

## Classification of bounded Baire class $\xi$ functions

by

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**Abstract.** Kechris and Louveau showed that each real-valued bounded Baire class 1 function defined on a compact metric space can be written as an alternating sum of a decreasing countable transfinite sequence of upper semicontinuous functions. Moreover, the length of the shortest such sequence is essentially the same as the value of certain natural ranks they defined on the Baire class 1 functions. They also introduced the notion of pseudouniform convergence to generate some classes of bounded Baire class 1 functions from others. The main aim of this paper is to generalize their results to Baire class  $\xi$  functions. For our proofs to go through, it was essential to first obtain similar results for Baire class 1 functions defined on not necessarily compact Polish spaces. Using these new classifications of bounded Baire class  $\xi$  functions, one can define natural ranks on these classes. We show that these ranks essentially coincide with those defined by Elekes et al. (2014).

**1. Introduction.** A real-valued function on a completely metrizable topological space is of *Baire class 1* if it is the pointwise limit of continuous functions. A *rank* on a class of functions is a map assigning an ordinal to each member of the class, typically measuring complexity.

Kechris and Louveau [8] investigated the properties of three natural ranks on Baire class 1 functions on compact metric spaces. We will recall their definitions in Section 2.1. They proved, among other things, that these ranks *essentially* coincide on bounded functions, showing that for a bounded Baire class 1 function  $f$  and an ordinal  $1 \leq \lambda < \omega_1$ , the value of one of these ranks on  $f$  is at most  $\omega^\lambda$  iff the same holds for the other ranks. This fact made it possible to define a hierarchy of those functions: for a bounded Baire class 1 function  $f$ , let  $f \in \mathcal{B}_1^\lambda$  if the value of one (or equivalently all) of these ranks on  $f$  is at most  $\omega^\lambda$ .

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They also proved that every bounded Baire class 1 function  $f$  can be written as the alternating sum of a decreasing transfinite sequence of upper semicontinuous (USC) functions. (Recall that a function  $g : X \rightarrow \mathbb{R}$  is USC if  $\{x \in X : g(x) < c\}$  is open in  $X$  for every  $c \in \mathbb{R}$ .) Moreover, they showed that the length of the shortest such sequence is at most  $\omega^\lambda$  if and only if  $f \in \mathcal{B}_1^\lambda$ . Hence, if we consider the length of the shortest such sequence as the rank of the function  $f$ , we obtain a new rank on the bounded Baire class 1 functions that coincides essentially with the three ranks investigated by Kechris and Louveau.

They also introduced the notion of pseudouniform convergence, and showed that  $\mathcal{B}_1^{\lambda+1}$  contains exactly those bounded Baire class 1 functions that can be written as the pseudouniform limit of a sequence of functions from  $\mathcal{B}_1^\lambda$ . For limit  $\lambda$ , they proved that  $f \in \mathcal{B}_1^\lambda$  if and only if  $f$  is the uniform limit of functions from  $\bigcup_{\eta < \lambda} \mathcal{B}_1^\eta$ .

Elekes, Kiss and Vidnyánszky [2] generalized their results concerning ranks to functions defined on general Polish spaces. They showed that most of the results proved by Kechris and Louveau remain true in this general setting. They defined analogous ranks on the Baire class  $\xi$  functions. A function is of *Baire class*  $\xi$  for a countable ordinal  $\xi > 1$  if it can be written as the pointwise limit of functions from smaller classes. Similarly to the Baire class 1 case, for a bounded Baire class  $\xi$  function  $f$  and an ordinal  $1 \leq \lambda < \omega_1$ , the value of one of these ranks on  $f$  is at most  $\omega^\lambda$  iff the same holds for the other ranks. We again denote by  $\mathcal{B}_\xi^\lambda$  the set of those bounded Baire class  $\xi$  functions with value of one (or equivalently, all) of these ranks at most  $\omega^\lambda$ .

The motivation for investigating ranks on Baire class  $\xi$  functions came from calculating the so called solvability cardinal of systems of difference equations (see [3]) that are connected to paradoxical geometric decompositions (see e.g. [9, 10]).

This paper is a continuation of the research started in [2]. The main aim is to generalize the results of Kechris and Louveau concerning bounded Baire class 1 functions to the Baire class  $\xi$  case. We show that a bounded Baire class  $\xi$  function  $f$  can be written as the alternating sum of a decreasing transfinite sequence  $(f_\eta)_\eta$  of non-negative *semi-Borel class*  $\xi$  functions (i.e.  $\{x : f_\eta(x) < c\} \in \Sigma_\xi^0$  for all  $c \in \mathbb{R}$  and  $\eta$ ). As in the Baire class 1 case, one can define a rank by assigning to the function  $f$  the length of the shortest such sequence. We show that this rank is essentially equal to those defined in [2]. We also show a method of generating the family  $\mathcal{B}_\xi^{\lambda+1}$  from  $\bigcup_{\eta < \lambda} \mathcal{B}_\xi^{\eta+1}$ .

Our approach is based on topology refinements. Therefore, it was essential to obtain the results of Kechris and Louveau for Baire class 1 functions

defined on general Polish spaces. Our proofs build on ideas of Kechris and Louveau; however, since they relied on the compactness of the space (they used for example the facts that the rank of a characteristic function is always a successor ordinal and that a decreasing sequence of USC functions converging pointwise to 0 converges uniformly), it was necessary to reprove their results.

**2. Preliminaries.** Most of the following basic notation and facts can be found in [7].

Throughout this paper,  $(X, \tau)$  is an uncountable Polish space, i.e., a separable and completely metrizable topological space.

For a set  $H$  we denote the characteristic function, closure and complement of  $H$  by  $\chi_H$ ,  $\overline{H}$  and  $H^c$ , respectively.

We write  $\Sigma_\xi^0$ ,  $\Pi_\xi^0$  and  $\Delta_\xi^0$  for the  $\xi$ th additive, multiplicative and ambiguous classes of the Borel hierarchy, i.e.,  $\Sigma_1^0 = \tau$ ,  $\Pi_1^0 = \{G^c : G \in \tau\}$ ,

$$\Sigma_\xi^0 = \left( \bigcup_{\lambda < \xi} \Pi_\lambda^0 \right)_\sigma, \quad \Pi_\xi^0 = \left( \bigcup_{\lambda < \xi} \Sigma_\lambda^0 \right)_\delta, \quad \Delta_\xi^0 = \Sigma_\xi^0 \cap \Pi_\xi^0,$$

where  $\mathcal{H}_\sigma = \{\bigcup_{n \in \mathbb{N}} H_n : H_n \in \mathcal{H}\}$  and  $\mathcal{H}_\delta = \{\bigcap_{n \in \mathbb{N}} H_n : H_n \in \mathcal{H}\}$ .

For a function  $f : X \rightarrow \mathbb{R}$  we write  $\|f\| = \sup_{x \in X} |f(x)|$ , whereas  $|f|$  denotes the function  $x \mapsto |f(x)|$ . If  $c \in \mathbb{R}$  then we let  $\{f < c\} = \{x \in X : f(x) < c\}$ . We use the notation  $\{f > c\}$ ,  $\{f \leq c\}$  and  $\{f \geq c\}$  similarly.

