ANNALES
POLONICI MATHEMATICI
117.3 (2016)

Finite determinacy of non-isolated singularities

Piotr Migus, Tomasz Rodak and Stanisław Spodzieja (Łódź)

Abstract. We give, in terms of the Łojasiewicz inequality, a sufficient condition for C^k mapping germs of non-isolated singularity at zero to be isotopic.

1. Introduction and results. Let $F : (\mathbb{R}^n, a) \to \mathbb{R}^m$ denote a mapping defined in a neighbourhood of $a \in \mathbb{R}^n$ with values in $\mathbb{R}^m$. If $F(a) = b$, we write $F : (\mathbb{R}^n, a) \to (\mathbb{R}^m, b)$. We denote by ∇f the gradient of a C^1-function $f : (\mathbb{R}^n, a) \to \mathbb{R}$. We set $V(\nabla f) = \{x : \nabla f(x) = 0\}$. Let $|\cdot|$ denote a norm in $\mathbb{R}^n$, and $\operatorname{dist}(x, V)$ the distance of a point $x \in \mathbb{R}^n$ to a set $V \subset \mathbb{R}^n$ (with $\operatorname{dist}(x, V) = 1$ if $V = \emptyset$).

By a *k-jet* at $a \in \mathbb{R}^n$ in the class C^l, we mean a family of C^l functions $(\mathbb{R}^n, a) \to \mathbb{R}$, called C^l*-realisations* of this jet, possessing the same Taylor polynomial of degree k at a. The k-jet is said to be C^r*-sufficient* (respectively *v-sufficient*) in the class C^l if for any two of his C^l-realisations f and g there exists a C^r diffeomorphism $\varphi : (\mathbb{R}^n, a) \to (\mathbb{R}^n, a)$ such that $f \circ \varphi = g$ in a neighbourhood of a (respectively the germs of $f^{-1}(0)$ and $g^{-1}(0)$ at 0 are homeomorphic) (R. Thom [26]).

In the paper we will consider k-jets in the class C^k, which we call briefly k-jets.

A classical result on sufficiency of jets is the following:

Theorem 1.1 (Kuiper, Kuo, Bochnak–Łojasiewicz). *Let w be a k-jet at $0 \in \mathbb{R}^n$ and let f be its C^k realisation. If $f(0) = 0$ then the following conditions are equivalent:*

(a) *w is C^0-sufficient in the class C^k,*
(b) *w is v-sufficient in the class C^k,*
(c) *$|\nabla f(x)| \geq C|x|^{k-1}$ as $x \to 0$ for some constant $C > 0$.*

2010 *Mathematics Subject Classification*: Primary 14P20; Secondary 14P10, 32C07.
Key words and phrases: isotopy, isotopic mappings, Łojasiewicz exponent, polynomial mapping, C^2 mapping.
Received 8 July 2015; revised 23 May 2016.
Published online 26 August 2016.

DOI: 10.4064/ap3835-5-2016

The implication (c)⇒(a) was proved by N. H. Kuiper [10] and T. C. Kuo [11], (b)⇒(c) by J. Bochnak and S. Łojasiewicz [2], and the implication (a)⇒(b) is obvious (see also [16], [23]). In the complex case the analogous result was proved by S. H. Chang and Y. C. Lu [4], B. Teissier [25] and J. Bochnak and W. Kucharz [1]. Similar considerations can be carried out for functions in a neighbourhood of infinity (see [3], [22], [19]).

Theorem 1.1 concerns the isolated singularity of f at 0, i.e. the point 0 is an isolated zero of ∇f. The case of non-isolated singularities of real functions was investigated by many authors, for instance by J. Damon and T. Gaffney [5], T. Fukui and E. Yoshinaga [7], V. Grandjean [8], X. Xu [28], and for complex functions, by D. Siersma [20, 21] and R. Pellikaan [17].

The purpose of this article is to generalise the above results to C^k mappings in a neighbourhood of zero with non-isolated singularity at zero. Recall the definition of a k-Z-jet in the class of functions with non-isolated singularity at zero (cf. [28]).

We denote by $\mathcal{C}_a^k(n,m)$ the set of C^k mappings $(\mathbb{R}^n,a)\to\mathbb{R}^m$. For $f\in\mathcal{C}_a^k(n,1)$, we denote by $j^kf(a)$ the k-jet at a (in the class C^k) determined by f. For $F=(f_1,\ldots,f_m)\in\mathcal{C}_a^k(n,m)$ we set $j^kF(a)=(j^kf_1(a),\ldots,j^kf_m(a))$.

