to Y$. Define $Y_p$ to be the set of all points $y \in Y_s$ such that $\hat{f}(\beta(y)) = 0 \Rightarrow Tf(y) = 0$ for any $f \in A(X)$.

LEMMA 2.2. *Suppose that $y \in Y_p$. Then there exists $h(y) \in \mathbb{R}$ such that $Tf(y) = h(y)\hat{f}(\beta(y))$ for any $f \in A(X)$ such that $\hat{f}(\beta(y)) \in \mathbb{R}$.*

*Proof.* Define $h(y) = T1(y)$ for all $y \in Y_p$. Let $f \in A(X)$ be such that $\hat{f}(\beta(y)) \in \mathbb{R}$. Set $g = f - \hat{f}(\beta(y))1 \in A(X)$. Since $\hat{g}(\beta(y)) = 0$ and $y \in Y_p$, $Tg(y) = 0$. Thus $Tf(y) = h(y)\hat{f}(\beta(y))$. ∎

PROPOSITION 2.3. *Let $T : A(X) \to A(Y)$ be a disjointness preserving linear isomorphism. Suppose that $\beta(Y_p)$ is dense in $\mathcal{A}X$. Let $h : Y_p \to \mathbb{R}$ be the function given by Lemma 2.2. Then for any nonempty open set $V$ in $\mathcal{A}X$, there exists $y \in Y_p$ such that $\beta(y) \in V$ and $h(y) \neq 0$.*

*Proof.* Otherwise, there exists a nonempty open set $V$ in $\mathcal{A}X$ such that $h(y) = 0$ for all $y \in Y_p$ with $\beta(y) \in V$. Let $f \in A(X)$ be such that $Tf = 1$. Suppose that $y \in Y_p$ with $\beta(y) \in V$. If $\hat{f}(\beta(y)) \in \mathbb{R}$, then $Tf(y) = h(y)\hat{f}(\beta(y)) = 0$, which is absurd. Thus $\hat{f}(x) = \infty$ for all $x \in \beta(Y_p) \cap V$. Since $\beta(Y_p) \cap V$ is dense in $V$, by continuity of $\hat{f}$, $\hat{f}(x) = \infty$ for all $x \in V$. But this is impossible since $V \cap X \neq \emptyset$. ∎

The next result gives a sufficient condition for $T^{-1}$ to be disjointness preserving.

PROPOSITION 2.4. *Let $T : A(X) \to A(Y)$ be a linear disjointness preserving bijection. If $\beta(Y_p)$ is dense in $\mathcal{A}X$, then $T^{-1}$ is disjointness preserving.*

*Proof.* Let $f, g \in A(X)$ be such that $Tf \cdot Tg = 0$. Assume, if possible, that there exists $x_0 \in X$ with $f(x_0)g(x_0) \neq 0$. Set

$$V = \{x \in \mathcal{A}X : |\hat{f}(x) - f(x_0)| < |f(x_0)|,\ |\hat{g}(x) - g(x_0)| < |g(x_0)|\}.$$

Then $V$ is an open neighborhood of $x_0$ in $\mathcal{A}X$. Let $h : Y_p \to \mathbb{R}$ be the function given by Lemma 2.2. By Proposition 2.3, there exists $y \in Y_p$ such that $\beta(y) \in V$ and $h(y) \neq 0$. Since $\hat{f}(\beta(y)), \hat{g}(\beta(y)) \in \mathbb{R}$,

$$0 = Tf(y) \cdot Tg(y) = h(y)^2 \hat{f}(\beta(y))\hat{g}(\beta(y)).$$

Thus $\hat{f}(\beta(y)) \cdot \hat{g}(\beta(y)) = 0$. But this is impossible since $\beta(y) \in V$ implies that neither $\hat{f}(\beta(y))$ nor $\hat{g}(\beta(y))$ is 0. ∎

**3. Uniformly closed subspaces of $C(X)$.** In this and the subsequent sections, we apply Proposition 2.4 to prove that a linear disjointness preserving isomorphism has a disjointness preserving inverse in various situations. A vector subspace $A(X)$ of $C(X)$ is said to be *uniformly closed* if for any sequence $(f_n)$ in $A(X)$ that converges uniformly on $X$ to a function $f$, that is, $\lim_{n\to\infty} \sup_{x\in X} |f_n(x) - f(x)| = 0$, we have $f \in A(X)$. For a topological space $Y$, denote by $C_b(Y)$ the space of bounded real-valued continuous functions.

The next theorem is a generalization in one respect of a result of Jarosz [J]. The proof uses essentially the same ideas.

THEOREM 3.1. *Let $X$ be a Hausdorff topological space and let $A(X)$ be a uniformly closed subspace of $C(X)$ that separates points from closed sets and is closed under $C^\infty_*$ operations. Assume that $A(Y)$ is a vector subspace of $C_b(Y)$ that contains constants, where $Y$ is a Hausdorff topological space. Suppose that $T : A(X) \to A(Y)$ is a linear disjointness preserving bijection. Then $T^{-1}$ is disjointness preserving.*

*Proof.* By Proposition 2.4, it suffices to show that $\beta(Y_p)$ is dense in $\mathcal{A}X$. By Proposition 2.1, $Y = Y_s$. First we show that if $y_0 \in Y = Y_s$ and $x_0 = \beta(y_0)$ is an isolated point in $\mathcal{A}X$, then $y_0 \in Y_p$. Indeed, in this case, since $X$

is dense in $\mathcal{A}X$, $x_0$ must be a point in $X$. Suppose that $f \in A(X)$ and that $f(x_0) = 0$. Then $f = 0$ on the open neighborhood $\{x_0\}$ of $\beta(y_0)$ in $\mathcal{A}X$. By definition of $\beta$, $Tf(y_0) = 0$. This proves that $y_0 \in Y_p$.

Next, we claim that $\beta(Y) \setminus \beta(Y_p)$ is a finite set. Suppose on the contrary that there is an infinite sequence $(x_n)$ of distinct points in $\beta(Y) \setminus \beta(Y_p)$. Let $x_n = \beta(y_n)$ for some $y_n \in Y \setminus Y_p$. For each $n$, there exists $f_n \in A(X)$ such that $\hat{f}_n(x_n) = 0$ and $Tf_n(y_n) > n$. By using a subsequence if necessary, we may assume that there are open sets $U_n$ and $V_n$ in $\mathcal{A}X$ such that

$$x_n \in V_n \subseteq \overline{V_n}^{\mathcal{A}X} \subseteq U_n, \qquad \overline{U_n}^{\mathcal{A}X} \cap \overline{U_m}^{\mathcal{A}X} = \emptyset \quad \text{if } n \neq m$$

and that $|\hat{f}_n| \leq 1/n$ on $U_n$. Let $P_n = X \setminus \overline{U_n}^{\mathcal{A}X}$ and $Q_n = V_n \cap X$. Then $\overline{P_n}^{\mathcal{A}X} \cap \overline{Q_n}^{\mathcal{A}X} = \emptyset$. By Proposition 1.1, there exists $\varphi_n \in A(X)$ such that $\varphi_n = 0$ on $P_n$ and $\varphi_n = 1$ on $Q_n$. If necessary, replace $\varphi_n$ by the composition $\Phi \circ \varphi_n$ for a suitable $C^\infty$ function $\Phi$ with bounded derivatives to guarantee additionally that $\|\varphi_n\|_\infty \leq 1$. Let $\Psi : \mathbb{R}^2 \to \mathbb{R}$ be a $C^\infty$ function with bounded derivatives such that

$$\Psi(s,t) = st \quad \text{if } |s|, |t| \leq 1 \quad \text{and} \quad \Psi(0,t) = 0 \quad \text{for all } t \in \mathbb{R}.$$

Set $g_n(x) = \Psi(\varphi_n(x), f_n(x))$ for all $x \in X$. Then $g_n \in A(X)$. If $x \in U_n \cap X$, then $|\varphi_n(x)| \leq 1$ and $|f_n(x)| \leq 1/n$. Hence $|g_n(x)| = |\varphi_n(x) f_n(x)| \leq 1/n$. By continuity, $|\hat{g}_n| \leq 1/n$ on $\overline{U_n}^{\mathcal{A}X}$. On the other hand, if $x \in X \setminus \overline{U_n}^{\mathcal{A}X}$, then $\varphi_n(x) = 0$ and hence $g_n(x) = 0$. By continuity, $\hat{g}_n = 0$ on $\mathcal{A}X \setminus \overline{U_n}^{\mathcal{A}X}$. Thus $\|g_n\|_\infty \leq 1/n$ and $(g_n)$ is a pairwise disjoint sequence. Since $A(X)$ is a uniformly closed subspace of $C(X)$, the sum $g = \sum g_n$ belongs to $A(X)$. If $x \in V_n \cap X$, then $\varphi_n(x) = 1$ and $|f_n(x)| \leq 1$. Hence $g(x) = g_n(x) = \varphi_n(x) f_n(x) = f_n(x)$. By continuity, $\widehat{g - f_n} = 0$ on $V_n$. By definition of $\beta$, $T(g - f_n)(y_n) = 0$; hence $Tg(y_n) = Tf_n(y_n) > n$ for all $n$. This contradicts the fact that $Tg$ is a bounded function. Thus the claimed is proved.

