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A SYSTEM OF VOLTERRA INTEGRAL EQUATIONS WITH BLOWING UP SOLUTIONS

BY

WOJCIECH MYDLARCZYK (Wrocław)

Abstract. A system of nonlinear integral equations related to combustion problems is considered. Necessary and sufficient conditions for the existence and explosion of positive solutions are given. Also the uniqueness of positive solutions is shown. The main results are obtained by monotonicity methods.

1. Introduction. Nonlinear integral equations arise in models of ignition and explosive behaviour in diffusive media. In applications, solutions can describe a variety of processes including solid fuel combustion processes. There are many papers related to this topic, e.g. [5], [12], [14].

Some simple models of combustion can be studied with the help of the equation

$$u(x)=\int_0^x (x-s)^{\alpha-1} g(u(s))\,ds \quad (\alpha>0), \tag{1.1}$$

where g is a nondecreasing continuous function such that $g(0)=0$. Obviously, $u\equiv 0$ is the trivial solution to (1.1). This equation describes ignition in media if there exists a nontrivial continuous solution u positive for $x>0$. It was shown in [1], [3], [4], [6], [7], [11] that a necessary and sufficient condition for ignition is

$$\int_0^\delta \left[\frac{u}{g(u)}\right]^{1/\alpha}\frac{du}{u}<\infty, \tag{1.2}$$

where δ is any finite positive number. A solution u to (1.1) has explosive behaviour if there exists a finite blow-up time T, that is, $u(x)\to\infty$ as $x\to T^-$. A necessary and sufficient condition for the existence of a finite

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blow-up time is

$$\int_0^\infty \left[\frac{u}{g(u)}\right]^{1/\alpha} \frac{du}{u} < \infty \tag{1.3}$$

(see [1], [2], [8]).

However in more complicated models of combustion, systems of integral equations appear [12], [13]. Inspired by considerations in those papers, we study ignition and blow-up criteria for the system of equations

$$u(x) = \int_0^x (x-s)^{\alpha-1}[v(s)]^\gamma \, ds, \tag{1.4}$$

$$v(x) = \int_0^x (x-s)^{\beta-1} g(u(s)) \, ds, \tag{1.5}$$

where $\alpha, \beta \ge 1$, $\gamma > 0$ and the function g is assumed to be as in (1.1).

Obviously $u \equiv 0$, $v \equiv 0$ is the trivial solution to (1.4), (1.5). But we shall ask about nontrivial solutions, i.e. nonnegative continuous functions u and v. Since they are nondecreasing, it follows from the convolution form of (1.4) and (1.5) that either they are simultaneously positive for $x > 0$ or there exists a constant $c > 0$ such that $u(x) = v(x) = 0$ for $0 \le x \le c$ and $u(x), v(x) > 0$ for $x > c$. We show that if the system (1.4)–(1.5) has nontrivial solutions then it has a unique solution with u, v positive for $x > 0$, and the components of any other solution are respectively translations of u and v of the form $u((x-c)_+)$, $v((x-c)_+)$, where $c > 0$ is any constant and $(x-c)_+ = 0$ for $0 \le x \le c$ and equals $x - c$ otherwise. From now on we deal only with nontrivial solutions with components positive for $x > 0$. We are also interested in the existence of a finite blow-up time T for such solutions.

The system (1.4), (1.5) of integral equations was studied in [9], [10]. Unfortunately, the methods used there allowed one to consider only the case of integer exponents α and $\beta \ge 1$. We extend those results to real exponents using different methods and describe all the nontrivial solutions of the system.

2. Notation and the statement of results.

Throughout the paper we assume that $g : [0,\infty) \to [0,\infty)$ is a nondecreasing continuous function positive for $x > 0$ with $g(0) = 0$. Furthermore, we make the following technical assumption:

(2.1) *there exists a constant* $c_g > 0$ *such that* $g(x) \le c_g g(x/2)$ *for* $x > 0$.