Let $Z\subset\mathbb{R}^n$ be a set with $0\in Z$ and let $k\in\mathbb{Z}$, $k>0$. By a *k-Z-jet in the class $\mathcal{C}_0^k(n,m)$*, or briefly just a *k-Z-jet*, we mean an equivalence class $w\subset\mathcal{C}_0^k(n,m)$ of the equivalence relation $\sim$ defined by $F\sim G$ iff for some neighbourhood $U\subset\mathbb{R}^n$ of the origin, $j^kF(a)=j^kG(a)$ for $a\in Z\cap U$ (cf. [28]). The mappings $F\in w$ are called *C^k-Z-realisations* of the jet w and we write $w=j_Z^kF$. The set of all jets j_Z^kF is denoted by $J_Z^k(n,m)$.

The k-Z-jet $w\in J_Z^k(n,m)$ is said to be *C^r-Z-sufficient* (resp. *Z-v-sufficient*) in the class C^k if for any two of its C^k-Z-realisations f and g there exist neighbourhoods $U_1,U_2\subset\mathbb{R}^n$ of 0 and a C^r diffeomorphism $\varphi:U_1\to U_2$ such that $f\circ\varphi=g$ in U_1 (resp. a homeomorphism $\varphi:[f^{-1}(0)\cup Z]\cap U_1\to[g^{-1}(0)\cup Z]\cap U_2$) with $\varphi(0)=0$ and $\varphi(Z\cap U_1)=Z\cap U_2$.

The following Kuiper and Kuo criterion (Theorem 1.1, (c)⇒(a)) for jets with non-isolated singularity was proved by X. Xu [28].

THEOREM 1.2. *Let $Z\subset\mathbb{R}^n$ be a closed set such that $0\in Z$. If $f\in\mathcal{C}^k(n,1)$ with $V(\nabla f)\subset Z$ satisfies the condition*

$$(1)\qquad |\nabla f(x)|\ge C\operatorname{dist}(x,Z)^{k-1}\quad \text{as } x\to 0 \text{ for some constant } C>0,$$

then the k-Z-jet of f is C^0-Z-sufficient.

The main result of this paper is Theorem 1.3 below. It is a generalisation of Theorem 1.2 to the case of mapping jets. Let us start with some definition. Let X,Y be Banach spaces over $\mathbb{R}$. Let $L(X,Y)$ denote the Banach space of continuous linear mappings from X to Y. For $A\in L(X,Y)$, A^* stands for the adjoint operator in $L(Y',X')$, where X' is the dual space of X. For

$A \in L(X,Y)$ we set

$$\nu(A) = \inf\{\|A^*\varphi\| : \varphi \in Y',\ \|\varphi\| = 1\}, \tag{2}$$

where $\|A\|$ is the norm of the linear mapping A (see [18]). If $f \in \mathcal{C}_0^k(n,1)$ we have $\nu(df) = |\nabla f|$, where df is the differential of f.

THEOREM 1.3. *Let $f\colon (\mathbb{R}^n,0) \to (\mathbb{R}^m,0)$, where $m \le n$, be a C^k-Z-realisation of a k-Z-jet $w \in J_Z^k(n,m)$, where $k > 1$, $Z = \{x \in \mathbb{R}^n : \nu(df(x)) = 0\}$ and $0 \in Z$. Assume that for a positive constant C,*

$$\nu(df(x)) \ge C \operatorname{dist}(x,Z)^{k-1} \quad \text{as } x \to 0. \tag{3}$$

Then the jet w is C^0-Z-sufficient in the class C^k. Moreover, for any C^k-Z-realisations f_1, f_2 of w, the deformation $f_1 + t(f_2 - f_1)$, $t \in \mathbb{R}$, is topologically trivial along $[0,1]$. In particular the mappings f_1 and f_2 are isotopic at zero.

For the definition of isotopy and topological triviality see Subsection 2.3. By Lemmas 2.2 and 2.7, Theorem 1.3 is also true for holomorphic mappings. It is not clear to the authors whether the inverse to Theorem 1.3 holds. In the proof of Theorem 1.3, given in Section 2, we use the method of proof of [19, Theorem 1].