The two paragraphs above show that $\beta(Y) \setminus \beta(Y_p)$ is a finite set of non-isolated points in $\mathcal{A}X$. By Proposition 2.1, $\beta(Y) = \beta(Y_p) \cup [\beta(Y) \setminus \beta(Y_p)]$ is dense in $\mathcal{A}X$. It follows that $\beta(Y_p)$ is dense in $\mathcal{A}X$. By Proposition 2.4, $T^{-1}$ is disjointness preserving. ∎

Corollary 3.2 (Jarosz [J]). *Let $X$ and $Y$ be compact Hausdorff spaces. If $T : C(X) \to C(Y)$ is a linear disjointness preserving bijection, then $T^{-1}$ is disjointness preserving.*

Corollary 3.3. *Let $X$ be a metric space and let $U(X)$ be the space of uniformly continuous real functions on $X$. Let $Y$ be a Hausdorff topological space and let $A(Y)$ be a vector subspace of $C_b(Y)$ that contains constants. If $T : U(X) \to A(Y)$ is a linear disjointness preserving bijection, then $T^{-1}$ is disjointness preserving.*

**4. Spaces of Lipschitz and uniformly continuous functions.** Let $X$ be a complete metric space and let $0 < \alpha < 1$. The space $\mathrm{Lip}(X)$ of *Lipschitz functions* consists of all real-valued functions $f$ on $X$ such that

$$L_1(f) = \sup\left\{\frac{|f(u)-f(v)|}{d(u,v)} : u,v \in X,\, u \neq v\right\} < \infty.$$

The space $\mathrm{lip}_\alpha(X)$ of *little Lipschitz functions of order* $\alpha$ consists of all real-valued functions $f$ on $X$ such that

$$L_\alpha(f) = \sup\left\{\frac{|f(u)-f(v)|}{d(u,v)^\alpha} : u,v \in X,\, u \neq v\right\} < \infty$$

and that

$$\lim_{r\to 0+} \sup\left\{\frac{|f(u)-f(v)|}{d(u,v)^\alpha} : u,v \in X,\, 0 < d(u,v) < r\right\} = 0.$$

Let $A(X)$ be one of the spaces $\mathrm{Lip}(X)$, $\mathrm{lip}_\alpha(X)$, $0 < \alpha < 1$, or $U(X)$. Note that $A(X)$ separates points from closed sets of $X$ and is closed under $C_*^\infty$ operations. In [AD, Theorem 3.5], Araujo and Dubarbie showed that a disjointness preserving linear isomorphism $T : \mathrm{Lip}(X) \to \mathrm{Lip}(Y)$ is biseparating when $Y$ is a compact metric space and $X$ is a complete bounded metric space.

In this section, we prove that under some topological assumptions on the space $Y$, if $T : A(X) \to A(Y)$ is a disjointness preserving linear isomorphism, then $T^{-1}$ is disjointness preserving.

Proposition 4.1. *Suppose that $f \in A(X)$ and $a, b$ are distinct points in $\mathcal{A}X$ such that $\hat{f}(a), \hat{f}(b) \in \mathbb{R}$. Then for any $\varepsilon > 0$, there are open neighborhoods $U$ and $V$ of $a$ and $b$ respectively (in $\mathcal{A}X$) such that $d(U\cap X, V\cap X) = s > 0$ and*

$$|\hat{f}(x) - \hat{f}(a)| < s \wedge \varepsilon \quad \text{if } x \in U, \qquad |\hat{f}(x) - \hat{f}(b)| < s \wedge \varepsilon \quad \text{if } x \in V.$$

*Proof.* There are open neighborhoods $U'$ and $V'$ of $a$ and $b$ respectively in $\mathcal{A}X$ such that $\overline{U'}^{\mathcal{A}X} \cap \overline{V'}^{\mathcal{A}X} = \emptyset$. By Proposition 1.2, there exists $\varphi \in A(X)$ such that $\varphi = 0$ on $U' \cap X$ and $\varphi = 1$ on $V' \cap X$. Since $\varphi$ is uniformly continuous on $X$, $r = d(U' \cap X, V' \cap X) > 0$. By continuity of $\hat{f}$, there are open neighborhoods $U \subseteq U'$ and $V \subseteq V'$ of $a$ and $b$ respectively (in $\mathcal{A}X$) such that

$$|\hat{f}(x) - \hat{f}(a)| < r \wedge \varepsilon \quad \text{if } x \in U, \qquad |\hat{f}(x) - \hat{f}(b)| < r \wedge \varepsilon \quad \text{if } x \in V.$$

Since $U \cap X \subseteq U' \cap X$ and $V \cap X \subseteq V' \cap X$,

$$s = d(U \cap X, V \cap X) \geq r. \blacksquare$$

Proposition 4.2. *Suppose that $f \in A(X)$ and $W$ is a subset of $X$. Let $t > 0$ be given. Assume that*

$$\sup\{|f(x)| : d(x,W) < t\} = M < \infty.$$

*Define $\varphi : X \to \mathbb{R}$ by*

$$\varphi(x) = \left(1 - \frac{d(x,W)}{t}\right)^{+}.$$

*Then for any $u, v \in X$, we have*

$$|(\varphi f)(u) - (\varphi f)(v)| \leq \left(\frac{d(u,v)}{t} \wedge 1\right) M + |f(u) - f(v)|.$$

*Furthermore, $\|\varphi f\|_\infty \leq M$.*

*Proof.* Since $\varphi(x) = 0$ if $d(x,W) \geq t$ and $\|\varphi\|_\infty \leq 1$, it is clear that $\|\varphi f\|_\infty \leq M$. If $d(u,W) < t$, then for any $v$,

$$\begin{aligned} |(\varphi f)(u) - (\varphi f)(v)| &\leq |\varphi(u) - \varphi(v)|\,|f(u)| + |\varphi(v)|\,|f(u) - f(v)| \\ &\leq \left(\frac{d(u,v)}{t} \wedge 1\right) M + |f(u) - f(v)|. \end{aligned}$$

A similar estimate holds if $d(v,W) < t$, by symmetry. If $d(u,W), d(v,W) \geq t$, then of course $(\varphi f)(u) = (\varphi f)(v) = 0$. ∎

For the rest of this section, assume that $Y$ is a first countable Hausdorff topological space and let $A(Y)$ be a vector subspace of $C(Y)$ that contains constants. Let $T : A(X) \to A(Y)$ be a disjointness preserving operator with support map $\beta : Y_s \to \mathcal{A}X$, where $Y_s$ and $\beta$ are as in Proposition 1.2.

Proposition 4.3. *Let $(y_n)$ be a sequence in $Y_s$ converging to $y_0 \in Y_s$. Assume that $(x_n)$ is a distinct sequence and $x_n \neq x_0$ for all $n \in \mathbb{N}$, where $x_n = \beta(y_n)$, $n \geq 0$. Let $f \in A(X)$ be a function such that $\hat{f}(x_0) = 0$. Then $Tf(y_0) = 0$.*

*Proof.* We will show that there exists a function $g \in A(X)$ such that $\widehat{g - f} = 0$ on an open neighborhood of $x_{2n}$ in $\mathcal{A}X$ and $\hat{g} = 0$ on an open neighborhood of $x_{2n+1}$ in $\mathcal{A}X$, for all $n \in \mathbb{N}$. Then $Tg(y_{2n}) = Tf(y_{2n})$ and $Tg(y_{2n+1}) = 0$ for all $n$. By continuity of $Tf$ and $Tg$, we see that $Tf(y_0) = 0$.

In the proof below, take $A(X) = U(X)$ if $\alpha = 0$, $A(X) = \mathrm{lip}_\alpha(X)$ if $0 < \alpha < 1$ and $A(X) = \mathrm{Lip}(X)$ if $\alpha = 1$. For any $r > 0$, let

$$\omega_f(r) = \sup\{|f(u) - f(v)| : d(u,v) \leq r\}.$$

Since $f$ is uniformly continuous, there exists $r_0 > 0$ such that $\omega_f(r_0) \leq 1$. Note that $(x_n)$ converges to $x_0$ and $(\hat{f}(x_n))$ converges to 0. In particular, we may assume that $|\hat{f}(x_n)| \leq 1$ for all $n$. Use Proposition 4.1 to choose open neighborhoods $U_n$ and $V_n$ of $x_0$ and $x_n$ respectively in $\mathcal{A}X$ such that

$s_n = d(U_n \cap X, V_n \cap X) > 0$ and

$$|\hat{f}(x) - \hat{f}(x_0)| < s_n \wedge 1 \quad \text{if } x \in U_n, \qquad |\hat{f}(x) - \hat{f}(x_n)| < s_n \wedge 1 \quad \text{if } x \in V_n.$$

We may also assume that $U_{n+1}, V_{n+1} \subseteq U_n$ for all $n$. By taking a subsequence if necessary, we may reduce to considering one of the following cases:

CASE 1: $s_n \downarrow 0$. In this case, set $t_n = s_n/4$ and assume additionally that $8t_n \leq r_0$.
CASE 2: $\inf s_n = \gamma > 0$. In this case, set $t_n = (\gamma \wedge r_0)/4$.