We denote the family of real valued functions defined on  $X$  that are of Baire class  $\xi$  by  $\mathcal{B}_\xi$ . It is well-known that a function  $f$  is of Baire class  $\xi$  iff  $f^{-1}(U) \in \Sigma_{\xi+1}^0$  for every  $U \subseteq \mathbb{R}$  open iff  $\{f < c\}, \{f > c\} \in \Sigma_{\xi+1}^0$  for every  $c \in \mathbb{R}$ . We use the abbreviation USC for upper semicontinuous functions, i.e., a function  $f : X \rightarrow \mathbb{R}$  is USC if  $\{f < c\}$  is open for every  $c \in \mathbb{R}$ . As an analogue, a function  $f$  is of semi-Borel class  $\xi$  if  $\{f < c\} \in \Sigma_\xi^0$  for every  $c \in \mathbb{R}$ . Note that the pointwise infimum of an arbitrary class of non-negative USC functions is USC.

For a countable ordinal  $\xi \geq 1$  we denote by  $\text{DUSB}_\xi$  the set of non-negative, bounded, transfinite decreasing sequences of semi-Borel class  $\xi$  functions  $(f_\eta)_{\eta < \lambda}$  with  $\lambda < \omega_1$  and  $f_\eta \rightarrow 0$  as  $\eta \rightarrow \lambda$  for limit  $\lambda$ . The length of a sequence  $(f_\eta)_{\eta < \lambda} \in \text{DUSB}_\xi$  is  $\text{length}((f_\eta)_{\eta < \lambda}) = \lambda$ .

If  $\tau'$  is a topology on  $X$  then we denote the set of Baire class  $\xi$  functions with respect to  $\tau'$  by  $\mathcal{B}_\xi(\tau')$ . Analogously,  $\Sigma_\xi^0(\tau')$  stands for the  $\xi$ th additive class of  $(X, \tau')$ , and similarly for  $\Pi_\xi^0(\tau')$  and  $\Delta_\xi^0(\tau')$ . Moreover, we will use the notation  $\text{DUSB}_\xi(\tau')$  analogously.

**2.1. Short introduction to ranks.** A rank on a class of functions  $\mathcal{F}$  is a map assigning an ordinal to each  $f \in \mathcal{F}$ . In this section we give the basic definitions for ranks on the Baire class  $\xi$  functions that we will need.

For more on ranks defined on the Baire class 1 functions on a compact space see [8], and for generalizations to Baire class  $\xi$  functions on Polish spaces see [2].

**2.1.1. Derivatives.** The definition of some ranks will use the notion of a *derivative operation*. A *derivative* on the closed subsets of  $X$  is a map  $D : \mathbf{\Pi}_1^0 \rightarrow \mathbf{\Pi}_1^0$  such that  $D(A) \subseteq A$  and  $A \subseteq B \Rightarrow D(A) \subseteq D(B)$  for all  $A, B \in \mathbf{\Pi}_1^0$ . In the definition below, every derivative operation will satisfy these conditions. However, we omit the proofs of these easy facts; for a more thorough introduction consult the above references.

For a derivative  $D$  we define the *iterated derivatives* of the closed set  $F$  as follows:

$$\begin{aligned} D^0(F) &= F, \\ D^{\theta+1}(F) &= D(D^\theta(F)), \\ D^\theta(F) &= \bigcap_{\eta < \theta} D^\eta(F) \quad \text{if } \theta \text{ is limit.} \end{aligned}$$

The *rank* of  $D$  is the smallest ordinal  $\theta$  such that  $D^\theta(X) = \emptyset$  if such an ordinal exists, and  $\omega_1$  otherwise. We denote the rank of  $D$  by  $\text{rk}(D)$ .

**2.1.2. Ranks on Baire class 1 functions.** Now we look at ranks on the Baire class 1 functions. The *separation rank* has been first introduced by Bourgain [1]. Let  $A$  and  $B$  be two subsets of  $X$ . We associate a derivative with them by  $D_{A,B}(F) = \overline{F \cap A} \cap \overline{F \cap B}$  and denote the rank of this derivative by  $\alpha(A, B)$ . The *separation rank* of a Baire class 1 function  $f$  is

$$\alpha(f) = \sup_{\substack{p < q \\ p, q \in \mathbb{Q}}} \alpha(\{f \leq p\}, \{f \geq q\}).$$

The *oscillation rank* was investigated by many authors (see e.g. [6]). The *oscillation* of a function  $f : X \rightarrow \mathbb{R}$  at a point  $x \in X$  restricted to a closed set  $F \subseteq X$  is

$$\omega(f, x, F) = \inf \left\{ \sup_{x_1, x_2 \in U \cap F} |f(x_1) - f(x_2)| : U \text{ open, } x \in U \right\}.$$

For each  $\varepsilon > 0$  consider the derivative  $D_{f,\varepsilon}(F) = \{x \in F : \omega(f, x, F) \geq \varepsilon\}$ . The *oscillation rank* of a function  $f$  is

$$(2.1) \quad \beta(f) = \sup_{\varepsilon > 0} \text{rk}(D_{f,\varepsilon}).$$

Next we define the *convergence rank* (see e.g. Zalcwasser [11] and Gillespie and Hurwitz [5]). Let  $(f_n)_{n \in \mathbb{N}}$  be a sequence of real valued functions on  $X$ . The *oscillation* of this sequence at a point  $x$  restricted to a closed set  $F \subseteq X$  is

$$\omega((f_n)_{n \in \mathbb{N}}, x, F) = \inf_{\substack{x \in U \\ U \text{ open}}} \inf_{N \in \mathbb{N}} \sup \{ |f_m(y) - f_n(y)| : n, m \geq N, y \in U \cap F \}.$$

Consider a sequence  $(f_n)_{n \in \mathbb{N}}$  of functions, and for each  $\varepsilon > 0$ , let a derivative be defined by  $D_{(f_n)_{n \in \mathbb{N}}, \varepsilon}(F) = \{x \in F : \omega((f_n)_{n \in \mathbb{N}}, x, F) \geq \varepsilon\}$ . Again, for a sequence  $(f_n)_{n \in \mathbb{N}}$  let

$$(2.2) \quad \gamma((f_n)_{n \in \mathbb{N}}) = \sup_{\varepsilon > 0} \text{rk}(D_{(f_n)_{n \in \mathbb{N}}, \varepsilon}).$$

For a Baire class 1 function  $f$  let the *convergence rank* of  $f$  be defined by

$$(2.3) \quad \gamma(f) = \min\{\gamma((f_n)_{n \in \mathbb{N}}) : \text{each } f_n \text{ is continuous and } f_n \rightarrow f \text{ pointwise}\}.$$

**2.1.3. Ranks on Baire class  $\xi$  functions.** Let  $(F_\eta)_{\eta < \lambda}$  be a continuous (i.e., for a limit ordinal  $\theta < \lambda$ ,  $\bigcap_{\eta < \theta} F_\eta = F_\theta$ ) decreasing sequence of  $\Pi_\xi^0$  sets for some  $\lambda < \omega_1$  with  $F_0 = X$  and  $\bigcap_{\eta < \lambda} F_\eta = \emptyset$  if  $\lambda$  is limit. We say that the sets  $A$  and  $B$  can be separated by the transfinite difference of this sequence if

$$A \subseteq \bigcup_{\substack{\eta < \lambda \\ \eta \text{ even}}} F_\eta \setminus F_{\eta+1} \subseteq B^c,$$

where  $F_\eta = \emptyset$  if  $\eta \geq \lambda$ . We denote by  $\alpha_\xi(A, B)$  the length of the shortest such sequence if there is any, otherwise we let  $\alpha_\xi(A, B) = \omega_1$ . We define the *modified separation rank* of a Baire class  $\xi$  function  $f$  as

$$\alpha_\xi(f) = \sup_{\substack{p < q \\ p, q \in \mathbb{Q}}} \alpha_\xi(\{f \leq p\}, \{f \geq q\}).$$

Now we introduce one of the methods used in [2] to construct ranks on the Baire class  $\xi$  functions from existing ranks on the Baire class 1 functions.