When Z is an algebraic or analytic set, some algebraic conditions for finite determinacy of a smooth function jet were obtained by L. Kushner [13] and L. Kushner and B. Terra Leme [14]. In those two papers, the authors use the idea of J. Mather [15] and J.-C. Tougeron [27]. For non-degenerate analytic functions f, g, some conditions for topological triviality of the deformation $f + tg$, $t \in [0,1]$, in terms of Newton polyhedra were obtained by J. Damon and T. Gaffney [5], and for blow analytic triviality by T. Fukui and E. Yoshinaga [7] (see also [24], [29]).

From the proof of Theorem 1.3 we obtain a version of the theorem for C^1 functions with locally Lipschitz differentials.

COROLLARY 1.4. *Let $f, f_1 : (\mathbb{R}^n,0) \to (\mathbb{R}^m,0)$ be differentiable mappings with locally Lipschitz differentials $df, df_1 : (\mathbb{R}^n,0) \to L(\mathbb{R}^n,\mathbb{R}^m)$, let $Z = \{x \in \mathbb{R}^n : \nu(df(x)) = 0\}$, and suppose $0 \in Z$. If*

$$\nu(df(x)) \ge C \operatorname{dist}(x,Z), \tag{4}$$

$$|f(x) - f_1(x)| \le C_1 \nu(df(x))^2, \tag{5}$$

$$\|df(x) - df_1(x)\| \le C_2 \nu(df(x)) \tag{6}$$

as $x \to 0$ for some constants $C, C_1, C_2 > 0$, $C_2 < 1/2$, then the deformation $f + t(f_1 - f)$ is topologically trivial along $[0,1]$. In particular f and f_1 are isotopic at zero.

The proof of the above corollary is given in Subsection 2.5.

In Section 3 we prove the following theorem of Bochnak–Łojasiewicz type (cf. implication (b)⇒(c) in Theorem 1.1), that Z-v-sufficiency of jets implies the Łojasiewicz inequality, provided $f(0) = 0$ for C^k-Z-realisations f of the jet:

THEOREM 1.5. *Let $Z \subset \mathbb{R}^n$ with $0 \in Z$, let w be a k-Z-jet, $k > 1$, and let f be its C^k-Z-realisation. If w is Z-v-sufficient in the class C^k, $f(0) = 0$ and $V(\nabla f) \subset Z$, then*

$$|\nabla f(x)| \geq C \operatorname{dist}(x, Z)^{k-1} \quad \text{as } x \to 0 \text{ for some constant } C > 0. \tag{7}$$

It is obvious that a C^0-Z-sufficient jet is also Z-v-sufficient, so in a sense Theorem 1.5 is inverse to Theorem 1.2.

2. Proof of Theorem 1.3

2.1. Differential equations. Let us start by recalling the following

LEMMA 2.1. *Let $G \subset \mathbb{R} \times \mathbb{R}^n$ be an open set, $W : G \to \mathbb{R}^n$ a continuous mapping and $V \subset \mathbb{R}^n$ a closed set. If in $G \backslash (\mathbb{R} \times V)$ the system*

$$\frac{dy}{dt} = W(t, y) \tag{8}$$

has global unique solutions and there exists a neighbourhood $U \subset G$ of the set $(\mathbb{R} \times V) \cap G$ and a positive constant C such that

$$|W(t, x)| \leq C \operatorname{dist}(x, V) \quad \text{for } (t, x) \in U, \tag{9}$$

then the system (8) in G has global unique solutions.

2.2. The Rabier function. Let X, Y be Banach spaces over $\mathbb{K}$, where $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$. We begin by recalling some properties of the Rabier function $\nu(\cdot)$ (cf. [19]).

LEMMA 2.2 ([12]). *Let Σ be the set of operators $A \in L(X, Y)$ such that $A(X) \subsetneq Y$. Then*

$$\nu(A) = \operatorname{dist}(A, \Sigma), \qquad A \in L(X, Y).$$

LEMMA 2.3 ([18]). *Let $A, B \in L(X, Y)$. Then*

$$|\nu(A) - \nu(B)| \leq \|A - B\|.$$

In particular $\nu : L(X, Y) \to \mathbb{R}$ is Lipschitz.