Note that in both instances, $(t_n)$ is nonincreasing and $s_n \geq 4t_n$ for all $n$. Define

$$M_n = \sup\{|f(x)| : x \in X,\, d(x, V_n \cap X) < t_n\}.$$

Suppose that $x \in X$ with $d(x, V_n \cap X) < t_n$. Choose $u_n \in U_n \cap X$ and $v_n, w_n \in V_n \cap X$ such that

$$d(u_n, v_n) < 2s_n \quad \text{and} \quad d(x, w_n) < t_n.$$

Since $\hat{f}(x_0) = 0$, in Case 1,

$$\begin{aligned} |f(x)| &\leq |f(x) - f(w_n)| + |f(w_n) - \hat{f}(x_n)| + |\hat{f}(x_n) - f(v_n)| \\ &\quad + |f(v_n) - f(u_n)| + |f(u_n) - \hat{f}(x_0)| \\ &\leq \omega_f(t_n) + 3s_n + \omega_f(2s_n) \leq 2\omega_f(8t_n) + 12t_n. \end{aligned} \tag{4.1}$$

Thus $M_n \leq 2\omega_f(8t_n) + 12t_n$. In Case 2,

$$\begin{aligned} |f(x)| &\leq |f(x) - f(w_n)| + |f(w_n) - \hat{f}(x_n)| + |\hat{f}(x_n)| \\ &\leq \omega_f(t_n) + 2 \leq \omega_f(\gamma \wedge r_0) + 2 \leq 3. \end{aligned} \tag{4.2}$$

Thus, $M_n \leq 3$. For each $n$, define $\varphi_n : X \to \mathbb{R}$ by

$$\varphi_n(x) = \left(1 - \frac{d(x, V_n \cap X)}{t_n}\right)^+.$$

Note that $(\varphi_n f)(x) = 0$ if $d(x, V_n \cap X) \geq t_n$. If $m < n$, then $V_n \cap X \subseteq U_m \cap X$ and thus

$$d(V_n \cap X, V_m \cap X) \geq d(U_m \cap X, V_m \cap X) = s_m \geq 4t_m > t_m + t_n. \tag{4.3}$$

In particular, the functions $\varphi_n f$ are disjoint. Let $g$ be the pointwise sum $\sum_{n=1}^{\infty} \varphi_{2n} f$. We will prove that $g \in A(X)$. Let $u, v \in X$ be distinct points.

Consider first the case where there are $m < n$ with $d(u, V_{2m} \cap X) < t_{2m}$ and $d(v, V_{2n} \cap X) < t_{2n}$. By (4.3),

$$d(u, v) \geq d(V_{2m} \cap X, V_{2n} \cap X) - t_{2m} - t_{2n} \geq 3t_{2m} - t_{2n} \geq 2t_{2m}.$$

By Proposition 4.2, $\|\varphi_n f\|_\infty \leq M_n$ for all $n$. In Case 1, since also $t_{2n} \leq r_0/8$,

$$\begin{aligned}
\frac{|g(u)-g(v)|}{d(u,v)^\alpha} &\le \frac{\|\varphi_{2m}f\|_\infty + \|\varphi_{2n}f\|_\infty}{d(u,v)^\alpha} \\
&\le \frac{2\omega_f(8t_{2m}) + 12t_{2m} + 2\omega_f(8t_{2n}) + 12t_{2n}}{d(u,v)^\alpha} \\
&\le \frac{4\omega_f(4d(u,v)) + 12d(u,v) \wedge 3r_0}{d(u,v)^\alpha},
\end{aligned} \tag{4.4}$$

while in Case 2, since $d(u,v) \ge 2t_{2m} = (\gamma \wedge r_0)/2$, we get

$$\begin{aligned}
\frac{|g(u)-g(v)|}{d(u,v)^\alpha} &\le \frac{\|\varphi_{2m}f\|_\infty + \|\varphi_{2n}f\|_\infty}{d(u,v)^\alpha} \\
&\le \frac{6}{d(u,v)^\alpha} \le \frac{12}{\gamma \wedge r_0} d(u,v)^{1-\alpha} \wedge 6\left(\frac{2}{\gamma \wedge r_0}\right)^\alpha.
\end{aligned} \tag{4.5}$$

In the remaining situation, there must be some $n \in \mathbb{N}$ such that $g(u) = (\varphi_{2n}f)(u)$ and $g(v) = (\varphi_{2n}f)(v)$. In Case 1, by Proposition 4.2 and (4.1),

$$\begin{aligned}
\frac{|g(u)-g(v)|}{d(u,v)^\alpha} &= \frac{|(\varphi_{2n}f)(u) - (\varphi_{2n}f)(v)|}{d(u,v)^\alpha} \\
&\le \begin{cases}
\dfrac{2\omega_f(8t_{2n})}{t_{2n}^\alpha} \cdot \left(\dfrac{d(u,v)}{t_{2n}}\right)^{1-\alpha} \\
\qquad + 12d(u,v)^{1-\alpha} + \dfrac{|f(u)-f(v)|}{d(u,v)^\alpha} & \text{if } d(u,v) \le t_{2n}, \\
\dfrac{2\omega_f(8t_{2n})}{t_{2n}^\alpha} \cdot \left(\dfrac{t_{2n}}{d(u,v)}\right)^{\alpha} \\
\qquad + 12t_{2n}^{1-\alpha} + \dfrac{|f(u)-f(v)|}{d(u,v)^\alpha} & \text{if } t_{2n} < d(u,v).
\end{cases}
\end{aligned} \tag{4.6}$$

Similarly, in Case 2, by Proposition 4.2 and (4.2),

$$\begin{aligned}
\frac{|g(u)-g(v)|}{d(u,v)^\alpha} &= \frac{|(\varphi_{2n}f)(u) - (\varphi_{2n}f)(v)|}{d(u,v)^\alpha} \\
&\le \frac{12}{\gamma \wedge r_0} \cdot d(u,v)^{1-\alpha} \wedge \frac{3}{d(u,v)^\alpha} + \frac{|f(u)-f(v)|}{d(u,v)^\alpha}.
\end{aligned} \tag{4.7}$$

If $0 < \alpha \le 1$, there is a finite constant $C$ such that $\omega_f(r) \le Cr^\alpha$. Then it follows from (4.4)–(4.7) that $L_\alpha(g) < \infty$.

It remains to show that if $0 \le \alpha < 1$, then

$$\lim_{r \to 0+} \sup\left\{\frac{|g(u)-g(v)|}{d(u,v)^\alpha} : 0 < d(u,v) < r\right\} = 0.$$

This is clear if $u, v$ fall under one of (4.4), (4.5) or (4.7). Finally, suppose $u, v$ satisfy (4.6). Obviously,

$$\lim_{r \to 0+} \sup\left\{12d(u,v)^{1-\alpha} + \frac{|f(u)-f(v)|}{d(u,v)^\alpha} : 0 < d(u,v) < r\right\} = 0.$$

Given $\varepsilon > 0$, since $t_n \to 0$ in the situation under consideration, there exists $N$ such that $\omega_f(8t_{2n}) \le \varepsilon t_{2n}^\alpha$ for all $n \ge N$. Thus, if $n \ge N$, then the first term of (4.6) is $\le \varepsilon$, for both the cases $d(u,v) \le t_{2n}$ and $t_{2n} < d(u,v)$. On the other hand, suppose that $n < N$. As observed above, for $0 < \alpha < 1$, there is a finite constant $C$ such that $\omega_f(r) \le Cr^\alpha$ for all $r$. By choice of $r_0$, the inequality also holds for $r \le r_0$ when $\alpha = 0$. In particular, it holds for $r = t_n$ for any $n$.

Now assume that $d(u,v) < t_{2N}$. Since $n < N$, we must have $d(u,v) \le t_{2n}$. Then the first term of (4.6) is

$$\frac{2\omega_f(8t_{2n})}{t_{2n}^\alpha} \cdot \left(\frac{d(u,v)}{t_{2n}}\right)^{1-\alpha} \le \frac{2 \cdot 8^\alpha \cdot C}{t_{2N}^{1-\alpha}} d(u,v)^{1-\alpha}.$$

Since $0 \le \alpha < 1$,

$$\lim_{r\to 0+} \sup\left\{\frac{2 \cdot 8^\alpha \cdot C}{t_{2N}^{1-\alpha}} d(u,v)^{1-\alpha} : 0 < d(u,v) < r\right\} = 0.$$

This completes the proof that $g \in A(X)$.

By definition of $g$, and the fact that the functions $(\varphi_n f)$ are disjoint, $g = f$ on $V_{2n} \cap X$ and $g = 0$ on $V_{2n-1} \cap X$ for all $n$. Therefore, $g$ has the properties enunciated in the first paragraph of the proof. ∎

PROPOSITION 4.4. *Suppose that $y_0 \in Y_s$. If $\beta^{-1}\{\beta(y_0)\} \cap Y_p = \emptyset$, then $\beta^{-1}\{\beta(y_0)\}$ is a clopen subset of $Y_s$.*

*Proof.* Clearly $\beta^{-1}\{\beta(y_0)\}$ is a closed subset of $Y_s$. Suppose that it is not open in $Y_s$. Since $Y$ is first countable, there exists a sequence $(y_n)$ in $Y_s \setminus \beta^{-1}\{\beta(y_0)\}$ converging to a point $z_0 \in \beta^{-1}\{\beta(y_0)\}$. By the assumption, $z_0 \notin Y_p$. Note that $(\beta(y_n))$ converges to $\beta(z_0) = \beta(y_0)$, and $\beta(y_n) \ne \beta(y_0)$ for all $n$. By Proposition 4.3, if $\hat{f}(\beta(y_0)) = 0$, then $Tf(y_0) = 0$. But this means that $y_0 \in Y_p$. Since $y_0 \in \beta^{-1}\{\beta(y_0)\}$, we have a contradiction to the assumption. ∎

PROPOSITION 4.5. *Let $T : A(X) \to A(Y)$ be a disjointness preserving linear isomorphism. Suppose that $Y$ has finitely many connnected components. Then $\mathcal{A}X \setminus \overline{\beta(Y_p)}^{\mathcal{A}X}$ is a finite set of points in $X$, each of which is isolated in $\mathcal{A}X$.*

*Proof.* By Proposition 2.1, $Y_s = Y$ and $\beta(Y)$ is dense in $\mathcal{A}X$. Suppose that $x \in \beta(Y) \setminus \beta(Y_p)$. Let $y \in Y$ be such that $\beta(y) = x$. Then $\beta^{-1}\{\beta(y)\} \cap Y_p = \emptyset$. By Proposition 4.4, $\beta^{-1}\{x\} = \beta^{-1}\{\beta(y)\}$ is a clopen subset of $Y$. It follows from the assumption that $\beta(Y) \setminus \beta(Y_p)$ is finite.