Let  $f$  be of Baire class  $\xi$ . Let

$$(2.4) \quad T_{f, \xi} = \{\tau' : \tau' \supseteq \tau \text{ Polish, } \tau' \subseteq \Sigma_\xi^0(\tau), f \in \mathcal{B}_1(\tau')\}.$$

Let  $\rho$  be a rank on the Baire class 1 functions and let

$$\rho_\xi^*(f) = \min_{\tau' \in T_{f, \xi}} \rho_{\tau'}(f),$$

where  $\rho_{\tau'}(f)$  is just the  $\rho$  rank of  $f$  in the topology  $\tau'$ . This method yields the rank  $\rho_\xi^*$  on the Baire class  $\xi$  functions.

We use the notation

$$\mathcal{B}_\xi^\lambda = \{f \in \mathcal{B}_\xi : f \text{ is bounded and } \alpha_\xi(f) \leq \omega^\lambda\}.$$

We also use the notation  $\mathcal{B}_\xi^\lambda(\tau')$  for the corresponding class with respect to the topology  $\tau'$  on  $X$ . Note that by [2, 3.14] and [2, 3.35], for a bounded Baire class 1 function  $f$  we have  $f \in \mathcal{B}_1^\lambda \Leftrightarrow \alpha(f) \leq \omega^\lambda \Leftrightarrow \beta(f) \leq \omega^\lambda \Leftrightarrow \gamma(f) \leq \omega^\lambda$ , and by [2, 5.7], for a bounded function  $f \in \mathcal{B}_\xi$  we have  $f \in \mathcal{B}_\xi^\lambda \Leftrightarrow \alpha_\xi^*(f) \leq \omega^\lambda \Leftrightarrow \beta_\xi^*(f) \leq \omega^\lambda \Leftrightarrow \gamma_\xi^*(f) \leq \omega^\lambda$ .

REMARK 2.1. For a function  $f$ ,  $f \in \mathcal{B}_\xi^\lambda$  if and only if there exists a topology  $\tau' \in T_{f,\xi}$  such that  $f \in \mathcal{B}_1^\lambda(\tau')$ . This can be easily seen:  $f \in \mathcal{B}_\xi^\lambda \Leftrightarrow \alpha_\xi^*(f) \leq \omega^\lambda \Leftrightarrow \exists \tau' \in T_{f,\xi} (\alpha_{\tau'}(f) \leq \omega^\lambda) \Leftrightarrow \exists \tau' \in T_{f,\xi} (f \in \mathcal{B}_1^\lambda(\tau'))$ .

Now we prove three lemmas about these ranks to be used later on.

LEMMA 2.2. For a characteristic function  $\chi_A \in \mathcal{B}_1$ ,  $\alpha(f) = \beta(f)$ .

*Proof.* It is enough to prove that for every  $\varepsilon < 1$  and  $F \subseteq X$  closed, we have  $D_{\{\chi_A \leq 0\}, \{\chi_A \geq 1\}}(F) = D_{\chi_A, \varepsilon}(F)$ . Let  $x \in X$ . Then  $x \in D_{\chi_A, \varepsilon}(F) \Leftrightarrow \omega(f, x, F) \geq \varepsilon \Leftrightarrow (x \in U \text{ is open} \Rightarrow \exists y, z \in U \cap F (y \in A \wedge z \notin A)) \Leftrightarrow x \in \overline{F} \cap A \cap \overline{F} \cap A^c \Leftrightarrow x \in D_{\{\chi_A \leq 0\}, \{\chi_A \geq 1\}}(F)$ . ■

LEMMA 2.3. Let  $(f_n)_{n \in \mathbb{N}}, (g_n)_{n \in \mathbb{N}}$  be two sequences of functions such that  $\gamma((f_n)_{n \in \mathbb{N}}), \gamma((g_n)_{n \in \mathbb{N}}) \leq \omega^\lambda$  for some  $\lambda < \omega_1$ . Then  $\gamma((f_n + g_n)_{n \in \mathbb{N}}) \leq \omega^\lambda$ .

*Proof.* By [2, Theorem 3.29], the rank  $\gamma$  defined on  $\mathcal{B}_1$  satisfies  $\gamma(f+g) \leq \omega^\lambda$  whenever  $\gamma(f), \gamma(g) \leq \omega^\lambda$ . But they actually prove the statement of this lemma and derive the theorem from this fact. ■

LEMMA 2.4. If  $f : X \rightarrow \mathbb{R}$  is any function and  $g : \mathbb{R} \rightarrow \mathbb{R}$  is a Lipschitz map then  $\beta(g \circ f) \leq \beta(f)$ .

*Proof.* Let the Lipschitz constant of  $g$  be  $c$ . Then one can easily see that  $\omega(g \circ f, x, F) \leq c \cdot \omega(f, x, F)$  for every  $x \in X$  and  $F \subseteq X$  closed, hence  $\text{rk}(D_{g \circ f, c \cdot \varepsilon}) \leq \text{rk}(D_{f, \varepsilon})$ , showing that  $\beta(g \circ f) \leq \beta(f)$ . ■

**3. The alternating sums of semi-Borel class  $\xi$  functions.** Now we define the notion of an alternating sum of a transfinite sequence of semi-Borel class  $\xi$  functions. It is a generalization of the alternating sum of USC functions defined by A. S. Kechris and A. Louveau [8].

DEFINITION 3.1. Let  $\lambda$  be a countable ordinal and let  $(f_\eta)_{\eta < \lambda} \in \text{DUSB}_\xi$ . The function  $\sum_{\eta < \theta}^* (-1)^\eta f_\eta$  is defined inductively on  $\theta \leq \lambda$  by

$$\sum_{\eta < \theta+1}^* (-1)^\eta f_\eta = \sum_{\eta < \theta}^* (-1)^\eta f_\eta + (-1)^\theta f_\theta,$$

where  $(-1)^\theta = 1$  if  $\theta$  is even and  $-1$  if  $\theta$  is odd, and for limit  $\theta \leq \lambda$ ,

$$\sum_{\eta < \theta}^* (-1)^\eta f_\eta = \sup \left\{ \sum_{\eta < \zeta}^* (-1)^\eta f_\eta : \zeta \text{ is even, } \zeta < \theta \right\}.$$

For a function  $f$ , if  $(f_\eta)_{\eta < \lambda} \in \text{DUSB}_\xi$  is a sequence with  $f = c + \sum_{\eta < \lambda}^* (-1)^\eta f_\eta$  for some  $c \in \mathbb{R}$ , then we say that  $f$  is the sum of a constant and the alternating sequence  $(f_\eta)_{\eta < \lambda}$  of length  $\lambda$ . We use the notation

$$\text{length}_\xi(f) = \inf \left\{ \lambda : \exists (f_\eta)_{\eta < \lambda} \in \text{DUSB}_\xi, c \in \mathbb{R} \left( f = c + \sum_{\eta < \lambda}^* (-1)^\eta f_\eta \right) \right\},$$

where we define  $\text{length}_\xi(f)$  to be  $\omega_1$  if  $f$  is not the sum of a constant and an alternating sequence from  $\text{DUSB}_\xi$ .