From Lemma 2.3 we have

LEMMA 2.4. *If $A, B \in L(X, Y)$ then*

$$\nu(A + B) \geq \nu(A) - \|B\|.$$

DEFINITION 1 ([9]). Let $\mathbf{a} = [a_{ij}]$ be the matrix of $A \in L(\mathbb{K}^n, \mathbb{K}^m)$, $n \geq m$. For a subsequence $I = (i_1, \ldots, i_m)$ of $(1, \ldots, n)$, we denote by $M_I(A)$ the $m \times m$ minor of $\mathbf{a}$ given by the columns indexed by I. Moreover, if $J = (j_1, \ldots, j_{m-1})$ is a subsequence of $(1, \ldots, n)$ and $j \in \{1, \ldots, m\}$, then $M_J(j)(A)$ denotes the $(m-1) \times (m-1)$ minor of $\mathbf{a}$ given by the columns indexed by J and with the jth row deleted (if $m=1$ we set $M_J(j)(A)=1$). Let

$$h_I(A) = \max\{|M_J(j)(A)| : J \subset I,\ j=1,\dots,m\},$$
$$g'(A) = \max_I \frac{|M_I(A)|}{h_I(A)}.$$

Here we set $0/0=0$. If $m=n$, we set $h_I = h$.

LEMMA 2.5 ([9]). *There exist $C_1, C_2 > 0$ such that for any $A \in L(\mathbb{K}^n, \mathbb{K}^m)$ we have*

$$C_1 g'(A) \le \nu(A) \le C_2 g'(A).$$

COROLLARY 2.6 ([19]). *The function g' is continuous.*

LEMMA 2.7 ([12]). *Assume that X, Y are complex Banach spaces. Let $\Sigma_{\mathbb{C}}$ (resp. $\Sigma_{\mathbb{R}}$) be the set of non-surjective $\mathbb{C}$-linear (resp. $\mathbb{R}$-linear) continuous maps from X to Y. Then for any continuous $\mathbb{C}$-linear map $A\colon X \to Y$,*

$$\operatorname{dist}(A, \Sigma_{\mathbb{C}}) = \operatorname{dist}(A, \Sigma_{\mathbb{R}}).$$

2.3. Isotopy and triviality. Let $\Omega \subset \mathbb{R}^n$ be a neighbourhood of $0 \in \mathbb{R}^n$ and let $Z \subset \mathbb{R}^n$ be a set such that $0 \in Z$.

We will say that a continuous mapping $H\colon \Omega \times [0,1] \to \mathbb{R}^n$ is an *isotopy near Z at zero*, or briefly a $(Z,0)$-isotopy, if

(a) $H_0(x) = x$ for $x \in \Omega$ and $H_t(x) = x$ for $t \in [0,1]$ and $x \in \Omega \cap Z$,
(b) for any t the mapping H_t is a homeomorphism onto $H_t(\Omega)$,

where the mapping $H_t : \Omega \to \mathbb{R}^n$ is defined by $H_t(x) = H(x,t)$ for $x \in \Omega$ and $t \in [0,1]$.

Let $f\colon \Omega_1 \to \mathbb{R}^m$ and $g\colon \Omega_2 \to \mathbb{R}^m$ where $\Omega_1, \Omega_2 \subset \mathbb{R}^n$ are neighbourhoods of $0 \in \mathbb{R}^n$, and let $Z \subset \mathbb{R}^n$ with $0 \in Z$. We call f and g *isotopic near Z at zero* if there exists a $(Z,0)$-isotopy $H : \Omega \times [0,1] \to \mathbb{R}^n$ with $\Omega \subset \Omega_1 \cap \Omega_2$ such that $f(H_1(x)) = g(x)$ for all $x \in \Omega$.

Let $h : \Omega_3 \to \mathbb{R}^m$, where $\Omega_3 \subset \mathbb{R}^n$ is a neighbourhood of $0 \in \mathbb{R}^n$. We say that a deformation $f + th$ is *topologically trivial near Z* along $[0,1]$ if there exists a $(Z,0)$-isotopy $H : \Omega \times [0,1] \to \mathbb{R}^m$, $\Omega \subset \Omega_1 \cap \Omega_2$, such that $f(H(t,x)) + th(H(t,x))$ does not depend on t.

2.4. Proof of Theorem 1.3. Let dP denote the differential of P, and $dP(x)$ the differential of P at the point x. By d_xP we denote the differential of P with respect to the variables x.

Let $f, f_1 \in w$ and let $P = f_1 - f = (P_1, \dots, P_m)$. Then $j^kP(a) = 0$ for $a \in Z \cap U$, for some neighbourhood $U \subset \mathbb{R}^n$ of 0. Consequently, decreasing U if necessary, we may assume that

$$(10) \qquad |P(x)| \le \frac{C}{3}\operatorname{dist}(x,Z)^k \quad \text{and} \quad \|dP(x)\| \le \frac{C}{3}\operatorname{dist}(x,Z)^{k-1}$$

for $x \in U$.