Next, we claim that

$$\mathcal{A}X \setminus \overline{\beta(Y_p)}^{\mathcal{A}X} = \beta(Y) \setminus \beta(Y_p).$$

Indeed, suppose that $x_0 \in \mathcal{A}X \setminus \overline{\beta(Y_p)}^{\mathcal{A}X}$. Let $U$ be an open neighborhood of $x_0$ in $\mathcal{A}X$. Then $U \setminus \overline{\beta(Y_p)}^{\mathcal{A}X}$ is an open neighborhood of $x_0$ and hence meets $\beta(Y)$. This proves that $U \cap (\beta(Y) \setminus \beta(Y_p)) \neq \emptyset$. Since $U$ is an arbitrary open neighborhood of $x_0$, we see that $x_0 \in \overline{\beta(Y) \setminus \beta(Y_p)}^{\mathcal{A}X}$. As $\beta(Y) \setminus \beta(Y_p)$ is a finite set, $x_0 \in \beta(Y) \setminus \beta(Y_p)$. This completes the proof of the claim.

It follows that $\mathcal{A}X \setminus \overline{\beta(Y_p)}^{\mathcal{A}X} = \beta(Y) \setminus \beta(Y_p)$ is an open set that contains only finitely many points. Therefore, each point in $\mathcal{A}X \setminus \overline{\beta(Y_p)}^{\mathcal{A}X}$ is an isolated point in $\mathcal{A}X$. In particular, each point in $\mathcal{A}X \setminus \overline{\beta(Y_p)}^{\mathcal{A}X}$ must belong to $X$. ∎

Corollary 4.6. *Let $T : A(X) \to A(Y)$ be a disjointness preserving linear isomorphism. Suppose that $Y$ has finitely many connnected components. Then $\beta(Y_p)$ is dense in $\mathcal{A}X$.*

*Proof.* Let $x_0 \in \mathcal{A}X \setminus \overline{\beta(Y_p)}^{\mathcal{A}X}$. By Proposition 4.5, $x_0 \in X$ and is an isolated point in $\mathcal{A}X$. Thus $\chi_{\{x_0\}} \in A(X)$. Let $g = T\chi_{\{x_0\}}$. There exists $y_0 \in Y$ such that $g(y_0) \neq 0$. Since $\hat{\chi}_{\{x_0\}} = 0$ on the open set $\mathcal{A}X \setminus \{x_0\}$, we have $g = 0$ on $\beta^{-1}(\mathcal{A}X \setminus \{x_0\})$. Thus $\beta(y_0) = x_0$. Let $f \in A(X)$ with $f(x_0) = 0$. Then $f$ and $\chi_{\{x_0\}}$ are disjoint. Hence so are $Tf$ and $g$. Thus $Tf(y_0) = 0$. This shows that $y_0 \in Y_p$, which implies $x_0 = \beta(y_0) \in \beta(Y_p)$, contrary to the choice of $x_0$. Therefore, $\mathcal{A}X \setminus \overline{\beta(Y_p)}^{\mathcal{A}X}$ must be empty. ∎

The main result of this section follows immediately from Proposition 2.4 and Corollary 4.6.

Theorem 4.7. *Let $X$ be a complete metric space and let $Y$ be a first countable Hausdorff topological space that has only finitely many connected components. Suppose $A(X)$ is one of the spaces* $\mathrm{Lip}(X)$, $\mathrm{lip}_\alpha(X)$, $0 < \alpha < 1$, *or $U(X)$, and $A(Y)$ is a vector subspace of $C(Y)$ that contains all constant functions. Let $T : A(X) \to A(Y)$ be a disjointness preserving linear isomorphism. Then $T^{-1}$ is disjointness preserving.*

**5. Spaces of differentiable functions: Representation of disjointness preserving operators.** In the final two sections, we focus on the case of differentiable functions. See [KN] for some previous results in this direction. Let $p \in \mathbb{N}$ and let $G$ be a separable Banach space that supports a $C^p(G)$ bump function with bounded derivatives. That is, there is a nonzero function $\varphi \in C^p(G)$ with bounded support such that $\sup_{x\in G} \|D^k\varphi(x)\| < \infty$ for $0 \leq k \leq p$. It is easy to see that in this case, for any $r > 0$, there exist $0 < s < r$ and a function $f \in C^p(G)$ with bounded derivatives such that $f = 1$ on $B(0, s)$ and $f = 0$ outside $B(0, r)$.

Let $X$ be an open subset of $G$ and let $E$ be a Banach space. Let $Y$ be a first countable Hausdorff topological space, $F$ be a Banach space and let $A(Y,F)$ be a vector subspace of $C(Y,F)$.

The main result in this section is a representation theorem for a disjointness preserving linear map $T : C^p(X,E) \to A(Y,F)$.

Since $G$ supports a $C^p(G)$ bump function, $C^p(X)$ separates points from closed sets. It is clear that $C^p(X)$ is closed under $C^\infty_*$ operations.

For the rest of the section, let $T : C^p(X,E) \to A(Y,F)$ be a linear disjointness preserving operator.

We will also assume that $T$ is *nowhere trivial*: for any $y \in Y$, there exists $f \in C^p(X,E)$ such that $Tf(y) \neq 0$. (This is always achievable by cutting down on the space $Y$.)

As in Section 1, let $\mathcal{A}X$ be the compactification of $X$ constructed using $A(X) = C^p(X)$. By [LW, Proposition 4], as in Proposition 1.2, there is a continuous function $\beta : Y \to \mathcal{A}X$ such that if $f \in C^p(X,E)$ and

$$\beta(y) \notin \overline{C}(f) = \overline{\{x \in X : f(x) \neq 0\}}^{\mathcal{A}X},$$

then $Tf(y) = 0$. If $S$ is a linear operator from $C^p(X,E)$ into a vector space, we say that $S$ is *diffuse* if $S$ is nonzero and there exists $\gamma > 0$ with $Tf = 0$ for any $f$ such that

$$\{f \neq 0\} = \{x \in X : f(x) \neq 0\}$$

has diameter $\leq \gamma$. This terminology is borrowed from the vector lattice setting (see, e.g., [HD1]). An operator that is not diffuse is said to be *focused.*

Proposition 5.1. *Suppose that $y_0 \in Y$ and $x_0 = \beta(y_0) \notin X$. Then there exists a sequence $(W_n)$ of open neighborhoods of $x_0$ in $\mathcal{A}X$ such that $(\bigcap W_n) \cap X = \emptyset$.*

*Proof.* First consider the case where $\delta_{y_0} \circ T$ is diffuse. There exists $\gamma > 0$ such that if $g \in C^p(X,E)$ and $\operatorname{diam}\{g \neq 0\} \leq \gamma$, then $Tg(y_0) = 0$. Let $(u_n)$ be a countable dense sequence in $X$. There exists $0 < r_1 < \gamma/2$ such that for each $n$, there is a $\psi_n \in C^p(X)$ satisfying $\psi_n = 0$ on $B(u_n, r_1)$ and $\psi_n = 1$ outside $B(u_n, \gamma/2)$. Choose $f \in C^p(X,E)$ such that $Tf(y_0) \neq 0$. Since $(1-\psi_n)f \in C^p(X,E)$ and $\operatorname{diam}\{(1-\psi_n)f \neq 0\} \leq \gamma$, we have $T((1-\psi_n)f)(y_0) = 0$. Thus $T(\psi_n f)(y_0) \neq 0$. By definition of $\beta$, $x_0 \in \overline{C}(\psi_n f) \subseteq \overline{B(u_n, r_1)^c}^{\mathcal{A}X}$. Let $0 < r_2 < r_1$ be such that for each $n$, there exists $\varphi_n \in C^p(X)$ with $\varphi_n = 0$ on $B(u_n, r_2)$ and $\varphi_n = 1$ outside $B(u_n, r_1)$. By continuity of $\hat{\varphi}_n$, $\hat{\varphi}_n(x_0) = 1$. Let $W_n = \{\hat{\varphi}_n > 1/2\}$. Then $W_n$ is an open neighborhood of $x_0$ in $\mathcal{A}X$. Suppose that $x \in (\bigcap W_n) \cap X$. Then $\varphi_n(x) > 1/2$ for all $n$. Thus $x \notin B(u_n, r_2)$ for all $n$. This is impossible since $(u_n)$ is dense in $X$. Therefore, $(\bigcap W_n) \cap X = \emptyset$.

Now, consider the case where $\delta_{y_0} \circ T$ is focused. For each $n$, there exists $f_n \in C^p(X, E)$ such that $\operatorname{diam}\{f_n \neq 0\} \to 0$ and $Tf_n(y_0) \neq 0$. Choose $x_n \in X$ such that $f_n(x_n) \neq 0$. If $(x_n)$ has no Cauchy subsequence, then, by using a subsequence if necessary, we may assume that there exists $r > 0$ such that $\|x_n - x_m\| > r$ if $n \neq m$. For sufficiently large $n$ and $m$, $f_n$ and $f_m$ are disjoint. However, $Tf_m(y_0)$ and $Tf_n(y_0)$ are both nonzero, contrary to the fact that $T$ is disjointness preserving. This proves that, by using a subsequence if necessary, we may assume that $(x_n)$ is a Cauchy sequence. Then $(x_n)$ converges to a point $z_0 \in G$.