We say that a solution (u, v) of the system (1.4)–(1.5) is *nontrivial* if the functions $u(x)$ and $v(x)$ are continuous and positive for $x > 0$. Such a nontrivial solution will be called *blowing up* at some $T > 0$ if $u(x) \to \infty$, and consequently $v(x) \to \infty$, as $x \to T^-$. We emphasize that the functions $u(x)$, $v(x)$ are nondecreasing. Integrating by parts in (1.4), (1.5) one can see that the derivatives $u'(x)$ and $v'(x)$ are also nondecreasing.

The main results of the paper are the following two theorems.

THEOREM 2.1. *The system* (1.4)–(1.5) *has a unique nontrivial solution if and only if*

$$\int_0^\delta \left[\frac{u}{g(u)^\gamma}\right]^{\frac{1}{\alpha+\beta\gamma}} \frac{du}{u} < \infty. \tag{2.2}$$

THEOREM 2.2. *The system* (1.4)–(1.5) *has a blowing up solution if and only if*

$$\int_0^\infty \left[\frac{u}{g(u)^\gamma}\right]^{\frac{1}{\alpha+\beta\gamma}} \frac{du}{u} < \infty. \tag{2.3}$$

Our first step in the study of the system (1.4)–(1.5) relies on its reduction to the following single nonlinear integral equation:

$$u(x) = \int_0^x (x-s)^{\alpha-1} \left[\int_0^s (s-t)^{\beta-1} g(u(t))\, dt\right]^\gamma ds. \tag{2.4}$$

REMARK 2.3. If $\gamma = 1$, then both the system (1.4)–(1.5) and the equation (2.4) reduce to the equation (1.1) with exponent $\alpha + \beta - 1$. In this case the conditions (2.2) and (2.3) are equivalent to (1.2) and (1.3), respectively.

We introduce the function $w(x) = u'(u^{-1}(x)) = 1/(u^{-1})'(x)$, where u^{-1} is the inverse function to u. To find a convenient relation for w we begin with the observation that integration by parts and then the substitution $z = u(t)$ gives

$$\int_0^s (s-t)^{\beta-1} g(u(t))\, dt = \frac{1}{\beta}\int_0^s (s-t)^\beta\, dg(u(t)) = \frac{1}{\beta}\int_0^{u(s)} (s-u^{-1}(z))^\beta\, dg(z).$$

Thus (2.4) can be rewritten in the form

$$u(x) = \frac{1}{\beta^\gamma}\int_0^x (x-s)^{\alpha-1} G(u(s))^\gamma\, ds = \frac{1}{\alpha\beta^\gamma}\int_0^x (x-s)^\alpha\, d[G(u(s))]^\gamma, \tag{2.5}$$

where

$$G(z) = \int_0^z (u^{-1}(z) - u^{-1}(t))^\beta\, dg(t) = \int_0^z \left(\int_t^z \frac{1}{w(z)}\, dz\right)^\beta dg(t).$$

The substitution $z = u(s)$ in (2.5) gives

$$u(x) = \frac{1}{\alpha\beta^\gamma}\int_0^{u(x)} (x - u^{-1}(z))^\alpha\, d[G(z)]^\gamma,$$

or equivalently

$$x = \frac{1}{\alpha\beta^\gamma}\int_0^x (u^{-1}(x) - u^{-1}(s))^\alpha \, d[G(s)]^\gamma. \tag{2.6}$$

Differentiating both sides of (2.6) we get the desired relation

$$w(x) = \frac{1}{\beta^\gamma}\int_0^x \Big(\int_s^x \frac{1}{w(z)}\, dz\Big)^{\alpha-1} d[G(s)]^\gamma. \tag{2.7}$$

3. Auxiliary lemmas. In this section, we obtain a priori estimates for the nontrivial solution $u(x)$ of (2.4). They will be expressed in terms of the function

$$\varPhi(x) = x\left[\frac{g(x)^\gamma}{x}\right]^{\frac{1}{\alpha+\beta\gamma}} \quad \text{for } x > 0 \quad \text{and} \quad \varPhi(0) = 0.$$