Consider the mapping $F\colon \mathbb{R}\times\mathbb{R}^n \to \mathbb{R}^m$ defined by

$$F(\xi,x) = f(x) + \xi P(x).$$

Fix $\xi \in (-2,2)$. By (10) and Lemma 2.4 we get

$$\nu(d_x F(\xi,x)) \ge \nu(df(x)) - |\xi|\,\|dP(x)\| \ge \frac{C}{3}\operatorname{dist}(x,Z)^{k-1}, \qquad x\in U.$$

Thus by Lemma 2.5 there exists $C' > 0$ such that

$$g'(d_x F(\xi,x)) \ge C' \operatorname{dist}(x,Z)^{k-1}, \quad \xi\in(-2,2),\ x\in U. \tag{11}$$

Set $G = \{(\xi,x)\in\mathbb{R}\times U : |\xi| < 2\}$. In the notation of Definition 1 we set

$$A_I = \left\{(\xi,x)\in G : \frac{|M_I(d_xF(\xi,x))|}{h_I(d_xF(\xi,x))} \le \frac{C'}{2}\operatorname{dist}(x,Z)^{k-1}\right\}.$$

By Corollary 2.6 the sets A_I are closed in G and $(\mathbb{R}\times Z)\cap G \subset A_I$. From (11) we see that $\{G\setminus A_I\}_I$ is an open covering of $G\setminus(\mathbb{R}\times Z)$. Let $\{\delta_I\}_I$ be a C^∞ partition of unity associated to this covering.

Consider the system of linear equations

$$(d_xF(\xi,x))W(\xi,x)^T = -P(x)^T \tag{12}$$

with unknowns $W(\xi,x) = (W_1(\xi,x),\dots,W_n(\xi,x))$ and parameters $(\xi,x) \in G$. Take any subsequence $I = (i_1,\dots,i_m)$ of $(1,\dots,n)$. For simplicity of notation we assume that $I = (1,\dots,m)$. For all $(\xi,x)\in G$ such that $M_I(d_xF(\xi,x)) \neq 0$ we define $W^I(\xi,x) = (W^I_1(\xi,x),\dots,W^I_n(\xi,x))$ by

$$W^I_l(\xi,x) = \sum_{j=1}^{m}(-P_j(x))(-1)^{l+j}\frac{M_{I\setminus l}(j)(d_xF(\xi,x))}{M_I(d_xF(\xi,x))}, \quad l = 1,\dots,m,$$

$$W^I_l(\xi,x) = 0, \quad l = m+1,\dots,n,$$

where $I\setminus l = (1,\dots,l-1,l+1,\dots,m)$ for $l = 1,\dots,m$. Cramer's rule implies

$$(d_xF(\xi,x))W^I(\xi,x)^T = -P(x)^T.$$

Since $k>1$, $\delta_I W^I$ is a C^1 mapping on $G\setminus(\mathbb{R}\times Z)$ (after suitable extension). Hence $W = \sum_I \delta_I W^I$ is also a C^1 mapping on $G\setminus(\mathbb{R}\times Z)$. We set $W(\xi,x) = 0$ for $(\xi,x)\in(\mathbb{R}\times Z)\cap G$. It is easy to see that W satisfies equation (12).

Observe that

$$\|W(\xi,x)\| \le C''\operatorname{dist}(x,Z), \quad \xi\in(-2,2),\ x\in U, \tag{13}$$

where $C'' = 2mC\sqrt{n}/(3C')$. Indeed from (10), the definition of A_I, the choice of P and the above construction we get

$$\begin{aligned}\|W(\xi,x)\| &\le \sum_{\{I\,:\,\delta_I(\xi,x)\neq 0\}} \delta_I(\xi,x)\|W^I(\xi,x)\| \\ &\le \sum_{\{I\,:\,\delta_I(\xi,x)\neq 0\}} \delta_I(\xi,x)\sqrt{n}\max_{l=1}^{m}|W^I_l(\xi,x)| \\ &\le \sum_{\{I\,:\,\delta_I(\xi,x)\neq 0\}} \delta_I(\xi,x)\sqrt{n}\sum_{j\in I}|P_j(x)|\frac{h_I(d_xF(\xi,x))}{|M_I(d_xF(\xi,x))|} \\ &\le \sum_{\{I\,:\,\delta_I(\xi,x)\neq 0\}} \delta_I(\xi,x)\sqrt{n}\sum_{j\in I}\frac{C}{3}\operatorname{dist}(x,Z)^k\frac{2}{C'}\frac{1}{\operatorname{dist}(x,Z)^{k-1}} \\ &= m\sqrt{n}\,\frac{C}{3}\,\frac{2}{C'}\operatorname{dist}(x,Z).\end{aligned}$$

Consider the system of differential equations

$$y' = W(t,y). \tag{14}$$

Since W is at least of class C^1 on $G\setminus(\mathbb{R}\times Z)$, it is a locally lipschitzian vector field. As a consequence, the above system has uniqueness of solutions in $G\setminus(\mathbb{R}\times Z)$. Hence, inequality (13) and Lemma 2.1 imply the global uniqueness of solutions of (14) in G.