Claim. *For any $r > 0$, $x_0 \in \overline{B_G(z_0, r) \cap X}^{\mathcal{A}X}$. Here, the ball $B_G(z_0, r)$ is taken in $G$.*

Let $r > 0$. Since $(x_n)$ converges to $z_0$, $f_n(x_n) \neq 0$ and $\operatorname{diam}\{f_n \neq 0\} \to 0$, there exists $n$ such that $\{f_n \neq 0\} \subseteq B_G(z_0, r/2) \cap X$. By choice, $Tf_n(y_0) \neq 0$. By definition of $\beta$, $x_0 \in \overline{C}(f_n) \subseteq \overline{B_G(z_0, r) \cap X}^{\mathcal{A}X}$. This completes the proof of the Claim.

Suppose, if possible, that $z_0 \in X$. Choose $r_0 > 0$ such that $B_G(z_0, r_0) \subseteq X$. Let $g$ be a function in $C^p(X)$. For any $\varepsilon > 0$, there exists $0 < r < r_0$ such that $|g(x) - g(z_0)| < \varepsilon$ for all $x \in B_G(z_0, r)$. By the Claim, $x_0 \in \overline{B_G(z_0, r)}^{\mathcal{A}X}$. By continuity of $\hat{g}$, $|\hat{g}(x_0) - g(z_0)| \leq \varepsilon$. This shows that $\hat{g}(x_0) = g(z_0)$. Since this holds for all $g \in C^p(X)$, we get $x_0 = z_0 \in X$, contrary to the assumption. Therefore, $z_0 \notin X$.

For any $n \in \mathbb{N}$, choose $0 < r_n < 1/n$ and $\varphi_n \in C^p(G)$ to be such that $\varphi_n = 1$ on $B_G(z_0, r_n)$ and $\varphi_n = 0$ outside $B_G(z_0, 1/n)$. We will also regard $\varphi_n$ as a function in $C^p(X)$ by restricting its domain to $X$. Since $\varphi_n = 1$ on $B_G(z_0, r_n) \cap X$, $\hat{\varphi}_n(x_0) = 1$ by the Claim. Let $W_n = \{\hat{\varphi}_n > 0\}$. Then $W_n$ is an open neighborhood of $x_0$ in $\mathcal{A}X$. Moreover, $W_n \cap X \subseteq B_G(z_0, 1/n)$. Thus $(\bigcap W_n) \cap X = \emptyset$. ∎

Proposition 5.2. *Suppose that $y_0 \in Y$ and $x_0 = \beta(y_0) \notin X$. Then $\beta^{-1}\{x_0\}$ is a clopen subset of $Y$.*

*Proof.* Clearly $\beta^{-1}\{x_0\}$ is a closed subset of $Y$. If it is not open, there exists a sequence $(y_n)$ in $Y$ converging to some $y_* \in \beta^{-1}\{x_0\}$ such that $y_n \notin \beta^{-1}\{x_0\}$ for all $n$. Let $x_n = \beta(y_n)$, $n \in \mathbb{N}$. We may assume that $(x_n)$ is a sequence of distinct points, all of which are different from $x_0$. By Proposition 5.1, there is a sequence $(W_n)$ of open neighborhoods of $x_0$ in $\mathcal{A}X$ such that $(\bigcap W_n) \cap X = \emptyset$. We may further assume that $\overline{W_{n+1}}^{\mathcal{A}X} \subseteq W_n$ for all $n$. Since $(x_n)$ converges to $x_0$, by replacing it with a subsequence if necessary, we may choose open neighborhoods $U_n$ and $V_n$ of $x_n$ in $\mathcal{A}X$ such that

$$x_n \in V_n \subseteq \overline{V_n}^{\mathcal{A}X} \subseteq U_n \subseteq W_n, \quad \overline{U_m}^{\mathcal{A}X} \cap \overline{U_n}^{\mathcal{A}X} = \emptyset \quad \text{if } m \neq n,$$

and $x_0 \notin \overline{U_n}^{\mathcal{A}X}$. By Proposition 1.1, there exists $\varphi_n \in C^p(X)$ such that $\varphi_n = 1$ on $V_n \cap X$ and $\varphi_n = 0$ on $X \setminus U_n$. The functions $\varphi_n, n \in \mathbb{N}$, are pairwise disjoint.

Let $f \in C^p(X,E)$ be such that $Tf(y_0) \neq 0$ and let $g$ be the pointwise sum $\sum \varphi_{2n} f$. For any $x \in X$, there exists $n_0$ such that $x \notin W_{2n_0+1}$. Thus $x \notin \overline{\bigcup_{n=2n_0+2}^{\infty}(U_n \cap X)}$ (closure in $X$). Hence $g = \sum_{n=1}^{n_0} \varphi_{2n} f$ on a neighborhood of $x$ in $X$. This proves that $g \in C^p(X,E)$.

For any $n$ we have $g = f$ on $V_{2n} \cap X$ and thus $\widehat{g-f} = 0$ on $V_{2n}$. Hence $T(g-f)(y_{2n}) = 0$, i.e., $Tg(y_{2n}) = Tf(y_{2n})$. Suppose that $x \in V_{2n-1} \cap X$. By the above, there exists $n_0$ such that $g = \sum_{n=1}^{n_0} \varphi_{2n} f$ on a neighborhood of $x$. But $x \in U_{2n-1} \cap X$ and $U_{2n-1} \cap U_{2k} = \emptyset$ for all $k$. Hence $g(x) = 0$. This proves that $g = 0$ on $V_{2n-1} \cap X$. Therefore, $\hat{g} = 0$ on $V_{2n-1}$, and hence $Tg(y_{2n-1}) = 0$. Taking limits as $n \to \infty$, we see that

$$Tf(y_0) = \lim Tf(y_{2n}) = \lim Tg(y_{2n}) = Tg(y_0) = \lim Tg(y_{2n-1}) = 0,$$

contrary to the choice of $f$. This proves that $\beta^{-1}\{x_0\}$ is open in $Y$. ∎

Let $Y_r$ be the set of all $y \in \beta^{-1}(X)$ such that if $f \in C^p(X,E)$ and $D^k f(\beta(y)) = 0$ for $0 \le k \le p$, then $Tf(y) = 0$. There is an obvious resemblance between $Y_r$ and the set $Y_p$ from Section 2. For $1 \le k \le p$, let $\mathcal{S}^k(G,E)$ be the Banach space of all bounded symmetric $k$-linear operators from $G$ to $E$, with the operator norm. Also let $\mathcal{S}^0(G,E) = E$. If $f \in C^p(X,E)$, then $D^k f(x) \in \mathcal{S}^k(G,E)$ for all $x \in X$.

Proposition 5.3. *Let $y_0 \in Y_r$. Then there are linear operators $\Phi_k(y_0,\cdot) : \mathcal{S}^k(G,E) \to F$, $0 \le k \le p$, such that*

$$Tf(y_0) = \sum_{k=0}^{p} \Phi_k\big(y_0, D^k f(\beta(y_0))\big) \quad \textit{for all } f \in C^p(X,E).$$

*Proof.* Let $x_0 = \beta(y_0) \in X$. For any $S \in \mathcal{S}^k(G,E)$, the function $f_S : X \to E$ given by $f_S(x) = S(x-x_0,\dots,x-x_0)$ ($k$ components) belongs to $C^p(X,E)$. Define $\Phi_k(y_0,S) = Tf_S(y_0)/k!$. Then $\Phi_k(y_0,\cdot) : \mathcal{S}^k(G,E) \to F$ is a linear operator. Using the fact that $y_0 \in Y_r$, one may verify by direct computation that $Tf(y_0) = \sum_{k=0}^{p} \Phi_k(y_0, D^k f(\beta(y_0)))$ for all $f \in C^p(X,E)$. Refer to the proof of [LW, Theorem 10] for details. ∎

Recall that we assume the existence of a function $\varphi \in C^p(G)$ with $\varphi = 1$ on a neighborhood of 0, $\varphi(x) = 0$ if $\|x\| \ge 1$, and $\sup_{x\in X} \|D^k\varphi(x)\| < \infty$ for $0 \le k \le p$.

Lemma 5.4. *Suppose that $y_0 \in Y$ and $x_0 = \beta(y_0) \in X$. If $f \in C^p(X,E)$ and $f = 0$ on a neighborhood $U$ of $x_0$ in $X$, then $Tf(y_0) = 0$.*

*Proof.* Choose $\psi \in C^p(X)$ such that $\psi(x_0) = 1$ and $\psi = 0$ outside $U$. The set $V = \{\hat{\psi} > 1/2\}$ is an open neighborhood of $x_0$ in $\mathcal{A}X$. Since $f = 0$ on $U \supseteq V \cap X$ and $V \cap X$ is dense in $V$, $\hat{f} = 0$ on $V$. Therefore, $Tf(y_0) = 0$ by definition of $\beta$. ∎

PROPOSITION 5.5. *Let $(y_n)$ be a sequence in $\beta^{-1}(X)$ that converges to a point $y_0 \in \beta^{-1}(X) \setminus Y_r$. Then $\beta(y_n) = \beta(y_0)$ for all sufficiently large $n$.*

*Proof.* If the proposition fails, we can choose a sequence $(x_n) = (\beta(y_n))$ in $X$ converging to $x_0 = \beta(y_0) \in X$, with $x_n \neq x_0$ for all $n$, and a function $f \in C^p(X, E)$ such that $D^k f(x_0) = 0$, $0 \le k \le p$, but $Tf(y_0) \neq 0$. We may assume that $r_n = \|x_n - x_0\|$ satisfies $0 < 3r_{n+1} < r_n$ for all $n$. Set $\varphi_n(x) = \varphi\big(\frac{2}{r_n}(x - x_n)\big)$. The functions $\varphi_n$, $n \in \mathbb{N}$, are pairwise disjoint. Since $D^k f(x_0) = 0$ for $0 \le k \le p$, by Taylor's Theorem,

$$\lim_{r \to 0+} \sup_{x \in B(x_0, r)} \frac{\|D^k f(x)\|}{r^{p-k}} = 0 \quad \text{for } 0 \le k \le p.$$