REMARK 3.2. It is easy to prove by transfinite induction that for even ordinals  $\theta_1 \leq \theta_2$  we have

$$(3.1) \quad \sum_{\eta < \theta_1}^* (-1)^\eta f_\eta \leq \sum_{\eta < \theta_2}^* (-1)^\eta f_\eta.$$

From this fact, for limit  $\theta$ , if  $\theta_n \rightarrow \theta$  with all  $\theta_n < \theta$  even then

$$(3.2) \quad \sum_{\eta < \theta}^* (-1)^\eta f_\eta = \lim_{n \rightarrow \infty} \sum_{\eta < \theta_n}^* (-1)^\eta f_\eta.$$

We will use this fact to calculate  $\sum_{\eta < \theta}^* (-1)^\eta f_\eta$ .

REMARK 3.3. Let  $(f_\eta)_{\eta < \lambda} \in \text{DUSB}_\xi$  and  $\theta \leq \lambda$  with  $\theta$  even. We show by transfinite induction on  $\zeta$  that for every  $\theta \leq \zeta \leq \lambda$  even, we have

$$(3.3) \quad 0 \leq \sum_{\eta < \zeta}^* (-1)^\eta f_\eta - \sum_{\eta < \theta}^* (-1)^\eta f_\eta \leq f_\theta - f_\zeta.$$

For  $\zeta + 2$  we have

$$\begin{aligned} 0 &\leq \sum_{\eta < \zeta}^* (-1)^\eta f_\eta - \sum_{\eta < \theta}^* (-1)^\eta f_\eta \\ &\leq \sum_{\eta < \zeta}^* (-1)^\eta f_\eta + f_\zeta - f_{\zeta+1} - \sum_{\eta < \theta}^* (-1)^\eta f_\eta \\ &\leq f_\theta - f_\zeta + f_\zeta - f_{\zeta+1} \leq f_\theta - f_{\zeta+2}, \end{aligned}$$

where the expression in the middle equals

$$\sum_{\eta < \zeta+2}^* (-1)^\eta f_\eta - \sum_{\eta < \theta}^* (-1)^\eta f_\eta,$$

proving the successor case. For limit  $\zeta$ , (3.3) is an easy consequence of (3.2) and the monotonicity of the sequence  $(f_\eta)_{\eta < \lambda}$ .

Now let  $f = \sum_{\eta < \lambda}^* (-1)^\eta f_\eta$ . Since the alternating sum of a sequence does not change if we append 0 functions to it, we can suppose that  $\lambda$  is even. Hence we can substitute  $\zeta = \lambda$  to get

$$(3.4) \quad 0 \leq f - \sum_{\eta < \theta}^* (-1)^\eta f_\eta \leq f_\theta,$$

in particular,

$$(3.5) \quad 0 \leq f \leq f_0.$$

THEOREM 3.4. *Let  $f$  be a bounded Baire class 1 function. Then  $f \in \mathcal{B}_1^\lambda$  if and only if  $\text{length}_1(f) \leq \omega^\lambda$ .*

REMARK 3.5. A straightforward consequence of this theorem is that every bounded Baire class 1 function can be written as the sum of a constant and an alternating sequence from  $\text{DUSB}_1$  (as  $\alpha_1(f) < \omega_1$  for every Baire class 1 function  $f$ , see [2, 3.15]). For the other direction (if  $f$  can be written in this form then  $f$  is a bounded Baire class 1 function), see [4].

*Proof of Theorem 3.4.* It is easy to see that it is enough to prove the theorem for non-negative functions, since for any constant  $c$ ,  $f \in \mathcal{B}_1^\lambda \Leftrightarrow f + c \in \mathcal{B}_1^\lambda$  and  $\text{length}_1(f + c) = \text{length}_1(f)$ . We first show that if  $f \in \mathcal{B}_1^\lambda$  then  $\text{length}_1(f) \leq \omega^\lambda$ .

Let  $f \in \mathcal{B}_1^\lambda$  be a characteristic function, i.e.,  $f = \chi_A$  for some  $A \subseteq X$ . Using the definition of  $\mathcal{B}_1^\lambda$ , we can separate  $\{f \geq 1\} = A$  and  $\{f \leq 0\} = A^c$  with an appropriate sequence, hence  $A$  can be written as

$$A = \bigcup_{\substack{\eta < \omega^\lambda \\ \eta \text{ is even}}} F_\eta \setminus F_{\eta+1},$$

where  $(F_\eta)_{\eta < \omega^\lambda}$  is a decreasing, continuous sequence of closed sets with  $F_0 = X$  and  $\bigcap_{\eta < \omega^\lambda} F_\eta = \emptyset$ .

Now let  $f_\eta = \chi_{F_\eta}$ . It is easy to see that  $(f_\eta)_{\eta < \omega^\lambda}$  is a decreasing sequence of non-negative, bounded USC functions with  $f_\eta \rightarrow 0$  as  $\eta \rightarrow \omega^\lambda$ . From this  $(f_\eta)_{\eta < \omega^\lambda} \in \text{DUSB}_1$ , hence to prove that  $\text{length}_1(f) \leq \omega^\lambda$ , it is enough to prove that  $f = \sum_{\eta < \omega^\lambda}^* (-1)^\eta f_\eta$ . We do this by proving that for every  $\theta \leq \omega^\lambda$  even we have

$$\sum_{\eta < \theta}^* (-1)^\eta f_\eta = \chi_{\bigcup_{\text{even } \eta < \theta} F_\eta \setminus F_{\eta+1}}.$$

For  $\theta = 0$  this is obvious. Suppose this holds for  $\theta$ . Then

$$\begin{aligned} \sum_{\eta < \theta+2}^* (-1)^\eta f_\eta &= \sum_{\eta < \theta}^* (-1)^\eta f_\eta + f_\theta - f_{\theta+1} \\ &= \chi_{\bigcup_{\text{even } \eta < \theta} F_\eta \setminus F_{\eta+1}} + \chi_{F_\theta} - \chi_{F_{\theta+1}} = \chi_{\bigcup_{\text{even } \eta < \theta+2} F_\eta \setminus F_{\eta+1}}. \end{aligned}$$

For limit  $\theta$  let  $\theta_n \rightarrow \theta$  with all  $\theta_n < \theta$  even. Then

$$\begin{aligned} \sum_{\eta < \theta}^* (-1)^\eta f_\eta &= \lim_{n \rightarrow \infty} \sum_{\eta < \theta_n}^* (-1)^\eta f_\eta \\ &= \lim_{n \rightarrow \infty} \chi_{\bigcup_{\text{even } \eta < \theta_n} F_\eta \setminus F_{\eta+1}} = \chi_{\bigcup_{\text{even } \eta < \theta} F_\eta \setminus F_{\eta+1}}, \end{aligned}$$

proving  $\text{length}_1(f) \leq \omega^\lambda$  for the characteristic function  $f \in \mathcal{B}_1^\lambda$ .