Choose $(\xi,x)\in G$ and define $\varphi_{(\xi,x)}$ to be the maximal solution of (14) such that $\varphi_{(\xi,x)}(\xi)=x$. Set $\Omega_0=\{x\in\mathbb{R}^n : \|x\|<r_0\}$ and $\Omega_1=\{x\in\mathbb{R}^n : \|x\|<r_1\}$, where $r_0,r_1>0$. Since $0\in Z$, the mapping $\varphi(\xi)=0$, $\xi\in\mathbb{R}$, is a solution of (14). Hence for sufficiently small r_0,r_1, for any $x\in\Omega_0$, the solution $\varphi_{(0,x)}$ is defined on $[0,1]$ and $\varphi_{(0,x)}(t)\in\Omega_1$, for all $t\in[0,1]$; moreover, for any $x\in\Omega_1$, the solution $\varphi_{(1,x)}$ is also defined on $[0,1]$. Let $H,\widetilde{H}\colon\Omega_0\times[0,1]\to\Omega_1$ be given by

$$H(x,t)=\varphi_{(0,x)}(t),\quad \widetilde{H}(y,t)=\varphi_{(t,y)}(0).$$

The mappings $H,\widetilde{H}$ are well defined. Moreover, one can extend them to continuous mappings on some open neighbourhood of $\Omega_0\times[0,1]$. Set $\Omega=\Omega_1$ and $\Omega^t=\{y\in\mathbb{R}^n : \widetilde{H}(y,t)\in\Omega\}$, $t\in[0,1]$. By uniqueness of solutions of (14), for any t we have $\widetilde{H}(H(x,t),t)=x$, $H(x,0)=x$, $x\in\Omega$, and $H(\widetilde{H}(y,t))=y$, $y\in\Omega^t$. Moreover there exists a neighbourhood $\Omega'\subset\mathbb{R}^n$ of 0 such that $\Omega'\subset\Omega^t$ for any t.

Finally, by (12) we have

$$\frac{d}{dt}F(t,\varphi_{(\xi,x)}(t))^T = P(x)^T + (d_xF)(t,\varphi_{(\xi,x)}(t))W(t,\varphi_{(\xi,x)}(t))^T = 0,$$

so $F(t,\varphi_{(0,x)}(t))=f(x)$, and consequently $f(H(x,1))+tP(H(x,1))=f(x)$ for $t\in[0,1]$ and $x\in\Omega'$. This ends the proof. ∎

2.5. Proof of Corollary 1.4. Under the notation of the proof of Theorem 1.3, by (4), (6) and Lemma 2.3 we obtain $\nu(d_xF(\xi,x)) = \nu(df(x) + \xi dP(x)) \geq \nu(df(x)) - |\xi|\,\|dP(x)\| \geq C(1-2C_2)\operatorname{dist}(x,Z)$, $x \in U$. Obviously $C(1-2C_2) > 0$. Then there exists $C' > 0$ such that

$$g'(d_xF(\xi,x)) \geq C'\operatorname{dist}(x,Z), \quad \xi \in (-2,2),\ x \in U. \tag{15}$$

So, we will use (15) instead of (11). By (5) we obtain (13). Moreover, the assumption that df and df_1 are locally Lipschitz mappings implies that the mapping W is locally Lipschitz outside $(-2,2) \times Z$. Then, by the same argument as in the proof of Theorem 1.3, we deduce the assertion. ■

3. Proof of Theorem 1.5. We will use the idea from [2]. It suffices to prove the theorem for $Z = V(\nabla f)$. Suppose to the contrary that for any neighbourhood U of 0 and for any constant $C > 0$ there exists $x \in U$ such that

$$|\nabla f(x)| < C\operatorname{dist}(x,Z)^{k-1}.$$

Then for some sequence $(a_\mathrm{v}) \subset \mathbb{R}^n \backslash Z$ such that $a_\mathrm{v} \to 0$ when $\mathrm{v} \to \infty$ we have

$$|\nabla f(a_\mathrm{v})| \leq \frac{1}{\mathrm{v}}\operatorname{dist}(a_\mathrm{v},Z)^{k-1} \quad \text{for } \mathrm{v} \in \mathbb{N}. \tag{16}$$