This function is continuous, nondecreasing, and in view of (2.1) there exists a constant $c > 0$ such that

$$\varPhi(x) \le c\varPhi(x/2) \qquad \text{for } x \ge 0. \tag{3.1}$$

We define an operator S related to the right hand side of (2.7) as follows:

$$Sw(x) = \frac{1}{\beta^\gamma}\int_0^x \Big(\int_s^x \frac{1}{w(z)}\, dz\Big)^{\alpha-1} d[F_w(s)]^\gamma \quad (x > 0),$$

where

$$F_w(s) = \int_0^s \Big(\int_t^s \frac{1}{w(z)}\, dz\Big)^{\beta} dg(t).$$

It is defined for any continuous function $w(x)$ positive for $x > 0$ such that the Stieltjes integrals $F_w(s)$ and

$$\int_0^x \Big(\int_s^x \frac{1}{w(z)}\, dz\Big)^{\alpha-1} d[F_w(s)]^\gamma$$

are convergent.

LEMMA 3.1. *There exist constants* $c_\varPhi^1, c_\varPhi^2 > 0$ *such that*

$$c_\varPhi^1 \varPhi(x) \le S\varPhi(x) \le c_\varPhi^2 \varPhi(x) \qquad \textit{for } x > 0. \tag{3.2}$$

Proof. To prove the upper estimate we consider the integrals

$$V_k(s) = \int_0^s \Big(\int_t^s \frac{1}{\varPhi(z)}\, dz\Big)^{\beta-k} dg(t)$$

for $k = 0, 1, \ldots, n$, where n is the integer such that $n < \beta \le n+1$. We show by induction that there exist constants $c_0, c_1, \ldots, c_n > 0$ such that

$$V_k(s) \le c_k g(s)^{1-\frac{(\beta-k)\gamma}{\alpha+\beta\gamma}} s^{\frac{\beta-k}{\alpha+\beta\gamma}} \tag{3.3}$$

for $s > 0$, $k = 0, 1, \ldots, n$. Starting with $k = n$ and integrating by parts we get

$$V_n(s) = J_1(t)|_0^s + (\beta - n) J_2(s),$$

where

$$J_1(t) = g(t)\bigg(\int_t^s \frac{1}{\Phi(z)}\,dz\bigg)^{\beta-n} \quad \text{and} \quad J_2(s) = \int_0^s \bigg(\int_t^s \frac{1}{\Phi(z)}\,dz\bigg)^{\beta-n-1} \frac{1}{\Phi(t)} g(t)\,dt.$$

Since the function g is nondecreasing, we have

$$g(t)\bigg(\int_t^s \frac{1}{\Phi(z)}\,dz\bigg)^{\beta-n} \le (\alpha+\beta\gamma)^{\beta-n} g(t)^{1-\frac{(\beta-n)\gamma}{\alpha+\beta\gamma}} \big(s^{\frac{1}{\alpha+\beta\gamma}} - t^{\frac{1}{\alpha+\beta\gamma}}\big)^{\beta-n},$$

which shows that $J_1(t) = 0$ both at $t = 0$ and at $t = s$. Since Φ is nondecreasing and $\beta - n - 1 \le 0$, we observe that

$$\begin{aligned} J_2(s) &\le \frac{1}{\Phi(s)^{\beta-n-1}} \int_0^s (s-t)^{\beta-n-1} t^{-1+\frac{1}{\alpha+\beta\gamma}} g(t)^{1-\frac{\gamma}{\alpha+\beta\gamma}}\,dt \\ &\le \frac{1}{\Phi(s)^{\beta-n-1}} g(s)^{1-\frac{\gamma}{\alpha+\beta\gamma}} \int_0^s (s-t)^{\beta-n-1} t^{-1+\frac{1}{\alpha+\beta\gamma}}\,dt \le c g(s)^{1-\frac{(\beta-n)\gamma}{\alpha+\beta\gamma}} s^{\frac{\beta-n}{\alpha+\beta\gamma}}, \end{aligned}$$