Now let  $f \in \mathcal{B}_1^\lambda$  be a non-negative step function, that is, a linear combination of characteristic functions. Such a function can be written as  $f = \sum_{i=1}^n c_i \chi_{A_i}$  where the  $c_i$ 's are distinct, non-negative real numbers and the  $A_i$ 's form a partition of  $X$  with  $A_i \in \Delta_2^0$  for each  $i$ . By the above statement, each  $\chi_{A_i}$  can be written as  $\chi_{A_i} = \sum_{\eta < \omega^\lambda}^* (-1)^\eta f_\eta^i$ , where  $(f_\eta^i)_{\eta < \omega^\lambda} \in \text{DUSB}_1$ ,

since  $\alpha_1(\chi_{A_i}) \leq \omega^\lambda$  (see [2, 3.38] and [2, 3.14]). Now let  $f_\eta = \sum_{i=1}^n c_i \cdot f_\eta^i$ . It is easy to see that  $(f_\eta)_{\eta < \omega^\lambda} \in \text{DUSB}_1$  and  $f = \sum_{\eta < \omega^\lambda}^* (-1)^\eta f_\eta$ , showing that  $\text{length}_1(f) \leq \omega^\lambda$  for step functions  $f \in \mathcal{B}_1^\lambda$ . Moreover, this construction shows that the  $f_\eta$ 's can be chosen in such a way that

$$(3.6) \quad \|f_\eta\| \leq \|f\|.$$

Now we turn to the case of arbitrary non-negative bounded functions.

LEMMA 3.6. *If  $f \in \mathcal{B}_1^\lambda$  then there exists a sequence  $(g^k)_{k \in \mathbb{N}}$  of non-negative step functions  $g^k \in \mathcal{B}_1^\lambda$  such that  $\inf f + \sum_{k \in \mathbb{N}} g^k = f$  and  $\|g^k\| \leq 1/2^k$  for  $k \geq 1$ .*

*Proof.* It is enough to show that there exists such a sequence with  $\sum_{k \in \mathbb{N}} g^k = f$  for a non-negative function  $f$ , since  $f - \inf f \in \mathcal{B}_1^\lambda$  is always non-negative.

So let  $f \in \mathcal{B}_1^\lambda$  be non-negative. Then there exists a sequence  $(f^k)_{k \in \mathbb{N}}$  of step functions converging uniformly to  $f$  with  $f^k \in \mathcal{B}_1^\lambda$  for every  $k \in \mathbb{N}$  (see [2, 3.40]). By taking a subsequence, we can suppose that  $\|f^k - f\| \leq 1/2^{k+5}$ . By replacing  $f^k$  with  $\max\{f^k - 1/2^{k+3}, 0\}$ , we can suppose moreover that  $(f^k)_{k \in \mathbb{N}}$  is an increasing sequence of non-negative functions now satisfying  $\|f^k - f\| \leq 1/2^{k+2}$ , and using Lemma 2.4, we still have  $f^k \in \mathcal{B}_1^\lambda$ .

Let  $g^0 = f^0$  and for  $k \geq 1$  let  $g^k = f^k - f^{k-1}$ . Then  $g^k \geq 0$ ,  $\|g^k\| \leq 1/2^k$  for  $k \geq 1$  and  $\sum_{k \in \mathbb{N}} g^k = f$ . By [2, 3.29],  $g^k \in \mathcal{B}_1^\lambda$ , proving the lemma. ■

Now let  $(g^k)_{k \in \mathbb{N}}$  be the sequence given by the lemma and replace  $g^0$  with  $g^0 + \inf f$ . Then  $g^0$  remains non-negative and now  $\sum_{k \in \mathbb{N}} g^k = f$ . Since  $g^k \in \mathcal{B}_1^\lambda$  is a step function for each  $k$ , we can write

$$g^k = \sum_{\eta < \omega^\lambda}^* (-1)^\eta g_\eta^k,$$

where  $(g_\eta^k)_{\eta < \omega^\lambda} \in \text{DUSB}_1$  and each  $g_\eta^k$  is chosen to satisfy (3.6), hence  $\|g_\eta^k\| \leq \|g^k\|$ .

For  $\eta < \omega^\lambda$  let

$$f_\eta = \sum_{k \in \mathbb{N}} g_\eta^k.$$

We claim that  $(f_\eta)_{\eta < \omega^\lambda} \in \text{DUSB}_1$  and

$$f = \sum_{\eta < \omega^\lambda}^* (-1)^\eta f_\eta.$$

It is enough to establish these claims to finish the proof of the implication  $f \in \mathcal{B}_1^\lambda \Rightarrow \text{length}_1(f) \leq \omega^\lambda$ .

Since  $\|g_\eta^k\| \leq 1/2^k$  for  $k \geq 1$ ,  $\|g_\eta^0\| \leq \|g^0\|$  and  $(g_\eta^k)_{\eta < \omega^\lambda} \in \text{DUSB}_1$ , the sequence  $(f_\eta)_{\eta < \omega^\lambda}$  is a non-negative, bounded, decreasing sequence of USC functions, as the finite sum and uniform limit of USC functions is USC.

Now we show that  $f_\eta \rightarrow 0$  as  $\eta \rightarrow \omega^\lambda$ . Let  $x \in X$  and  $\varepsilon > 0$  be fixed. There exists a  $k_0$  with  $\sum_{k \geq k_0} g_\eta^k(x) \leq \sum_{k \geq k_0} 1/2^k < \varepsilon/2$ . For this  $k_0$ , we can find an ordinal  $\lambda_0 < \omega^\lambda$  such that for every  $\lambda_0 \leq \eta < \omega^\lambda$  and  $k < k_0$ , we have  $g_\eta^k(x) < \varepsilon/(2k_0)$ , since  $g_\eta^k \rightarrow 0$  as  $\eta \rightarrow \omega^\lambda$  for each  $k$ . Hence for every  $\lambda_0 \leq \eta < \omega^\lambda$  we have  $f_\eta(x) \leq \varepsilon$ , showing that  $f_\eta \rightarrow 0$  as  $\eta \rightarrow \omega^\lambda$ , thus proving  $(f_\eta)_{\eta < \omega^\lambda} \in \text{DUSB}_1$ .