Choosing a subsequence of (a_v) if necessary, we can assume that

$$\operatorname{dist}(a_{\mathrm{v}+1},Z) < \frac{1}{2}\operatorname{dist}(a_\mathrm{v},Z) \quad \text{for } \mathrm{v} \in \mathbb{N}.$$

Choose $0 < r_\mathrm{v} \leq \operatorname{dist}(a_\mathrm{v},Z)/4$ and set

$$B_\mathrm{v} = \{x \in \mathbb{R}^n : |x - a_\mathrm{v}| \leq r_\mathrm{v}\}, \quad \mathrm{v} \in \mathbb{N},$$

Then $\{B_\mathrm{v}\}$ is a family of pairwise disjoint balls. Shrinking r_v if necessary, we may assume that

$$|\nabla f(x)| \leq \frac{2}{\mathrm{v}}\operatorname{dist}(x,Z)^{k-1} \quad \text{for } x \in B_\mathrm{v}. \tag{17}$$

Let $\alpha : \mathbb{R}^n \to \mathbb{R}$ be a C^∞ function such that $\alpha(x) = 0$ for $|x| \geq 1/4$ and $\alpha(x) = 1$ in some neighbourhood of 0. We are going to prove that $f - F$, where

$$F(x) = \begin{cases} \alpha\left(\dfrac{x-a_\mathrm{v}}{r_\mathrm{v}}\right)f(x), & x \in D_\mathrm{v},\ \mathrm{v} \in \mathbb{N}, \\ 0, & x \in \mathbb{R}^n \setminus \bigcup_{\mathrm{v}=1}^{\infty} D_\mathrm{v}, \end{cases}$$

is a C^k-Z-realisation of w.

By the above construction of F it is enough to check that

$$\lim_{x\to 0}\frac{F(x)}{|x|^k} = 0. \tag{18}$$

To this end write f in the form $f = T + R$ where T is the Taylor polynomial of degree k of f at 0 and R is the remainder. Observe that

$$\lim_{x\to 0} \frac{R(x)}{|x|^k} = 0 \quad \text{and} \quad \lim_{x\to 0} \frac{|\nabla R(x)|}{|x|^{k-1}} = 0. \tag{19}$$

On the other hand, since $T(0) = f(0) = 0$, by the Bochnak–Łojasiewicz inequality [2] there exists $C_1 > 0$ such that locally we have $|T(x)| \le C_1|x|\,|\nabla T(x)|$. Hence

$$|f(x)| \le C_1|x|\,|\nabla T(x)| + |R(x)| \le C_1|x|\,|\nabla f(x)| + C_1|x|\,|\nabla R(x)| + |R(x)|$$

in a neighbourhood of the origin. Using this, (17), (19) and the fact that α is bounded we get (18).

Now, for any sufficiently small neighbourhood U of $0 \in \mathbb{R}^n$ the set $f_{|U}^{-1}(0) \setminus Z$ is a smooth manifold of codimension 1 and $(f - F)_{|U}^{-1}(0) \setminus Z$ has non-empty interior. This contradicts the fact that the jet w is Z-v-sufficient.

Acknowledgements. This research was partially supported by the Polish National Science Centre, grant 2012/07/B/ST1/03293.

References

[1] J. Bochnak et W. Kucharz, *Sur les germes d'applications différentiables à singularités isolées*, Trans. Amer. Math. Soc. 252 (1979), 115–131.

[2] J. Bochnak and S. Łojasiewicz, *A converse of the Kuiper–Kuo Theorem*, in: Proceedings of Liverpool Singularities–Symposium, I (1969/70), Lecture Notes in Math. 192, Springer, Berlin, 1971, 254–261.

[3] P. Cassou-Noguès et H. H. Vui, *Théorème de Kuiper–Kuo–Bochnak–Łojasiewicz à l'infini*, Ann. Fac. Sci. Toulouse Math. (6) 5 (1996), 387–406.

[4] S. H. Chang and Y. C. Lu, *On C^0-sufficiency of complex jets*, Canad. J. Math. 25 (1973), 874–880.

[5] J. Damon and T. Gaffney, *Topological triviality of deformations of functions and Newton filtrations*, Invent. Math. 72 (1983), 335–358.

