COLLOQUIUM MATHEMATICUM

VOL. 146 2017 NO. 2

ON MINIMAL ZERO-SUM SEQUENCES OF LENGTH FOUR OVER CYCLIC GROUPS

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Abstract. Let G be a cyclic group of order n. It has been conjectured that if $\gcd(n, 6) = 1$, then every minimal zero-sum sequence S of length 4 over G has index 1, that is, $S = (n_1 g) \cdot (n_2 g) \cdot (n_3 g) \cdot (n_4 g)$ for some generator $g \in G$ and some integers $n_1, n_2, n_3, n_4 \in [1, n]$ with $n_1 + n_2 + n_3 + n_4 = n$. This conjecture has been confirmed recently for the case when $\langle \mathrm{supp}(S) \rangle = G$ and S contains at least one element g with $\langle g \rangle \neq G$. We show that if $\gcd(n, 30) = 1$ and any element of S is a generator of G, then this conjecture is true. Together with other known results, this conjecture is thus settled in the affirmative when $\gcd(n, 30) = 1$.

1. Introduction. Throughout the paper, let G be an additively written cyclic group of order n. By a *sequence* over G we mean a finite sequence of terms from G which is unordered and repetition of terms is allowed. We view sequences over G as elements of the free abelian monoid $\mathcal{F}(G)$ and write $S \in \mathcal{F}(G)$ in the form

$$S = g_1 \cdot \ldots \cdot g_s = \prod_{g \in G} g^{\mathsf{v}_g(S)},$$

where $s \in \mathbb{N}_0$ (the set of nonnegative integers), $g_1, \ldots, g_s \in G$ and $\mathsf{v}_g(S) \in \mathbb{N}_0$. We call $\mathsf{v}_g(S)$ the *multiplicity* of g in S, $|S| = s = \sum_{g \in G} \mathsf{v}_g(S)$ the *length* of S and $\mathrm{supp}(S) = \{g \in G : \mathsf{v}_g(S) > 0\}$ the *support* of S. A sequence S is called

- *zero-sum* if $\sum_{i=1}^{s} g_i = 0$,
- *minimal zero-sum* if S is a zero-sum sequence, but no proper nontrivial subsequence of S has sum zero.

In zero-sum theory, a well known result states that any minimal zero-sum sequence over G has length at most n. So it is natural to ask for the structure of minimal zero-sum sequences of length at most n. For this purpose, the following notion introduced by Chapman, Freeze and Smith [1] is useful.

2010 *Mathematics Subject Classification*: Primary 11B50; Secondary 20K01.

Key words and phrases: minimal zero-sum sequences, index of sequences.

Received 10 July 2015; revised 4 January 2016.

DOI: 10.4064/cm6702-1-2016

Definition 1.1. Let G be a cyclic group of order n, $g \in G$ and S be a sequence over G of the form

$$S = (n_1 g) \cdot \ldots \cdot (n_s g), \text{ where } s \geq 0 \text{ and } 1 \leq n_1, \ldots, n_s \leq \operatorname{ord}(g).$$

We define the *g-norm* of S by

$$\|S\|_g = \frac{n_1 + \cdots + n_s}{\operatorname{ord}(g)}$$

and the *index* of S by

$$\operatorname{Ind}(S) = \min\{\|S\|_g : g \in G \text{ with } G = \langle g \rangle\}.$$

Clearly, S is zero-sum if and only if $\operatorname{Ind}(S)$ is an integer. We note that there are also slightly different definitions of the index in the literature, but they are all equivalent. Indeed, by [5, Lemma 5.1.2],

$$\begin{aligned}\operatorname{Ind}(S) &= \min\{\|S\|_g : g \in G \text{ with } \langle \operatorname{supp}(S) \rangle = \langle g \rangle\} \\ &= \min\{\|S\|_g : g \in G \text{ with } \langle \operatorname{supp}(S) \rangle \subset \langle g \rangle\} \\ &= \min\{\|S\|_g : g \in G \text{ with } G = \langle g \rangle\}.\end{aligned}$$

Therefore we only need to consider the case $G = \langle \operatorname{supp}(S) \rangle$ if we want to calculate the index of S. From now on, we always assume that $G = \langle \operatorname{supp}(S) \rangle$.

The index of sequences has been studied since the 1980s. Because of its applications in factorization theory and other parts of combinatorial number theory (see [4–6] for example), it has received a great deal of attention (see, for example, [3, 7, 8, 12–18]).