where $c > 0$ is some constant. Thus we get (3.3) for $k = n$. For $0 \le k < n$, to avoid the difficulties connected with possible divergence of the integral $\int_t^x \frac{dz}{\Phi(z)}$ as $t \to 0$ we introduce the truncated integrals

$$V_k^\epsilon(x) = \int_\epsilon^x \bigg(\int_t^x \frac{1}{\Phi(z)}\,dz\bigg)^{\beta-k} dg(t)$$

defined for $0 \le k \le n$ and any $0 < \epsilon \le x$. We first note that

$$\begin{aligned} \frac{d}{ds} V_{k-1}^\epsilon(s) &= (\beta - k + 1) \frac{1}{\Phi(s)} \int_\epsilon^s \bigg(\int_t^s \frac{1}{\Phi(z)}\,dz\bigg)^{\beta-k} dg(t) \\ &= (\beta - k + 1) \frac{1}{\Phi(s)} V_k^\epsilon(s). \end{aligned}$$

Using this recurrence relation and the induction assumption we get the estimate

$$\begin{aligned} V_{k-1}^\epsilon(x) &= (\beta-k+1)\int_\epsilon^x \frac{1}{\Phi(s)} V_k^\epsilon(s)\,ds \le (\beta-k+1)\int_\epsilon^x \frac{1}{\Phi(s)} V_k(s)\,ds \\ &\le (\beta-k+1) c_k \int_\epsilon^x g(s)^{1-\frac{(\beta-k+1)\gamma}{\alpha+\beta\gamma}} s^{-1+\frac{\beta-k+1}{\alpha+\beta\gamma}}\,ds \\ &\le c_{k-1} g(x)^{1-\frac{(\beta-(k-1))\gamma}{\alpha+\beta\gamma}} x^{\frac{\beta-(k-1)}{\alpha+\beta\gamma}} \end{aligned} \tag{3.4}$$

with $c_{k-1} = (\alpha+\beta\gamma)c_k$, valid for any $0 < \epsilon \le x$. Letting $\epsilon \to 0$ in (3.4), we get the required estimate for $V_{k-1}(x)$, which by an induction argument ends the proof of (3.3).

We also need to discuss the integrals

$$U_k(x) = \int_0^x \Big(\int_s^x \frac{1}{\Phi(z)}\, dz \Big)^{\alpha-1-k} d[V_0(s)]^\gamma$$

for $\alpha > 1$ and $k = 0, 1, \ldots, n$, where n is the integer such that $0 \le n < \alpha - 1 \le n + 1$. For $\alpha = 1$, our discussion simplifies to considering $U_0(x) = V_0(x)^\gamma$ only. Due to the estimate of $V_0(s)$ in (3.3) we can argue as when estimating $V_k(s)$ and show that there exist constants $d_0, \ldots, d_n > 0$ such that

$$U_k(x) \le d_k g(x)^{\frac{(k+1)\gamma}{\alpha+\beta\gamma}} x^{1-\frac{k+1}{\alpha+\beta\gamma}} \tag{3.5}$$

for $x > 0$ and $k = 0, 1, \ldots, n$.

Now we are ready to justify the inequalities of our main interest.

Since $S\Phi(x) = \beta^{-\gamma} U_0(x)$, the upper estimate in (3.2) follows from the estimate for $U_0(x)$ given in (3.5).

To prove the lower estimate in (3.3) we first note that

$$\begin{aligned} S\Phi(x) &\ge \frac{1}{\beta^\gamma} \int_0^{x/2} \Big(\int_s^x \frac{1}{\Phi(z)}\, dz \Big)^{\alpha-1} d[V_0(s)]^\gamma \\ &\ge \frac{1}{\beta^\gamma} \frac{1}{\Phi(x)^{\alpha-1}} \Big(\frac{x}{2}\Big)^{\alpha-1} [V_0(x/2)]^\gamma \\ &\ge \frac{1}{\beta^\gamma} \frac{1}{\Phi(x)^{\alpha-1}} \Big(\frac{x}{2}\Big)^{\alpha-1} \frac{1}{\Phi(x/2)^{\beta\gamma}} \Big(\frac{x}{4}\Big)^{\beta\gamma} g(x/4)^\gamma \end{aligned}$$