It follows from [LW, Lemma 11] that the pointwise sum $g = \sum \varphi_{2n} f$ belongs to $C^p(X, E)$. Consider $m \in \mathbb{N}$. There is an open neighborhood $U_m$ of $x_m$ in $X$ such that $\varphi_m = 1$ on $U_m$. Then $g = f$ on $U_m$ if $m$ is even, and $g = 0$ on $U_m$ if $m$ is odd. By Lemma 5.4, $Tg(y_m) = Tf(y_m)$ if $m$ is even and $Tg(y_m) = 0$ if $m$ is odd. Therefore,

$$Tf(y_0) = \lim Tf(y_{2n}) = \lim Tg(y_{2n}) = Tg(y_0) = \lim Tg(y_{2n-1}) = 0,$$

contrary to the choice of $f$. ∎

THEOREM 5.6. *Let $G$ be a separable Banach space that supports a $C^p(G)$ bump function with bounded derivatives for some $1 \le p < \infty$, and let $X$ be an open subset of $G$. Suppose that $E$ and $F$ are Banach spaces and $Y$ is a Hausdorff first countable topological space. Let $A(Y, F)$ be a vector subspace of $C(Y, F)$. Assume that $T : C^p(X, E) \to A(Y, F)$ is a nowhere trivial disjointness preserving linear operator. Denote by $\beta : Y \to \mathcal{A}X$ the support map of $T$. Then there is a partition*

$$Y = \Big(\bigcup_{\alpha \in I} Y_\alpha\Big) \cup Y_r \cup \Big(\bigcup_{\alpha \in J} Y_\alpha\Big)$$

*such that:*

(1) *For each $\alpha \in I$, $Y_\alpha$ is a clopen subset of $Y$; there exists $x_\alpha \in \mathcal{A}X \setminus X$ such that $Y_\alpha = \beta^{-1}\{x_\alpha\}$.*
(2) *$\beta^{-1}(X) = Y_r \cup \bigcup_{\alpha \in J} Y_\alpha$, and $Y_r$ are closed subsets of $Y$.*
(3) *$y \in Y_r$ if and only if there are linear operators $\Phi_k(y, \cdot) : \mathcal{S}^k(G, E) \to F$, $0 \le k \le p$, with*

$$Tf(y) = \sum_{k=0}^{p} \Phi_k\big(y, D^k f(\beta(y))\big) \quad \text{for all } f \in C^p(X,E).$$

(4) *For each $\alpha \in J$, $Y_\alpha$ is a clopen subset of $\beta^{-1}(X)\setminus Y_r$ and there exists $x_\alpha \in X$ such that $Y_\alpha = \beta^{-1}\{x_\alpha\} \setminus Y_r$.*

*Proof.* Let $I = \beta(Y) \setminus X$. For each $x \in I$, let $Y_x = \beta^{-1}\{x\}$. Define $Y_r$ as in the paragraph preceding Proposition 5.3 and let $J = \beta(\beta^{-1}(X) \setminus Y_r)$. For each $x \in J$, let $Y_x = \beta^{-1}\{x\} \setminus Y_r$. Clearly, $\bigcup_{x\in I} Y_x = \beta^{-1}(\mathcal{A}X \setminus X)$ and $\bigcup_{x\in J} Y_x = \beta^{-1}(X) \setminus Y_r$. It follows easily that $(\bigcup_{x\in I} Y_x) \cup Y_r \cup \bigcup_{x\in J} Y_x$ is a partition of $Y$ and that $\beta^{-1}(X) = Y_r \cup \bigcup_{x\in J} Y_x$. By Proposition 5.2, $Y_x$ is a clopen subset of $Y$ for all $x \in I$. Thus, condition (1) holds. As a result, $\beta^{-1}(X) = Y \setminus \bigcup_{x\in I} Y_x$ is closed in $Y$. Condition (3) follows from Proposition 5.3.

Let us show that $Y_r$ is closed in $Y$. It suffices to prove that it is closed in $\beta^{-1}(X)$. Otherwise, there is a sequence $(y_n)$ in $Y_r$ that converges to a point $y_0 \in \beta^{-1}(X)\setminus Y_r$. By Proposition 5.5, there exists $N$ such that $\beta(y_n) = \beta(y_0)$ for all $n \geq N$. Let $f \in C^p(X,E)$ be such that $D^k f(\beta(y_0)) = 0$, $0 \leq k \leq p$. Then $D^k f(\beta(y_n)) = 0$, $0 \leq k \leq p$, for all $n \geq N$. Since $y_n \in Y_r$, $Tf(y_n) = 0$ for all $n \geq N$. Thus $Tf(y_0) = 0$. This proves that $y_0 \in Y_r$, contrary to the assumption. Thus $Y_r$ is closed in $Y$.

Finally, for each $x \in J$, $Y_x$ is clearly a closed subset of $\beta^{-1}(X) \setminus Y_r$. Let $(y_n)$ be a sequence in $\beta^{-1}(X) \setminus Y_r$ that converges to some $y_0 \in Y_x$. By Proposition 5.5, $\beta(y_n) = \beta(y_0)$ for all sufficiently large $n$. Thus $y_n \in Y_x$ for all sufficiently large $n$. This proves that $Y_x$ is open in $\beta^{-1}(X) \setminus Y_r$. ∎

We conclude this section by observing an additional property of the set $Y_r$ which will be used in the next section.

LEMMA 5.7. *Suppose that $y_0 \in \beta^{-1}(X) \setminus Y_r$ and $x_0 = \beta(y_0)$. For any $\varepsilon > 0$ and $M > 0$, there exist $r > 0$ and $f \in C^p(X,E)$ such that $f(x) = 0$ if $x \notin B(x_0, r)$ and*

$$\max_{0\leq k\leq p} \sup_{x\in X} \|(D^k f)(x)\| < \varepsilon \quad \textit{and} \quad \|Tf(y_0)\| > M.$$

*Proof.* Since $y_0 \notin Y_r$, there exists $f_0 \in C^p(X,E)$ such that $D^k f_0(x_0) = 0$, $0 \leq k \leq p$, and $\|Tf_0(y_0)\| > M$. By Taylor's Theorem,

$$\lim_{r\to 0+} \sup_{x\in B(x_0,r)} \frac{\|D^k f_0(x)\|}{r^{p-k}} = 0 \quad \text{for } 0 \leq k \leq p. \tag{5.1}$$

For any $r>0$, define the function $f_r$ by $f_r(x) = \varphi\big(\frac{x-x_0}{r}\big) f_0(x)$. Then $f_r(x) = 0$ if $x \notin B(x_0, r)$. Moreover, since $C = \sup_{0\leq k\leq p} \sup_{x\in X} \|D^k \varphi(x)\| < \infty$, for

$0 \le k \le p$ we have

$$\sup_{x\in X}\|D^k f_r(x)\| \le \sup_{x\in B(x_0,r)}\sum_{j=0}^{k}\binom{k}{j}\|D^j f_0(x)\|\cdot\frac{\|D^{k-j}\varphi(\frac{x-x_0}{r})\|}{r^{k-j}}$$
$$\le 2^k C \max_{0\le j\le k}\sup_{x\in B(x_0,r)}\frac{\|D^j f_0(x)\|}{r^{k-j}}.$$

In view of (5.1), there exists $r > 0$ such that $f_r(x) = 0$ if $x \notin B(x_0, r)$ and that $\max_{0\le k\le p}\sup_{x\in X}\|(D^k f_r)(x)\| < \varepsilon$. There is an open neighborhood $U$ of $x_0$ in $X$ with $f_r = f_0$ on $U$. By Lemma 5.4, $\|Tf_r(y_0)\| = \|Tf_0(y_0)\| > M$. ∎

PROPOSITION 5.8. *Let $(y_n)$ be a sequence in $\beta^{-1}(X)\setminus Y_r$ that converges to a point $y_0 \in Y$. Then $\beta(y_n) = \beta(y_0)$ for all sufficiently large $n$. Consequently, for any $y_0 \in Y$, there exists $\varepsilon > 0$ such that $\beta(y) = \beta(y_0)$ for all $y \in \partial Y_r \cap B(y_0,\varepsilon)$, where $\partial Y_r$ is the boundary of $Y_r$ in $\beta^{-1}(X)$.*

*Proof.* The second statement follows easily from the first. If the first statement fails, we can choose a sequence $(y_n)$ in $\beta^{-1}(X)\setminus Y_r$ converging to some $y_0 \in Y$ such that $(x_n) = (\beta(y_n))$ is a sequence of distinct points in $X$. Let $U_n$ be pairwise disjoint open sets in $X$ with $x_n \in U_n$ for all $n$. By Lemma 5.7, for each $n$, there exist $r_n > 0$ and $f_n \in C^p(X,E)$ such that $f_n(x) = 0$ for all $x \notin B(x_n,r_n)\cap U_n$, $\max_{0\le k\le p}\sup_{x\in X}\|D^k f_n(x)\| \le 1/2^n$ and $\|Tf_n(y_n)\| > n$. Clearly, $f = \sum f_n$ exists pointwise and $f \in C^p(X,E)$. For each $n$, $f = f_n$ on the neighborhood $B(x_n,r_n)\cap U_n$ of $x_n$. By Lemma 5.4, $\|Tf(y_n)\| = \|Tf_n(y_n)\| \ge n$ for all $n$. This is impossible since $(y_n)$ converges to $y_0$ and thus $(Tf(y_n))$ converges in $F$. ∎

**6. Spaces of differentiable functions: Finite-dimensional case.** In this section, let $1 \le p,q < \infty$, $X$ be an open set in $\mathbb{R}^m$, $Y$ be an open set in $\mathbb{R}^n$, and $E,F$ be finite-dimensional Banach spaces. We consider a linear disjointness preserving operator $T : C^p(X,E) \to C^q(Y,F)$ that is *strongly nontrivial*: for any finite subset $A$ of $Y$ and any $y \in Y\setminus A$, there exists $f \in C^p(X,E)$ such that $Tf = 0$ on $A$ and $Tf(y) \ne 0$. Clearly, under this assumption, $T$ is nowhere trivial in the sense of Section 5. In this case, of course, all results in Section 5 apply. As a special case, $T$ is strongly nontrivial if it maps onto $C^q(Y,F)$.