To show that  $f = \sum_{\eta < \omega^\lambda}^* (-1)^\eta f_\eta$ , we prove by transfinite induction that for every  $\theta \leq \omega^\lambda$ ,

$$\sum_{\eta < \theta}^* (-1)^\eta f_\eta = \sum_{k \in \mathbb{N}} \sum_{\eta < \theta}^* (-1)^\eta g_\eta^k.$$

Suppose this holds for  $\theta$ . Then

$$\begin{aligned} \sum_{\eta < \theta+1}^* (-1)^\eta f_\eta &= \sum_{\eta < \theta}^* (-1)^\eta f_\eta + (-1)^\theta f_\theta \\ &= \sum_{k \in \mathbb{N}} \sum_{\eta < \theta}^* (-1)^\eta g_\eta^k + \sum_{k \in \mathbb{N}} (-1)^\theta g_\theta^k = \sum_{k \in \mathbb{N}} \sum_{\eta < \theta+1}^* (-1)^\eta g_\eta^k. \end{aligned}$$

And for limit  $\theta$  let  $\theta_n \rightarrow \theta$  with all  $\theta_n < \theta$  even. Then

$$\begin{aligned} \sum_{\eta < \theta}^* (-1)^\eta f_\eta &= \lim_{n \rightarrow \infty} \sum_{\eta < \theta_n}^* (-1)^\eta f_\eta = \lim_{n \rightarrow \infty} \sum_{k \in \mathbb{N}} \sum_{\eta < \theta_n}^* (-1)^\eta g_\eta^k \\ &= \sum_{k \in \mathbb{N}} \lim_{n \rightarrow \infty} \sum_{\eta < \theta_n}^* (-1)^\eta g_\eta^k = \sum_{k \in \mathbb{N}} \sum_{\eta < \theta}^* (-1)^\eta g_\eta^k, \end{aligned}$$

where we have used the dominated convergence theorem to interchange the operators  $\lim$  and  $\sum$ : For a fixed  $x \in X$  let

$$h_n(k) = \sum_{\eta < \theta_n}^* (-1)^\eta g_\eta^k(x) \quad \text{and} \quad h(k) = \sum_{\eta < \theta}^* (-1)^\eta g_\eta^k(x).$$

Then  $h_n(k)$  converges to  $h(k)$  for every  $k$ , and for every  $n \in \mathbb{N}$  by (3.1) and (3.5) we have  $|h_n(k)| \leq H(k)$ , where  $H(k) = \|g_0^k\|$ . The function  $H(k)$  is summable, since  $H(k) \leq 1/2^k$  for  $k \geq 1$ , hence we can apply the dominated convergence theorem to get  $\lim_{n \rightarrow \infty} \sum_{k \in \mathbb{N}} h_n(k) = \sum_{k \in \mathbb{N}} \lim_{n \rightarrow \infty} h_n(k)$ . This finishes the proof of  $\text{length}_1(f) \leq \omega^\lambda$  for  $f \in \mathcal{B}_1^\lambda$ .

Now we prove the following two statements by transfinite induction on  $\lambda$ :

$$(3.7) \quad \text{if } f = \sum_{\eta < \omega^\lambda}^* (-1)^\eta f_\eta \text{ with } (f_\eta)_{\eta < \omega^\lambda} \in \text{DUSB}_1 \text{ then } f \in \mathcal{B}_1^\lambda,$$

$$(3.8) \quad \text{if } f = \sum_{\eta < \omega^\lambda}^* (-1)^\eta f_\eta \text{ with } (f_\eta)_{\eta < \omega^\lambda} \in \text{DUSB}'_1 \text{ then } f \in \mathcal{B}_1^{\lambda+1},$$

where  $\text{DUSB}'_1$  consists of all decreasing, transfinite sequences of bounded, non-negative USC functions of countable length, i.e., we do not assume that

$f_\eta \rightarrow 0$  as  $\eta \rightarrow \omega^\lambda$  for the sequence  $(f_\eta)_{\eta < \omega^\lambda} \in \text{DUSB}'_1$ . It is easy to see that (3.7) yields the second part of the theorem, hence it is enough to prove these two statements.

First we prove (3.7) for  $\lambda + 1$ , supposing (3.7) and (3.8) for  $\lambda$ . So let  $f = \sum_{\eta < \omega^{\lambda+1}}^* (-1)^\eta f_\eta$ , where  $(f_\eta)_{\eta < \omega^{\lambda+1}} \in \text{DUSB}_1$ . Let  $f^k = \sum_{\eta < \omega^{\lambda \cdot k}}^* (-1)^\eta f_\eta$ . By (3.2) we have  $f^k \rightarrow f$ .

CLAIM 3.7.  $\beta(f^k) \leq \omega^{\lambda+1}$ .

*Proof.* We prove this by induction on  $k$ . For  $k = 1$  this is (3.8) for  $\lambda$  as the sequence  $(f_\eta)_{\eta < \omega^\lambda}$  is in  $\text{DUSB}'_1$ . For  $k + 1$  we have  $f^{k+1} = f^k + g^k$ , where  $g^k = f^{k+1} - f^k$ . We have  $g^k = \sum_{\eta < \omega^\lambda}^* (-1)^\eta f'_\eta$ , where  $f'_\eta = f_{\omega^\lambda \cdot k + \eta}$  with  $(f'_\eta)_{\eta < \omega^\lambda} \in \text{DUSB}'_1$ . Now using (3.8) for  $g^k$  we have  $g^k \in \mathcal{B}_1^{\lambda+1}$ , hence  $f^{k+1} = f^k + g^k \in \mathcal{B}_1^{\lambda+1}$  using [2, 3.29] to show that  $\beta(f^k), \beta(g^k) \leq \omega^{\lambda+1}$  implies  $\beta(f^{k+1}) \leq \omega^{\lambda+1}$ . ■

Now we prove  $f \in \mathcal{B}_1^{\lambda+1}$  by showing that  $\beta(f) \leq \omega^{\lambda+1}$ . Let  $x \in X$ . It is enough to prove that  $x \notin D_{f,\varepsilon}^{\omega^{\lambda+1}}(X)$  for every  $\varepsilon > 0$ . By (3.4) we have  $0 \leq f - f^k \leq f_{\omega^\lambda \cdot k}$ , hence there exists a  $k$  such that  $|f(x) - f^k(x)| \leq f_{\omega^\lambda \cdot k}(x) \leq \varepsilon/5$ . Since  $f_{\omega^\lambda \cdot k}$  is USC, we have an open set  $x \in U$  with  $|f(y) - f^k(y)| \leq f_{\omega^\lambda \cdot k}(y) \leq \varepsilon/4$  for every  $y \in U$ . Now we need the following lemma.

LEMMA 3.8. *If  $f$  and  $g$  are Baire class 1 functions,  $U$  is open and  $F$  is closed with  $|f(y) - g(y)| \leq \varepsilon/4$  for every  $y \in F \cap U$  then for every  $\eta < \omega_1$ ,*

$$D_{f,\varepsilon}^\eta(F) \cap U \subseteq D_{g,\varepsilon/4}^\eta(F) \cap U.$$

*Proof.* The proof is by transfinite induction on  $\eta$ . For  $\eta = 0$  this is obvious from the definition of the derivative. Let  $x \in (D_{g,\varepsilon/4}^\eta(F) \cap U) \setminus D_{f,\varepsilon}^{\eta+1}(F)$ . We need to show that  $x \notin D_{f,\varepsilon}^{\eta+1}(F)$ . There is an open neighborhood  $x \in V \subseteq U$  such that  $|g(y) - g(z)| < \varepsilon/4$  for every  $y, z \in D_{g,\varepsilon/4}^\eta(F) \cap V$ . Then  $|f(y) - f(z)| < 3\varepsilon/4$  for all  $y, z \in D_{g,\varepsilon/4}^\eta(F) \cap V$ . By the induction hypothesis  $D_{f,\varepsilon}^\eta(F) \cap V \subseteq D_{g,\varepsilon/4}^\eta(F) \cap V$ , hence this holds for all  $y, z \in D_{f,\varepsilon}^\eta(F) \cap V$ , thus  $x \notin D_{f,\varepsilon}^{\eta+1}(F)$ . This shows the successor case, and for limit  $\eta$  the lemma is an easy consequence of the definition of the derivative. ■