for $x > 0$. Now using the estimates (2.1) and (3.1) we get our assertion. ∎

Remark 3.2. Since

$$\int_s^x \frac{1}{\Phi(z)}\, dz \le (\alpha + \beta\gamma) g(s)^{-\frac{\gamma}{\alpha+\beta\gamma}} (x^{\frac{1}{\alpha+\beta\gamma}} - s^{\frac{1}{\alpha+\beta\gamma}}),$$

and since in view of (3.3),

$$F_\Phi(s) = V_0(s) \le c_0 g(s)^{\frac{\alpha}{\alpha+\beta\gamma}} s^{\frac{\beta}{\alpha+\beta\gamma}},$$

we observe that in the case of $\alpha > 1$ the expression

$$J_1(s) = \Big(\int_s^x \frac{1}{\Phi(z)}\, dz \Big)^{\alpha-1} V_0(s)^\gamma$$

is equal to 0 both at $s = 0$ and at $s = x$. This shows that we can integrate by parts to obtain

$$S\Phi(x) = \frac{\alpha-1}{\beta^\gamma} \int_0^x \Big(\int_s^x \frac{1}{\Phi(z)}\, dz \Big)^{\alpha-2} \frac{1}{\Phi(s)} [V_0(s)]^\gamma\, ds.$$

For $\alpha = 1$, we have simply

$$S\Phi(x) = \frac{1}{\beta^\gamma} V_0(x)^\gamma.$$

LEMMA 3.3. *Let $w(x)$ be a continuous function and let $c_w > 0$ be a constant such that*

$$c_w \Phi(x) \le w(x) \quad \text{for } x > 0.$$

Then

$$Sw(x) \le c_w^{-(\alpha+\beta\gamma-1)} S\Phi(x) \quad \text{for } x > 0.$$

Proof. We first note that

$$F_w(s) \le c_w^{-\beta} F_\Phi(s) \quad \text{for } s > 0. \tag{3.6}$$

For $\alpha = 1$, we have $S\Phi(x) = \beta^{-\gamma} F_\Phi(x)^\gamma$ and $Sw(x) = \beta^{-\gamma} F_w(x)^\gamma$, therefore our assertion follows from (3.6) immediately.

For $\alpha > 1$, we first note that

$$Sw(x) \le c_w^{-(\alpha-1)} \beta^{-\gamma} \int_0^x \left(\int_s^x \frac{1}{\Phi(z)}\, dz \right)^{\alpha-1} d[F_w(s)]^\gamma.$$

Now integrating by parts in the outer integral and then using (3.6) we get

$$Sw(x) \le c_w^{-(\alpha+\beta\gamma-1)} \frac{\alpha-1}{\beta^\gamma} \int_0^x \left(\int_s^x \frac{1}{\Phi(z)}\, dz \right)^{\alpha-2} \frac{1}{\Phi(s)} F_\Phi(s)^\gamma\, ds.$$

Hence our assertion follows from Remark 3.2 immediately. ∎

COROLLARY 3.4. *Let u be a nontrivial solution to (2.4) and let $w(x) = u'(u^{-1}(x))$, where u^{-1} is the inverse to u. Then there exist constants $c_w^1, c_w^2 > 0$ such that*

$$c_w^1 \Phi(x) \le w(x) \le c_w^2 \Phi(x) \quad \text{for } x > 0.$$

Proof. To show the lower estimate we examine (2.7). Since w is nondecreasing, we have

$$\int_{x/2}^x \frac{1}{w(z)}\, dz \ge \frac{x}{2w(x)} \tag{3.7}$$

and

$$\begin{aligned} G(x/2) = \int_0^{x/2} \left(\int_t^{x/2} \frac{1}{w(z)}\, dz \right)^\beta dg(t) &\ge \left(\frac{x}{4w(x/2)} \right)^\beta g(x/4) \\ &\ge \left(\frac{x}{4w(x)} \right)^\beta g(x/4) \quad \text{for } x > 0. \end{aligned} \tag{3.8}$$