[6] A. van den Essen, A. Nowicki and A. Tyc, *Generalizations of a lemma of Freudenburg*, J. Pure Appl. Algebra 177 (2003), 43–47.

[7] T. Fukui and E. Yoshinaga, *The modified analytic trivialization of family of real analytic functions*, Invent. Math. 82 (1985), 467–477.

[8] V. Grandjean, *Finite determinacy relative to closed and finitely generated ideals*, Manuscripta Math. 103 (2000), 313–328.

[9] Z. Jelonek, *On the generalized critical values of a polynomial mapping*, Manuscripta Math. 110 (2003), 145–157.

[10] N. H. Kuiper, *C^1-equivalence of functions near isolated critical points*, in: Symposium on Infinite-Dimensional Topology (Baton Bouge, LA, 1967), Ann. of Math. Stud. 69, Princeton Univ. Press, Princeton, NJ, 1972, 199–218.

[11] T. C. Kuo, *On C^0-sufficiency of jets of potential functions*, Topology 8 (1969), 167–171.

[12] K. Kurdyka, P. Orro and S. Simon, *Semialgebraic Sard theorem for generalized critical values*, J. Differential Geom. 56 (2000), 67–92.

[13] L. Kushner, *Finite determination on algebraic sets*, Trans. Amer. Math. Soc. 331 (1992), 553–561.

[14] L. Kushner and B. Terra Leme, *Finite relative determination and relative stability*, Pacific J. Math. 192 (2000), 315–328.

[15] J. Mather, *Finitely determined map germs*, Publ. Math. Inst. Hautes Études Sci. 35 (1968), 127–156.

[16] B. Osińska-Ulrych, G. Skalski and S. Spodzieja, *On C^0-sufficiency of jets*, in: Analytic and Algebraic Geometry, Faculty of Mathematics and Computer Science, Univ. of Łódź, Łódź, 2013, 95–113.

[17] R. Pellikaan, *Finite determinacy of functions with non-isolated singularities*, Proc. London Math. Soc. (3) 57 (1988), 357–382.

[18] P. J. Rabier, *Ehresmann fibrations and Palais–Smale conditions for morphisms of Finsler manifolds*, Ann. of Math. 146 (1997), 647–691.

[19] T. Rodak and S. Spodzieja, *Equivalence of mappings at infinity*, Bull. Sci. Math. 136 (2012), 679–686.

[20] D. Siersma, *Singularities of functions on boundaries, corners, etc.*, Quart. J. Math. Oxford Ser. (2) 32 (1981), 119–127.

[21] D. Siersma, *Singularities with critical locus a 1-dimensional complete intersection and transversal type A_1*, Topology Appl. 27 (1987), 51–73.

[22] G. Skalski, *On analytic equivalence of functions at infinity*, Bull. Sci. Math. 135 (2011), 517–530.

[23] S. Spodzieja, *On C^0-sufficiency of jets*, in: Proc. 27th Instructional Conf. on Analytic and Algebraic Complex Geometry (Łódź, 2006), Wydawnictwo Uniwersytetu Łódzkiego, Łódź, 2006, 63–81 (in Polish).

[24] B. Teissier, *Cycles évanescents, sections planes et conditions de Whitney*, in: Singularités à Cargèse (Cargèse, 1972), Astérisque 7–8 (1973), 285–362.

[25] B. Teissier, *Variétés polaires. I. Invariants polaires des singularités d'hypersurfaces*, Invent. Math. 40 (1977), 267–292.

[26] R. Thom, *La stabilité topologique des applications polynomiales*, Enseign. Math. (2) (1962), 24–33.

[27] J.-C. Tougeron, *Idéaux de Fonctions Différentiables*, Ergeb. Math. Grenzgeb. 71, Springer, New York, 1972.

[28] X. Xu, *C^0-sufficiency, Kuiper–Kuo and Thom conditions for non-isolated singularity*, Acta Math. Sinica (English Ser.) 23 (2007), 1251–1256.

[29] E. Yoshinaga, *Topologically principal part of analytic functions*, Trans. Amer. Math. Soc. 314 (1989), 803–814.

Piotr Migus, Tomasz Rodak, Stanisław Spodzieja
Faculty of Mathematics and Computer Science
University of Łódź
S. Banacha 22
90-238 Łódź, Poland
E-mail: migus@math.uni.lodz.pl
rodakt@math.uni.lodz.pl
spodziej@math.uni.lodz.pl