If $\operatorname{Ind}(S) = 1$, then $S = (n_1 g) \cdot \ldots \cdot (n_s g)$ for some generator g and $1 \leq n_1, \ldots, n_s \leq n$ with $\sum_{i=1}^{s} n_i = n$. In this case S is a minimal zero-sum sequence and its structure is clear as $n_1, \ldots, n_s$ form a partition of n. So the main focus of the investigation of index is to determine minimal zero-sum sequences of index 1. In [2], Gao showed that any minimal zero-sum sequence of length roughly greater than $2n/3$ has index 1. Later, Yuan [17] and Savchev and Chen [11] independently extended the above result to the case of minimal zero-sum sequences of length greater than $n/2+1$. If $|S| \leq 3$, it was proved by Ponomarenko [10] that S has index 1.

In contrast to the above results, for each $k \in [5, n/2 + 1]$, there is a minimal zero-sum sequence S of length k with $\operatorname{Ind}(S) \geq 2$. Indeed, let

$$S = \begin{cases} g^{k-3} \cdot \left(\dfrac{n-1}{2} g\right) \cdot \left(\dfrac{n+3}{2} g\right) \cdot ((n+2-k)g) & \text{for odd } n \geq 9, \\ g^{k-3} \cdot \left(\dfrac{n}{2} g\right) \cdot \left(\dfrac{n+2}{2} g\right) \cdot ((n+2-k)g) & \text{for even } n \geq 8, \end{cases}$$

where g is a generator of G. Then S has index 2. For more examples, see [1, 9, 16].

It remains to consider the case when $|S| = 4$. When $\gcd(n,6) \neq 1$, Ponomarenko [10] gave examples of minimal zero-sum sequences of length 4 with indices at least 2. In the same paper, he also suggested the following conjecture and mentioned that it had been verified computationally for $n \leq 1000$.

Conjecture 1.2. *Let G be a cyclic group of order n with $\gcd(n,6) = 1$. Then every minimal zero-sum sequence S of length 4 over G has index 1.*

This conjecture was first confirmed for the case when n is a prime power by Li, Plyley, Yuan and the first author [8]. Then Li and Peng [7] proved that it holds for n being a product of two prime powers with the restriction that S contains a generator of G. Later Xia and Shen [15] considered the remaining situation of n a product of two prime powers. Thus these two papers together completely settled the case when n is a product of two prime powers. Recently, Shen, Xia and Li [13] tackled the general case and showed that the conjecture is true if $\langle \mathrm{supp}(S) \rangle = G$ and S contains at least one element g with $\langle g \rangle \neq G$.

In the present paper, we obtain the following main result.

Theorem 1.3. *Let G be a finite cyclic group of order n with $n \geq 1001$ and $\gcd(n,30) = 1$, and let $S = g_1g_2g_3g_4$ be a minimal zero-sum sequence over G where $\mathrm{ord}(g_i) = n$ for all $i \in [1,4]$. Then $\mathrm{Ind}(S) = 1$.*

Together with known results [5,7,10,13], we have the following corollary.

Corollary 1.4. *Let G be a finite cyclic group of order n with $\gcd(n,30) = 1$. Then every minimal zero-sum sequence S over G of length $|S| = 4$ has index 1.*

2. Proofs. For a real $Q \geq 3$, let M_Q denote the maximal integer $\leq Q$ of the form $2^\alpha 3^\beta 5^\gamma$, and O_Q denote the maximal integer $\leq Q$ of the form $3^\beta 5^\gamma$, where α, β and γ are nonnegative integers.

Lemma 2.1. *$\mathsf{M}_Q \geq 5Q/7$ and $\mathsf{O}_Q \geq 5Q/9$ for any $Q \geq 3$.*

Proof. Suppose that $\mathsf{M}_Q < 5Q/7$. Write $\mathsf{M}_Q = 2^\alpha 3^\beta 5^\gamma$.

If $\alpha \geq 2$, then $\mathsf{M}_Q < 2^{\alpha-2}3^\beta 5^{\gamma+1} < 5Q/7 \cdot 5/4 < Q$, contrary to the definition of M_Q.

If $\beta \geq 1$, then $\mathsf{M}_Q < 2^{\alpha+2}3^{\beta-1}5^\gamma < 5Q/7 * 4/3 < Q$, a contradiction.