Applying the lemma with  $g = f^k$ ,  $F = X$  and  $\eta = \omega^{\lambda+1}$ , we see that  $D_{f,\varepsilon}^{\omega^{\lambda+1}}(X) \cap U \subseteq D_{f^k,\varepsilon/4}^{\omega^{\lambda+1}}(X) \cap U = \emptyset$ , since  $\beta(f^k) \leq \omega^{\lambda+1}$ . This shows that  $x \notin D_{f,\varepsilon}^{\omega^{\lambda+1}}(X)$ , proving (3.7) for the successor case.

The proof of (3.7) for the limit case is similar. Let  $\lambda$  be a limit ordinal and let  $\lambda_k \rightarrow \lambda$  with  $\lambda_k < \lambda$ . Let

$$f^k = \sum_{\eta < \omega^{\lambda_k}}^* (-1)^\eta f_\eta.$$

By (3.8) for  $\lambda_k < \lambda$  we have  $f^k \in \mathcal{B}_1^{\lambda_k+1} \subseteq \mathcal{B}_1^\lambda$ . Again by (3.4),  $0 \leq f - f^k \leq f_{\omega^{\lambda_k}}$ , and using the fact that  $f_\eta \rightarrow 0$  and  $f_\eta$  is USC, for a fixed  $x \in X$  we get a neighborhood  $x \in U$  and a  $k$  such that  $|f(y) - f^k(y)| \leq \varepsilon/4$  for every  $y \in U$ . The application of Lemma 3.8 yields  $D_{f,\varepsilon}^{\omega^\lambda}(X) \cap U \subseteq D_{f^k,\varepsilon/4}^{\omega^\lambda}(X) \cap U = \emptyset$ , hence  $x \notin D_{f,\varepsilon}^{\omega^\lambda}(X)$ . As we started with an arbitrary  $x \in X$ , this shows  $D_{f,\varepsilon}^{\omega^\lambda}(X) = \emptyset$ , thus  $\beta(f) \leq \omega^\lambda$ , proving  $f \in \mathcal{B}_1^\lambda$ .

It remains to prove (3.8). Now we can use (3.7) for  $\lambda$  as we proved it using (3.8) only for smaller ordinals. Let  $(f_\eta)_{\eta < \omega^\lambda} \in \text{DUSB}'_1$  and  $\lambda_k \rightarrow \omega^\lambda$  with  $\lambda_k < \omega^\lambda$  even. Let

$$f = \sum_{\eta < \omega^\lambda}^* (-1)^\eta f_\eta \quad \text{and} \quad f^k = \sum_{\eta < \lambda_k}^* (-1)^\eta f_\eta.$$

Since we can extend the sequence  $(f_\eta)_{\eta < \lambda_k}$  by zero functions to a sequence in  $\text{DUSB}_1$  of length  $\omega^\lambda$ , using (3.7) we deduce that  $f^k \in \mathcal{B}_1^\lambda$ . By (3.2) we have  $f^k \rightarrow f$ ; moreover, (3.3) for the sequence  $(f_\eta)_{\eta < \omega^{\lambda+1}} \in \text{DUSB}_1$ , where  $f_{\omega^\lambda} = g = \inf_{\eta < \omega^\lambda} f_\eta$  is a USC function, yields

$$(3.9) \quad 0 \leq f - f^k \leq f_{\lambda_k} - g.$$

It is enough to prove that  $D_{f,\varepsilon}^{\omega^{\lambda+1}}(X) = \emptyset$  for every fixed  $\varepsilon > 0$ . To do so, let  $F_n = \{x \in X : g(x) \geq n \cdot \varepsilon/12\}$ . Note that  $g$  is USC, hence  $F_n$  is closed for every  $n \in \mathbb{N}$ . Since  $\bigcap_n F_n = \emptyset$ , it is enough to prove that

$$(3.10) \quad D_{f,\varepsilon}^{\omega^\lambda}(F_n) \subseteq F_{n+1},$$

since then by induction on  $n$  one can easily get  $D_{f,\varepsilon}^{\omega^{\lambda \cdot n}}(X) \subseteq F_n$ , hence

$$D_{f,\varepsilon}^{\omega^{\lambda+1}}(X) = \bigcap_{n \in \mathbb{N}} D_{f,\varepsilon}^{\omega^{\lambda \cdot n}}(X) \subseteq \bigcap_{n \in \mathbb{N}} F_n = \emptyset.$$

Let  $x \in F_n \setminus F_{n+1}$ . Since  $f_{\lambda_k} \rightarrow g$ , there exists a  $k$  such that

$$(3.11) \quad f_{\lambda_k}(x) - g(x) \leq \varepsilon/12.$$

Since  $f_{\lambda_k}$  is USC, there exists a neighborhood  $U \ni x$  such that  $f_{\lambda_k}(y) < f_{\lambda_k}(x) + \varepsilon/12$  for every  $y \in U$ . Using that  $x \in F_n \setminus F_{n+1}$ , we have  $g(x) - g(y) \leq \varepsilon/12$  for every  $y \in F_n$ . Using (3.9), the last two inequalities and (3.11), we find that for every  $y \in U \cap F_n$ ,

$$0 \leq f(y) - f^k(y) \leq f_{\lambda_k}(y) - g(y) \leq f_{\lambda_k}(x) + \varepsilon/12 - g(x) + \varepsilon/12 \leq \varepsilon/4.$$

Again applying Lemma 3.8 with  $g = f^k$ ,  $F = F_n$  and  $\eta = \omega^\lambda$ , we get  $D_{f,\varepsilon}^{\omega^\lambda}(F_n) \cap U \subseteq D_{f^k,\varepsilon/4}^{\omega^\lambda}(F_n) \cap U = \emptyset$ , hence  $x \notin D_{f,\varepsilon}^{\omega^\lambda}(F_n)$ . Since  $x \in F_n \setminus F_{n+1}$  was arbitrary, we get (3.10) as desired. This finishes the proof of (3.8) and also the proof of the theorem. ■

Now we prove an analogue of the previous theorem for the Baire class  $\xi$  case.

**THEOREM 3.9.** *Let  $f$  be a bounded Baire class  $\xi$  function. Then  $f \in \mathcal{B}_\xi^\lambda$  if and only if  $\text{length}_\xi(f) \leq \omega^\lambda$ .*

**REMARK 3.10.** If one considers  $\text{length}_\xi(f)$  as the rank of the function  $f$ , then the theorem says that this rank essentially coincides with  $\alpha_\xi^*$ ,  $\beta_\xi^*$  and  $\gamma_\xi^*$  on the bounded Baire class  $\xi$  functions.