It follows from (2.7) that

$$w(x) \geq \frac{1}{\beta^\gamma}\bigg(\int\limits_{x/2}^{x} \frac{1}{w(z)}\,dz\bigg)^{\alpha-1} G(x/2)^\gamma \quad \text{for } x > 0.$$

Now combining (3.7), (3.8) and using (2.1) we note that there exists a constant $c_w^1 > 0$ such that $w(x) \geq c_w^1 \Phi(x)$ for $x > 0$.

Now the upper estimate follows immediately from Lemma 3.3. ■

4. Proofs of theorems

Proof of Theorem 2.1. Uniqueness. We begin by showing that the system (1.4)–(1.5) has at most one nontrivial solution with u, v positive for $x > 0$. Let (u_i, v_i), $i = 1, 2$, be two nontrivial solutions of (1.4)–(1.5) with components positive for $x > 0$. Consider a shifted solution with $u_{2,c}(x) = u_2((x-c)_+)$ and $v_{2,c}(x) = v_2((x-c)_+)$, where $c > 0$ is a constant. We observe that $u_{2,c}(x) < u_1(x)$ and $v_{2,c}(x) < v_1(x)$ at least for $0 < x \leq c$. Moreover, the following implication holds: if $u_{2,c}(x) < u_1(x)$ and $v_{2,c}(x) < v_1(x)$ for $0 < x < a$, where $a \geq c$ is a constant, then also $u_{2,c}(a) < u_1(a)$ and $v_{2,c}(a) < v_1(a)$. Hence $u_{2,c}(x) \leq u_1(x)$ and $v_{2,c}(x) \leq v_1(x)$ on their common interval of existence. Letting $c \to 0$, we see that $u_2(x) \leq u_1(x)$ and $v_2(x) \leq v_1(x)$ for $x > 0$. Of course this is possible if and only if $u_2 = u_1$ and $v_2 = v_1$ for $x > 0$.

Sufficiency. First, we are going to construct a nondecreasing subsolution w of (2.4), that is, a nondecreasing function positive for $x > 0$ such that

$$w(x) \leq Tw(x) = \int\limits_0^x (x-s)^{\alpha-1}\Big[\int\limits_0^s (s-t)^{\beta-1} g(w(t))\,dt\Big]^\gamma ds. \tag{4.1}$$

Such functions w are equibounded at least on a fixed small interval $x \in [0, \delta]$. To see this we first note that for any subsolution w,

$$\frac{w(x)}{g(w(x))^\gamma} \leq \frac{1}{\beta^\gamma}\int\limits_0^x (x-s)^{\alpha-1} s^{\beta\gamma}\,ds = c(\alpha,\beta,\gamma)x^{\alpha+\beta\gamma} \quad (x > 0), \tag{4.2}$$

where $c(\alpha, \beta, \gamma) > 0$ is a constant. Hence

$$z/g(z)^\gamma \to 0 \quad \text{as } z \to 0.$$

Now, let $M > 0$ and define

$$h(z) = \inf\{s/g(s)^\gamma : z \leq s \leq M\} \quad \text{and} \quad \psi(z) = c(\alpha,\beta,\gamma)z^{\alpha+\beta\gamma} \quad \text{for } z \geq 0.$$

Since the function h is continuous and nondecreasing, we can choose $0 < M_0 < M$ such that $h(M_0) < h(M_0 + \epsilon)$ for $\epsilon > 0$. Due to (4.2) we have

$$w(x) \leq h^{-1}(\psi(x))$$

for $0 \le x \le \delta$, where h^{-1} is the inverse function to h and $\delta > 0$ is such that $\psi(x) \le h(M_0)$ for $0 \le x \le \delta$. Integrating by parts in the inner integral in (4.1) and then substituting $z = w(t)$ we get

$$\int_0^s (s-t)^{\beta-1} g(w(t))\,dt = \frac{1}{\beta}\int_0^s (s-t)^\beta\,dg(w(t)) = \frac{1}{\beta}\int_0^{w(s)} (s - w^{-1}(z))^\beta\,dg(z).$$