We will refine the representation theorem (Theorem 5.6) and prove that if $p = q$ and $T$ as above is a bijection, then $T^{-1}$ is disjointness preserving. We retain the notation from Theorem 5.6.

PROPOSITION 6.1. *Let $U$ be an infinite subset of $Y_r$. Then $\beta$ cannot be constant on $U$.*

*Proof.* If the proposition fails, there is an infinite sequence $(y_i)$ of distinct points in $Y_r$ and an $x_0 \in X$ such that $\beta(y_i) = x_0$ for all $i$. Consider $k$ with

$0 \le k \le p$. Since $E$ is finite-dimensional, so is $\mathcal{S}^k(\mathbb{R}^m, E)$. Let $(S_j)$ (a finite sequence) be a basis for $\mathcal{S}^k(\mathbb{R}^m, E)$. Define $g_j(y_i) = \Phi_k(y_i, S_j)$ for any $i$ and $j$. For any $f \in C^p(X, E)$, the function $y_i \mapsto \Phi_k(y_i, D^k f(x_0))$ (defined on $(y_i)$) lies in the span of $(g_j)$, a finite sequence. It follows from Theorem 5.6(3) that the space $\{Tf_{|(y_i)} : f \in C^p(X, E)\}$ is finite-dimensional. However, since $T$ is strongly nontrivial, there are functions $f_i$ in $C^p(X, E)$ with $Tf_i(y_l) = 0$ if $l < i$ and $Tf_i(y_i) \neq 0$. Obviously, the functions $Tf_i$ are linearly independent on $(y_i)$, contrary to what was established above. ∎

Since $X$ is locally compact by assumption, it is an open subset in any compactification [DJ, Theorem XI.8.3]. In particular, $X$ is open in $\mathcal{A}X$ and thus $\beta^{-1}(X)$ is open in $Y$. Since $\beta^{-1}(X)$ is also closed in $Y$ by Theorem 5.6, it is a clopen subset of $Y$. In particular, the boundary of $Y_r$ in $\beta^{-1}(X)$ agrees with its boundary in $Y$; we denote both by $\partial Y_r$. The interior of a set $A$ in a topological space is denoted by $\operatorname{int} A$.

Proposition 6.2. *Assume that $n > 1$. Then $\operatorname{int} Y_r$ is a clopen subset of $Y$.*

*Proof.* It suffices to show that $\operatorname{int} Y_r$ is closed in $Y$. Otherwise, there exists $y_0 \in \overline{\operatorname{int} Y_r}$ such that $y_0 \notin \operatorname{int} Y_r$. Since $Y_r$ is closed in $Y$, $y_0 \in Y_r$. Then $y_0 \in \partial Y_r$ and $\beta^{-1}(X)$ is an open set containing $y_0$. By Proposition 5.8, there exists $\varepsilon > 0$ such that $\beta(y) = \beta(y_0)$ for all $y \in \partial Y_r \cap B(y_0, \varepsilon)$. By Proposition 6.1, $\partial Y_r \cap B(y_0, \varepsilon)$ is a finite set. Since $Y$ is an open set in $\mathbb{R}^n$, we may assume that $B(y_0, \varepsilon)$ is a ball in $\mathbb{R}^n$. Therefore, $B(y_0, \varepsilon) \setminus (\partial Y_r \cap B(y_0, \varepsilon))$ is a connected set, since $n > 1$ and the set being subtracted is finite. But

$$B(y_0, \varepsilon) \setminus \partial Y_r = (B(y_0, \varepsilon) \cap \operatorname{int} Y_r) \cup (B(y_0, \varepsilon) \cap Y_r^c)$$

is a partition into two open sets. The sets are nonempty, since $y_0 \in \overline{\operatorname{int} Y_r}$ and $y_0 \in \partial Y_r$. This contradicts the connectedness of $B(y_0, \varepsilon) \setminus \partial Y_r$. ∎

Theorem 6.3. *Let $1 \le p, q < \infty$, $X$ and $Y$ be open sets in $\mathbb{R}^m$ and $\mathbb{R}^n$ respectively, and $E$ and $F$ be finite-dimensional Banach spaces. Suppose that $n > 1$ and that $T : C^p(X, E) \to C^q(Y, F)$ is a strongly nontrivial disjointness preserving linear operator. Denote by $\beta : Y \to \mathcal{A}X$ the support map of $T$. Then there is a partition of $Y$ into clopen sets*

$$Y = \Big(\bigcup_{\alpha \in I} Y_\alpha\Big) \cup \operatorname{int} Y_r \cup \bigcup_{\alpha \in J} Z_\alpha$$

*such that:*

(1) *For each $\alpha \in I$, there exists $x_\alpha \in \mathcal{A}X \setminus X$ such that $Y_\alpha = \beta^{-1}\{x_\alpha\}$.*

(2) *If* $y \in \operatorname{int} Y_r$, *then there are linear operators* $\Phi_k(y,\cdot) : \mathcal{S}^k(\mathbb{R}^m, E) \to F$, $0 \le k \le p$, *such that*

$$Tf(y) = \sum_{k=0}^{p} \Phi_k(y, D^k f(\beta(y))) \qquad \text{for all } f \in C^p(X,E).$$

(3) *For each* $\alpha \in J$, *there exists* $x_\alpha \in X$ *such that* $Z_\alpha = \beta^{-1}\{x_\alpha\} \setminus \operatorname{int} Y_r$.

*Proof.* Define $I$ and $Y_\alpha$, $\alpha \in I$, as in Theorem 5.6. By the same theorem and the observation in the paragraph preceding Proposition 6.2, $(\bigcup_{\alpha\in I} Y_\alpha) \cup \beta^{-1}(X)$ is a clopen partition of $Y$. Let $J = \{x \in X : \beta^{-1}\{x\} \not\subseteq \operatorname{int} Y_r\}$. For each $x \in J$, set $Z_x = \beta^{-1}\{x\} \setminus \operatorname{int} Y_r$. It follows readily from the definitions that $\operatorname{int} Y_r \cup \bigcup_{\alpha \in J} Z_\alpha$ is a partition of $\beta^{-1}(X)$. Since $\operatorname{int} Y_r$ is clopen in $Y$ by Proposition 6.2, to complete the proof it remains to show that $Z_x$ is a clopen set for each $x \in J$.

Obviously, each $Z_x$ is closed. If it is not open, there exists a sequence $(y_k)$ in $Z_x^c$ convergent to some $y_0 \in Z_x$. By Proposition 5.8 and the fact that $\beta^{-1}(X)$ is open, there exists $\varepsilon > 0$ such that $\beta(y) = \beta(y_0)$ for all $y \in \partial Y_r \cap B(y_0, \varepsilon)$. As $\operatorname{int} Y_r$ is clopen and $y_0 \notin \operatorname{int} Y_r$, we may assume that $y_k \notin \operatorname{int} Y_r$ for all $k$. Then $y_k \notin \beta^{-1}\{x\}$ for all $k$. Without loss of generality, we may assume that $(\beta(y_k))$ is a sequence of distinct points convergent to $\beta(y_0)$. By Proposition 5.8, we may assume that $y_k \in Y_r$ for all $k$. Since $y_k$ and $y_0$ are not in $\operatorname{int} Y_r$, we see that $y_k, y_0 \in \partial Y_r$. But then for sufficiently large $k$, $y_k \in \partial Y_r \cap B(y_0, \varepsilon)$ and yet $\beta(y_k) \ne \beta(y_0)$, contradicting what was established above. ∎

Corollary 6.4. *Let* $1 \le p, q < \infty$, $X$ *and* $Y$ *be open sets in* $\mathbb{R}^m$ *and* $\mathbb{R}^n$ *respectively, and* $E$ *and* $F$ *be finite-dimensional Banach spaces. Suppose that* $n > 1$ *and* $T : C^p(X,E) \to C^q(Y,F)$ *is a strongly nontrivial disjointness preserving linear operator. Denote by* $\beta : Y \to \mathcal{A}X$ *the support map of* $T$. *If* $Y$ *is connected and* $\beta(Y)$ *contains at least two points, then for each* $y \in Y$, *there are linear operators* $\Phi_k(y,\cdot) : \mathcal{S}^k(\mathbb{R}^m, E) \to F$, $0 \le k \le p$, *such that*

$$Tf(y) = \sum_{k=0}^{p} \Phi_k(y, D^k f(\beta(y))) \qquad \text{for all } f \in C^p(X,E).$$

Remark. In Corollary 6.4, $C^p(X,E)$ and $C^q(Y,F)$ are Fréchet spaces under their respective topologies of uniform convergence of derivatives of all orders ($\le p$ and $\le q$ respectively) on compact sets. In the given representation of $T$, the operators $\Phi_k(y,\cdot)$, $0 \le k \le p$, $y \in Y$, are bounded. A standard application of the Closed Graph Theorem shows that $T$ is a continuous operator.