If $\gamma \geq 1$, then $\mathsf{M}_Q < 2^{\alpha+1}3^{\beta+1}5^{\gamma-1} < 5Q/7 * 6/5 < Q$, a contradiction.

Therefore $\mathsf{M}_Q \leq 2$, which is impossible since $Q \geq 3$.

Suppose now that $\mathsf{O}_Q < 5Q/9$. Write $\mathsf{O}_Q = 3^\beta 5^\gamma$.

If $\beta \geq 1$, then $\mathsf{O}_Q < 3^{\beta-1}5^{\gamma+1} < 5Q/9 * 5/3 < Q$, a contradiction.

If $\gamma \geq 1$, then $\mathsf{O}_Q < 3^{\beta+2}5^{\gamma-1} < 5Q/9 * 9/5 = Q$, a contradiction.

Therefore $\mathsf{O}_Q \leq 1$, which is also impossible. ■

For $Q \geq 3$, there are a finite number, say k, of distinct rational numbers in $(0, 1/2]$ with denominators less than or equal to Q and of the form $2^\alpha 3^\beta 5^\gamma$. We order these rational numbers increasingly and denote by

$$\mathsf{A}_Q = (a_1/b_1, a_2/b_2, \ldots, a_k/b_k) \in \mathbb{Q}^k,$$

where $b_i \leq Q$ is a positive integer of the form $2^\alpha 3^\beta 5^\gamma$, $a_i \geq 1$ and $\gcd(a_i, b_i) = 1$ for any $i \in [1, k]$. It is clear that $a_1/b_1 = 1/\mathsf{M}_Q$ and $a_k/b_k = 1/2$. Let

$$\mathsf{L}_Q = \max\{a_i/b_i - a_{i-1}/b_{i-1} \mid i \in [2, k]\}.$$

Note that A_Q partitions $[0, 1/2]$ into the union of intervals,

$$[0, 1/2] = [0, 1/\mathsf{M}_Q] \cup \bigcup_{i=2}^{k} [a_{i-1}/b_{i-1}, a_i/b_i],$$

and L_Q is the maximal length of these intervals except the first one $[0, 1/\mathsf{M}_Q]$.

LEMMA 2.2. $\mathsf{L}_Q \leq 9/(10Q)$ *for any* $Q \geq 3$.

Proof. Since

$$[3, \infty) = \biguplus_{\ell=0}^{\infty} [3 \cdot 2^\ell, 3 \cdot 2^{\ell+1}),$$

it is equivalent to prove

$$\forall \ell \in \mathbb{N}_0, \forall Q \in [3 \cdot 2^\ell, 3 \cdot 2^{\ell+1}), \qquad \mathsf{L}_Q \leq \frac{9}{10Q}.$$

We prove this by induction on ℓ and start with $\ell = 0$.

If $3 \leq Q < 4$, then $\mathsf{A}_Q = (1/3, 1/2)$ and $L_Q = 1/6 \leq 9/(10Q)$.

If $4 \leq Q < 5$, then $\mathsf{A}_Q = (1/4, 1/3, 1/2)$ and $L_Q = 1/6 \leq 9/(10Q)$.

If $5 \leq Q < 6$, then $\mathsf{A}_Q = (1/5, 1/4, 1/3, 2/5, 1/2)$ and $L_Q = 1/10 \leq 9/(10Q)$.

Now let $\ell \geq 1$ and suppose that the assertion holds for $\ell - 1$. Let $Q \in [3 \cdot 2^\ell, 3 \cdot 2^{\ell+1})$, $Q' = Q/2 \in [3 \cdot 2^{\ell-1}, 3 \cdot 2^\ell)$ and

$$\mathsf{A}_{Q'} = (a_1/b_1, \ldots, a_{k'}/b_{k'}),$$

where $a_1/b_1 = 1/\mathsf{M}_{Q'}$ and $a_{k'}/b_{k'} = 1/2$. Let

$$B_{Q'} = (a_1/b_1, \ldots, a_{k'-1}/b_{k'-1}, a_{k'}/b_{k'}, 1 - a_{k'-1}/b_{k'-1}, \ldots, 1 - a_1/b_1).$$