Denote

$$F_w(z) = \frac{1}{\beta}\int_0^z (w^{-1}(z) - w^{-1}(s))^\beta\,dg(s).$$

Integrating by parts in the outer integral in (4.1) and then substituting $z = w(s)$ we get

$$Tw(x) = \frac{1}{\alpha\beta^\gamma}\int_0^x (x-s)^\alpha\,d[F_w(w(s))]^\gamma = \frac{1}{\alpha\beta^\gamma}\int_0^{w(x)} (x - w^{-1}(z))^\alpha\,d[F_w(z)]^\gamma.$$

To construct a subsolution we define an auxiliary function $w_0(x)$ by its inverse

$$w_0^{-1}(x) = \int_0^x \left[\frac{s}{g(s)^\gamma}\right]^{\frac{1}{\alpha+\beta\gamma}} \frac{ds}{s} \qquad \text{for } x > 0.$$

We note that

$$\int_0^x (w_0^{-1}(x) - w_0^{-1}(z))^\alpha\,d[F_{w_0}(z)]^\gamma \ge \int_0^{x/2} (w_0^{-1}(x) - w_0^{-1}(z))^\alpha\,d[F_{w_0}(z)]^\gamma \tag{4.3}$$

$$\ge (w_0^{-1}(x) - w_0^{-1}(x/2))^\alpha [F_{w_0}(x/2)]^\gamma,$$

$$w_0^{-1}(x) - w_0^{-1}(x/2) \ge c g(x/2)^{-\frac{\gamma}{\alpha+\beta\gamma}} x^{\frac{1}{\alpha+\beta\gamma}}, \tag{4.4}$$

where $c > 0$ is a constant and

$$F_{w_0}(x/2) = \frac{1}{\beta}\int_0^{x/2} (w_0^{-1}(x/2) - w_0^{-1}(s))^\beta\,dg(s) \tag{4.5}$$

$$\ge \frac{1}{\beta}\int_0^{x/4} (w_0^{-1}(x/2) - w_0^{-1}(s))^\beta dg(s) \ge (w_0^{-1}(x/2) - w_0^{-1}(x/4))^\beta g(x/4).$$

Now combining (4.3)–(4.5) and using (2.1) we see that there exists a constant $c > 0$ such that

$$\int_0^x (w_0^{-1}(x) - w_0^{-1}(z))^\alpha\,d[F_{w_0}(z)]^\gamma \ge \int_0^{x/2} (w_0^{-1}(x) - w_0^{-1}(z))^\alpha\,d[F_{w_0}(z)]^\gamma \ge cx$$

for $x > 0$. Now if take $\tilde{w}_0^{-1}(x) = c^{-\frac{1}{\alpha+\beta\gamma}} w_0^{-1}(x)$ then we get

$$\int_0^x (\tilde{w}_0^{-1}(x) - \tilde{w}_0^{-1}(z))^\alpha \, d[F_{\tilde{w}_0}(z)]^\gamma \ge x.$$

The substitution $z = \tilde{w}_0(x)$ shows that $T\tilde{w}_0(x) \ge \tilde{w}_0(x)$, which means that $\tilde{w}_0(x)$ is a sought subsolution. Moreover, the functions $T^n\tilde{w}_0$, $n = 1, 2, \ldots$, constitute a nondecreasing bounded sequence of subsolutions of (4.1). The required solution u can be obtained as the limit $u(x) = \lim_{n\to\infty} T^n\tilde{w}_0(x)$ for $0 \le x \le \delta$. Thus we get a nontrivial solution u of (2.4) defined on a small interval $[0, \delta]$. Now using standard arguments from the theory of Volterra integral equations [4] one can extend this solution to the maximal interval of existence.