Our final result is to show that if $T : C^p(X, \mathbb{R}^l) \to C^p(Y, \mathbb{R}^l)$ is a disjointness preserving linear bijection, where $X$ and $Y$ are open sets in $\mathbb{R}^m$

and $\mathbb{R}^n$ respectively, and $m, n, l \in \mathbb{N}$, then $T^{-1}$ is disjointness preserving. First we will prove an analog of Proposition 2.4.

PROPOSITION 6.5. *Suppose that* $T : C^p(X, \mathbb{R}^l) \to C^p(Y, \mathbb{R}^l)$ *is a disjointness preserving linear bijection, where* $X$ *and* $Y$ *are open sets in* $\mathbb{R}^m$ *and* $\mathbb{R}^n$ *respectively. If* $\beta(\operatorname{int} Y_r)$ *is dense in* $X$*, then* $T^{-1}$ *is disjointness preserving.*

*Proof.* Consider the operator $\widetilde{T} : C^p(X, \mathbb{R}^l) \to C^p(\operatorname{int} Y_r, \mathbb{R}^l)$ given by $\widetilde{T} f = T f_{|\operatorname{int} Y_r}$. By Proposition 5.3, for each $y \in \operatorname{int} Y_r$, there are linear operators $\Phi_k(y, \cdot) : \mathcal{S}^k(\mathbb{R}^m, \mathbb{R}^l) \to \mathbb{R}^l$, $0 \le k \le p$, such that

$$\widetilde{T} f(y) = \sum_{k=0}^{p} \Phi_k(y, D^k f(\beta(y))) \quad \text{for all } f \in C^p(X, E) \text{ and all } y \in \operatorname{int} Y_r.$$

As in the Remark after Corollary 6.4, it follows that $\widetilde{T}$ is a continuous linear operator. By [LW, Theorem 10], $\Phi_k(y, S)$ is a continuous function of $y$ for any fixed $k$ and $S$.

Suppose that there exists $y_0 \in \operatorname{int} Y_r$ and $k > 0$ such that $\Phi_k(y_0, \cdot) \ne 0$. There is a neighborhood $U$ of $y_0$ contained in $\operatorname{int} Y_r$ with $\Phi_k(y, \cdot) \ne 0$ for all $y \in U$. By [LW, Theorem 12], for all $z \in U$, there exist $\varepsilon > 0$ and $C < \infty$ such that $\|\beta(y) - \beta(z)\|^{p-k} \le C\|y - z\|^p$ for all $y \in B(z, \varepsilon)$. Then $\beta$ is differentiable at $z$ and $D\beta(z) = 0$. Since this holds for all $z$ in the open set $U$, $\beta$ is constant on $U$. This contradicts Proposition 6.1.

Therefore, $\widetilde{T} f(y) = \Phi_0(y, f(\beta(y)))$ for all $f \in C^p(X, \mathbb{R}^l)$ and all $y \in \operatorname{int} Y_r$. For all $v \in \mathbb{R}^l$, there exists $f \in C^p(X, \mathbb{R}^l)$ such that $Tf(y) = v$ for all $y \in Y$. In particular, $\Phi_0(y, f(\beta(y))) = v$ for all $y \in \operatorname{int} Y_r$. This shows that $\Phi_0(y, \cdot) : \mathbb{R}^l \to \mathbb{R}^l$ is onto for all $y \in \operatorname{int} Y_r$. Hence it is also one-to-one.

Now suppose that $f, g \in C^p(X, \mathbb{R}^l)$ are not disjoint. Since $\beta(\operatorname{int} Y_r)$ is dense in $X$, there exists $x_0 \in \beta(\operatorname{int} Y_r)$ such that $f(x_0), g(x_0) \ne 0$. Choose $y_0 \in \operatorname{int} Y_r$ with $\beta(y_0) = x_0$. Since $\Phi_0(y_0, \cdot)$ is one-to-one, $\Phi_0(y_0, f(x_0))$ and $\Phi_0(y_0, g(x_0))$ are not zero. Therefore, $Tf(y_0) = \widetilde{T} f(y_0) \ne 0$ and $Tg(y_0) = \widetilde{T} g(y_0) \ne 0$. This proves that $T^{-1}$ is disjointness preserving. ∎

THEOREM 6.6. *Suppose that* $T : C^p(X, \mathbb{R}^l) \to C^p(Y, \mathbb{R}^l)$ *is a disjointness preserving linear bijection, where* $X$ *and* $Y$ *are open sets in* $\mathbb{R}^m$ *and* $\mathbb{R}^n$ *respectively. If* $Y$ *has only finitely many connected components, then* $T^{-1}$ *is disjointness preserving.*

*Proof.* Using the same proof as for Proposition 2.1, one can show that $\beta(Y)$ is dense in $\mathcal{A}X$. Since $X$ is open in $\mathcal{A}X$, $\beta(Y) \cap X$ is dense in $X$.

First consider the case where $n > 1$. By Theorem 6.3 and the assumption that $Y$ has finitely many connected components, $\beta(Y) \cap X = \beta(\operatorname{int} Y_r) \cup A$, where $A$ is a finite subset of $X$. Taking closures in $X$, we have

$$X = \overline{\beta(\operatorname{int} Y_r)} \cup A.$$

Since $X$ is an open set in $\mathbb{R}^m$, it follows that $X = \overline{\beta(\operatorname{int} Y_r)}$. By Proposition 6.5, $T^{-1}$ is disjointness preserving.

Now, consider the case where $n = 1$. We will show that

$$(\beta(Y) \cap X) \setminus \overline{\beta(\operatorname{int} Y_r)}$$

is a finite set, where the closure is taken in $X$. Since it was observed above that $\beta(Y) \cap X$ is dense in $X$, and since $X$ has no isolated points, this will show that $\beta(\operatorname{int} Y_r)$ is dense in $X$. As $Y$ is an open set in $\mathbb{R}$ that has finitely many connected components, we may write $Y$ as a finite union $\bigcup_{i=1}^{j} H_i$ of pairwise disjoint open intervals.

Claim. *Suppose that $x_0 \in (\beta(Y) \cap X) \setminus \overline{\beta(\operatorname{int} Y_r)}$. Let $y_0 \in Y$ be such that $\beta(y_0) = x_0$ and let $y_0 \in H_i$ for some $1 \leq i \leq j$. Then $\beta(H_i) = \{x_0\}$.*

We will show that $K = H_i \cap \beta^{-1}\{x_0\}$ is a clopen subset of $H_i$. Since $H_i$ is connected and $K \supseteq \{y_0\}$ is nonempty, it will follow that $K = H_i$. Thus $\beta(H_i) = \{x_0\}$, as desired. Clearly $K$ is closed in $H_i$. Hence it suffices to show that it is open in $H_i$.

Assume otherwise. There is a sequence $(y_k)$ in $H_i \setminus K$ that converges to a point $z \in K$. By taking a subsequence if necessary, we may assume that $(y_k)$ is strictly monotone and that $(\beta(y_k))$ is a sequence of distinct points. By Proposition 5.8, all but finitely many $y_k$ belong to $Y_r$. Without loss of generality, we may assume that all $y_k$ are in $Y_r$. For each $k$, the open interval $U_k$ with end points $y_k$ and $y_{k+1}$ is contained in $H_i$. If $U_k \subseteq Y_r$, then clearly $y_k \in \overline{\operatorname{int} Y_r}$, and hence $\beta(y_k) \in \overline{\beta(\operatorname{int} Y_r)}$. If this occurs for infinitely many $k$, then $x_0 = \beta(z) = \lim \beta(y_k) \in \overline{\beta(\operatorname{int} Y_r)}$, contrary to the choice of $x_0$. Therefore, we may assume that $U_k \not\subseteq Y_r$ for all $k$. Choose $z_k \in U_k \setminus Y_r$ for each $k$. Since $\beta^{-1}(X)$ is clopen in $Y$ and $y_0 \in \beta^{-1}(X) \cap H_i$, we get $H_i \subseteq \beta^{-1}(X)$. Thus $(z_k) \subseteq \beta^{-1}(X) \setminus Y_r = \bigcup_{\alpha \in J} Y_\alpha$, in the notation of Theorem 5.6. Furthermore, as $(z_k)$ converges to $z$, by Proposition 5.8, we may assume that $\beta(z_k) = \beta(z) = x_0$ for all $k$. For each $k$, choose $\alpha_k \in J$ such that $z_k \in Y_{\alpha_k}$. Since $Y_{\alpha_k}$ is an open set in $\beta^{-1}(X)$, it is open in $\mathbb{R}$. Let $W_k$ be the maximal open interval in $Y_{\alpha_k}$ containing $z_k$ and let $w_k$ be the right end point of $W_k$. Clearly, $w_k$ lies between $y_k$ and $y_{k+1}$; hence $w_k \in H_i$ and $(w_k)$ converges to $z$. By the maximality of the interval $W_k$, we must have $w_k \in Y_r$. For each $k$, by the continuity of $\beta$, $\beta(w_k) = \beta(z_k) = x_0$. Since $(w_k)$ is a sequence of distinct points in $Y_r$, we have a contradiction with Proposition 6.1. This completes the proof of the Claim.

As there are only finitely many intervals $H_i$, it follows immediately from the Claim that $(\beta(Y) \cap X) \setminus \overline{\beta(\operatorname{int} Y_r)}$ is a finite set. This completes the proof of the theorem. ∎

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