By the inductive assumption and the fact that the numbers in $B_{Q'}$ are symmetric with respect to $a_{k'}/b_{k'} = 1/2$, it follows that $B_{Q'}$ partitions $[0, 1]$ into the union of intervals of maximal length $\mathsf{L}_{Q'} \leq 9/(10Q')$ except the first one $[0, a_1/b_1]$ and the last one $[1 - a_1/b_1, 1]$. So

$$\frac{1}{2} B_{Q'} := \left(\frac{a_1}{2b_1}, \ldots, \frac{a_{k'}}{2b_{k'}}, \frac{1}{2} - \frac{a_{k'-1}}{2b_{k'-1}}, \ldots, \frac{1}{2} - \frac{a_1}{2b_1} \right)$$

partitions $[0, 1/2]$ into the union of intervals of maximal length $\leq 9/(2 * 10Q') = 9/(10Q)$ except the first one $[0, a_1/(2b_1)]$ and the last one $[1/2 - a_1/(2b_1), 1/2]$. Consider

$$\tilde{B}_Q = \left(\frac{1}{\mathsf{M}_Q}, \frac{a_1}{2b_1}, \dots, \frac{a_{k'}}{2b_{k'}}, \frac{1}{2} - \frac{a_{k'-1}}{2b_{k'-1}}, \dots, \frac{1}{2} - \frac{a_1}{2b_1}, \frac{\mathsf{O}_Q - 1}{2\mathsf{O}_Q}, \frac{1}{2}\right).$$

By Lemma 2.1,

$$0 \le \frac{a_1}{2b_1} - \frac{1}{\mathsf{M}_Q} = \frac{1}{2\mathsf{M}_{Q'}} - \frac{1}{\mathsf{M}_Q} \le \frac{7}{2 * 5Q'} - \frac{1}{Q} \le \frac{9}{10Q},$$

$$\frac{\mathsf{O}_Q - 1}{2\mathsf{O}_Q} - \left(\frac{1}{2} - \frac{a_1}{2b_1}\right) = \frac{1}{2\mathsf{M}_{Q'}} - \frac{1}{2\mathsf{O}_Q} \le \frac{7}{2 * 5Q'} - \frac{1}{2Q} = \frac{9}{10Q}$$

and

$$\frac{1}{2} - \frac{\mathsf{O}_Q - 1}{2\mathsf{O}_Q} = \frac{1}{2\mathsf{O}_Q} \le \frac{9}{2 * 5Q} = \frac{9}{10Q}.$$

So $\tilde{B}_Q$ partitions $[0, 1/2]$ into the union of intervals of maximal length $\le 9/(10Q)$ except the first one $[0, 1/\mathsf{M}_Q]$. Note that all rational numbers in $\tilde{B}_Q$ come from A_Q. Hence A_Q partitions $[0, 1/2]$ into the union of intervals of maximal length $\mathsf{L}_Q \le 9/(10Q)$ except the first one $[0, 1/\mathsf{M}_Q]$, and we are done. ∎

Proof of Theorem 1.3. By assumption, we can write $g_i = t_i g_1$ with $1 \le t_i \le n-2$ for $i \in [2, 4]$, and so $S = g_1 \cdot (t_2 g_1) \cdot (t_3 g_1) \cdot (t_4 g_1)$. If two integers among t_2, t_3 and t_4, say t_2 and t_3, are smaller than $n/2$, then $1 + t_2 + t_3 + t_4 < 2n$ and hence $\mathrm{Ind}(S) = 1$. If $t_i \ge n/2$ for all $i \in [2, 4]$, then

$$\begin{aligned} 2S &:= (2g_1) \cdot (2t_2 g_1) \cdot (2t_3 g_1) \cdot (2t_4 g_1) \\ &= (2g_1) \cdot ((2t_2 - n) g_1) \cdot ((2t_3 - n) g_1) \cdot ((2t_4 - n) g_1) \end{aligned}$$

has index 1. Since multiplying S by an integer k with $\gcd(k, n) = 1$ does not change the index, we have $\mathrm{Ind}(S) = 1$. Hence we may assume that one of t_2, t_3, t_4 is in $[1, n/2)$, while the other two are in $[n/2, n-2]$. Then we may write $S = g_1 \cdot (c g_1) \cdot ((n-b) g_1) \cdot ((n-a) g_1)$ with $1 < a \le b < c < n/2$ and $c - b = a - 1$.