Necessity. Let u be the nontrivial solution of (2.4). Since

$$u^{-1}(x) = \int_0^x (u^{-1})'(s) \, ds < \infty \quad \text{for } x > 0,$$

our assertion follows from the upper estimate of $u'(u^{-1})$ in Corollary 3.4. ■

Proof of Theorem 2.2. Necessity. Let u be the nontrivial solution of (2.4) blowing up at $T < \infty$. Since

$$T = \lim_{x\to\infty} u^{-1}(x) = \int_0^\infty (u^{-1})'(s) \, ds < \infty,$$

our assertion follows from the upper estimate of $u'(u^{-1})$ in Corollary 3.4.

Sufficiency. Assume that (2.3) holds. We first note that a nondecreasing, nontrivial solution $u(x)$ of (2.4) is blowing up if and only if $u^{-1}(x)$ converges to some $0 < T < \infty$ as $x \to \infty$. Since by the left inequality in Corollary 3.4,

$$u^{-1}(x) \le (1/c_\Phi^1) \int_0^x \frac{dz}{\Phi(z)} \quad \text{for } x > 0,$$

it follows from (2.3) that

$$T = \lim_{x\to\infty} u^{-1}(x) \le (1/c_\Phi^1) \lim_{x\to\infty} \int_0^x \frac{dz}{\Phi(z)} < \infty,$$

which shows that $u(x)$ is blowing up at $T > 0$. Thus the proof is complete. ■

EXAMPLE 4.1. Assume $g(u) \sim u^r$ $(r > 0)$ as $u \to 0+$. In this case, $\Phi(u) \sim u^{1+(r\gamma-1)/(\alpha+\beta\gamma)}$ as $u \to 0+$. Therefore the ignition condition (2.2) is equivalent to

$$\int_0^\delta \frac{du}{u^{1+(r\gamma-1)/(\alpha+\beta\gamma)}} < \infty.$$

Thus ignition holds if and only if $r\gamma < 1$.

If $g(u) \sim u^r$ as $u \to \infty$, then $\varPhi(u) \sim u^{1+(r\gamma-1)/(\alpha+\beta\gamma)}$ as $u \to \infty$. Therefore the blow-up condition (2.3) is then equivalent to

$$\int_\delta^\infty \frac{du}{u^{1+(r\gamma-1)/(\alpha+\beta\gamma)}} < \infty \quad (\delta > 0).$$

Thus blow-up occurs if and only if $r\gamma > 1$.

EXAMPLE 4.2. Assume $g(u) \sim u^{1/\gamma}(-\ln u)^r$ $(r > 0)$ as $u \to 0+$. In this case, $\varPhi(u) \sim u(-\ln u)^{r\gamma/(\alpha+\beta\gamma)}$ as $u \to 0+$. Therefore the ignition condition (2.2) is equivalent to

$$\int_0^\delta \frac{du}{u(-\ln u)^{r\gamma/(\alpha+\beta\gamma)}} < \infty.$$

Thus ignition holds if and only if $r > \beta + \alpha/\gamma$.

If $g(u) \sim u^{1/\gamma}(\ln u)^r$ $(r > 0)$ as $u \to \infty$, then $\varPhi(u) \sim u(\ln)^{r\gamma/(\alpha+\beta\gamma)}$ as $u \to \infty$. Therefore the blow-up condition (2.3) is equivalent to

$$\int_\delta^\infty \frac{du}{u(-\ln u)^{r\gamma/(\alpha+\beta\gamma)}} < \infty \quad (\delta > 0).$$

Thus blow-up occurs if and only if $r > \beta + \alpha/\gamma$.

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Wojciech Mydlarczyk
Faculty of Pure and Applied Mathematics
Hugo Steinhaus Center
Wrocław University of Science and Technology
Wybrzeże Wyspiańskiego 27
50-370 Wrocław, Poland
E-mail: wojciech.mydlarczyk@pwr.edu.pl