First we consider the case $a \ge 10$. Let $Q = n/a$. It is clear that $2a \le a + b = c + 1 \le (n+1)/2$, and so $Q = n/a > 3$. By Lemma 2.2,

$$\mathsf{A}_Q = (a_1/b_1 = 1/\mathsf{M}_Q, a_2/b_2, \dots, a_k/b_k = 1/2)$$

partitions $[0, 1/2]$ into the union of intervals of maximal length $\mathsf{L}_Q \le 9/(10Q)$ except the first one $[0, 1/\mathsf{M}_Q]$. Hence

$$n\mathsf{A}_Q := (n/\mathsf{M}_Q, na_2/b_2, \dots, na_k/b_k)$$

partitions $[0, n/2]$ into the union of intervals of maximal length $n\mathsf{L}_Q \le 9n/(10Q) = 9a/10$ except the first one $[0, n/\mathsf{M}_Q]$. Note that $na_i/b_i \notin \mathbb{Z}$ because $\gcd(n, 30) = 1$. Hence no positive integer can be exactly at an endpoint of one of these subintervals, and any two integers in the same subinterval (except the first one) have distance $< 9a/10$.

We claim that b and c are in different subintervals which derive from $n\mathsf{A}_Q$. If $b \le n/\mathsf{M}_Q$, note that $n/\mathsf{M}_Q \le 7n/(5Q) < 2a-1 \le a+b-1 = c$ by Lemma 2.1, and we are done. If $b > n/\mathsf{M}_Q$, then $c-b = a-1 \ge 9a/10$, and hence b and c cannot be in the same subinterval. The claim is proved.

Now let $b < na_v/b_v < c$ for some $v \in [1, k-1]$. Since $cb_v - bb_v = (a-1)b_v < aQ = n$, we have $(a_v-1)n < b_v b < na_v < b_v c < (a_v+1)n$. Hence $b_v S = (b_v g_1) \cdot ((b_v c - na_v)g_1) \cdot ((na_v - b_v b)g_1) \cdot ((n - b_v a)g_1)$ with $b_v, b_v c - na_v, na_v - b_v b, n - b_v a \in [1, n-1]$. A direct calculation gives $n\|b_v S\|_{g_1} = b_v + b_v c - na_v + na_v - b_v b + n - b_v a = n$, that is, $\mathrm{Ind}(b_v S) = 1$. Since $\gcd(b_v, n) = 1$ by the definition of A_Q, we have $\mathrm{Ind}(S) = \mathrm{Ind}(b_v S) = 1$, and we are done.

Next we consider the case $2 \le a \le 9$. For convenience, we define $[a,b]g := \{kg \mid k \in [a,b]\}$ for $g \in G$ and $a, b \in \mathbb{Z}$. Replace g_1 by $(n-a)g_1$ and redo the proof above. It follows that one element of $\{g_1, cg_1, (n-b)g_1\}$ must be in $-[2,9](n-a)g_1 \subset [4,81]g_1 \subset [4,(n-1)/2]g_1$. We infer that it must be cg_1, and thus $c \in [4,81]$. Once again replace g_1 by cg_1 and redo the proof above. It follows that one element of $\{g_1, (n-b)g_1, (n-a)g_1\}$ must be in $-[2,9]cg_1 \subset [n-729, n-8]g_1 \subset [2, n-8]g_1$. We infer that it must be $(n-b)g_1$ or $(n-a)g_1$, and thus b or a is equal to sc for some $s \in [2,9]$. However, this is impossible because $a \le b < c$. This completes the proof of the theorem. ■

Proof of Corollary 1.4. Let G be a cyclic group of order n with $\gcd(n,30) = 1$, and $S = g_1 g_2 g_3 g_4 \in \mathcal{F}(G)$ be a minimal zero-sum sequence. By Definition 1.1 and [5, Lemma 5.1.2], we may assume that $G = \langle \mathrm{supp}(S) \rangle$.

The case when $n \le 1000$ has been verified computationally by V. Ponomarenko [10, p. 2] and double-checked by Peng and Wang by using a computer program (see [7, p. 847]).

When S contains at least one element whose order is not n, the conclusion has been proven by Shen, Xia and Li [13].

When the order of every element of S is n and $n > 1000$, the conclusion is exactly Theorem 1.3.

Combining these results, we obtain the corollary. ■

Acknowledgements. This research was supported in part by the National Natural Science Foundation of China (No. 11301556), the Guangdong Provincial Natural Science Foundation (No. S2013040013796), the Specialized Research Fund for the Doctoral Program of Higher Education (No. 20130171120008) and the Fundamental Research Funds for the Central Universities (No. 14lgpy30).

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