*Proof of Theorem 3.9.* First we prove that if  $f \in \mathcal{B}_\xi^\lambda$  then  $\text{length}_\xi(f) \leq \omega^\lambda$ . By Remark 2.1 we have a topology  $\tau' \in T_{f,\xi}$  such that  $f \in \mathcal{B}_1^\lambda(\tau')$ . Theorem 3.4 shows that there is a sequence  $(f_\eta)_{\eta < \omega^\lambda} \in \text{DUSB}_1(\tau')$  and  $c \in \mathbb{R}$  with

$$f = c + \sum_{\eta < \omega^\lambda}^* (-1)^\eta f_\eta.$$

The function  $f_\eta$  is USC in  $\tau'$  for each  $\eta$ , hence  $\{f_\eta < c\} \in \Sigma_1^0(\tau')$ , and since  $\tau' \in T_{f,\xi}$ , we have  $\Sigma_1^0(\tau') \subseteq \Sigma_\xi^0(\tau)$ , thus  $f_\eta$  is a semi-Borel class  $\xi$  function with respect to the original topology  $\tau$ . From this, one can easily conclude that  $(f_\eta)_{\eta < \omega^\lambda} \in \text{DUSB}_\xi(\tau)$  and consequently  $\text{length}_\xi(f) \leq \omega^\lambda$ , proving this part of the theorem.

For the other direction, suppose that  $\text{length}_\xi(f) \leq \omega^\lambda$ , and let

$$f = c + \sum_{\eta < \omega^\lambda}^* (-1)^\eta f_\eta,$$

where  $(f_\eta)_{\eta < \omega^\lambda} \in \text{DUSB}_\xi$ . Since  $\{f_\eta < q\} \in \Sigma_\xi^0$  for every  $q \in \mathbb{Q}$ , it can be written as  $\{f_\eta < q\} = \bigcup_n F_n^{\eta,q}$ , where  $F_n^{\eta,q} \in \bigcup_{\zeta < \xi} \Pi_\zeta^0 \subseteq \Delta_\xi^0$ . By Kuratowski's theorem (see e.g. [7, 22.18]), there exists a Polish refinement  $\tau' \supseteq \tau$  such that  $F_n^{\eta,q} \in \Delta_1^0(\tau')$  for every  $\eta, n$  and  $q \in \mathbb{Q}$ , and  $\tau' \subseteq \Sigma_\xi^0(\tau)$ .

Now  $\{f_\eta < q\} \in \Sigma_1^0(\tau')$  for every  $\eta$  and  $q \in \mathbb{Q}$ , hence  $f_\eta$  is USC in  $\tau'$ , since  $\{f_\eta < c\} = \bigcup_n \{f_\eta < q_n\}$  is open, where  $q_n \in \mathbb{Q}$ ,  $q_n \rightarrow c$ ,  $q_n < c$ . From this  $(f_\eta)_{\eta < \omega^\lambda} \in \text{DUSB}_1(\tau')$ , hence with the application of Theorem 3.4 for the space  $(X, \tau')$ , we get  $f \in \mathcal{B}_1^\lambda(\tau')$ . Note that  $\tau' \in T_{f,\xi}$ , hence Remark 2.1 yields  $f \in \mathcal{B}_\xi^\lambda(\tau)$ , completing the proof. ■

**4. Generating  $\mathcal{B}_\xi^\lambda$  from lower classes.** Kechris and Louveau introduced the notion of pseudouniform convergence.

**DEFINITION 4.1** ([8]). A sequence  $(f_n)_{n \in \mathbb{N}}$  of functions is *pseudouniformly convergent* if  $\gamma((f_n)_{n \in \mathbb{N}}) \leq \omega$ , as defined in (2.2).

**DEFINITION 4.2.** If  $\mathcal{F}$  is a class of bounded Baire class 1 functions then let  $\Phi(\mathcal{F})$  be the set of those bounded Baire class 1 functions that are the

pseudouniform limit of a sequence of functions from  $\mathcal{F}$ , i.e.,

$$\Phi(\mathcal{F}) = \{f \in \mathcal{B}_1 : f \text{ is bounded,} \\ \exists (f_n)_{n \in \mathbb{N}} \in \mathcal{F}^{\mathbb{N}} (\gamma((f_n)_{n \in \mathbb{N}}) \leq \omega \text{ and } f_n \rightarrow f \text{ pointwise})\}.$$

Now we define inductively the families  $\Phi_\lambda$  of functions by  $\Phi_0 = \mathcal{B}_1^1$  and for  $0 < \lambda < \omega_1$ ,

$$\Phi_\lambda = \Phi\left(\bigcup_{\eta < \lambda} \Phi_\eta\right).$$

**THEOREM 4.3.** *For every ordinal  $\lambda < \omega_1$ , we have  $\Phi_\lambda = \mathcal{B}_1^{\lambda+1}$ .*

**REMARK 4.4.** This is a nice analogue of the well-known theorem that a function is of Baire class  $\lambda$  if and only if it is Borel- $(\lambda + 1)$  (see e.g. [7, 24.3, 24.10]).

**REMARK 4.5.** The authors of [8] defined  $\Phi_\lambda$  for limit  $\lambda$  as the uniform limits of functions from the smaller classes, and they proved that in this case  $\Phi_\lambda = \mathcal{B}_1^\lambda$  (with  $\Phi_0 = \mathcal{B}_1^0$ ), if the space is compact. However, this is not the case for arbitrary Polish spaces. We sketch the proof of this.

First, for every  $\lambda < \omega_1$ , one can easily construct a countable closed set  $F_\lambda \subseteq \mathbb{R}$  and a subset  $A_\lambda \subseteq F_\lambda$  such that the  $\alpha$  rank of  $\chi_{A_\lambda}$  in the space  $F_\lambda$  is equal to  $\lambda$ . (Let  $F_\lambda$  be a set with Cantor–Bendixson rank  $\lambda$  (see [7, 6.12]). Then choose  $A_\lambda$  such that  $A_\lambda$  and  $F_\lambda \setminus A_\lambda$  are both “dense” in  $F_\lambda$ , meaning that if  $F_\lambda^\alpha \subseteq F_\lambda$  is the  $\alpha$ th iterated Cantor–Bendixson derivative of  $F_\lambda$  then the closures of both  $A_\lambda \cap F_\lambda^\alpha$  and  $F_\lambda^\alpha \setminus A_\lambda$  contain every limit point of  $F_\lambda^\alpha$ .) This step will not work in compact spaces as the  $\alpha$  rank of a characteristic function on a compact space is always a successor ordinal.

Then, it is easy to see that  $\chi_{A_{\omega^\omega}}$  cannot be the uniform limit of functions from  $\bigcup_{n < \omega} \mathcal{B}_1^n$ , since if  $\|f - \chi_{A_{\omega^\omega}}\| \leq 1/3$  then  $\alpha(f) \geq \alpha(\chi_{A_{\omega^\omega}}) = \omega^\omega$ .

*Proof of Theorem 4.3.* We use transfinite induction. For  $\lambda = 0$  the conclusion is exactly the definition of  $\Phi_0$ .

To prove that  $\Phi_\lambda \subseteq \mathcal{B}_1^{\lambda+1}$ , it is enough to show that

$$(4.1) \quad \Phi(\mathcal{B}_1^\lambda) \subseteq \mathcal{B}_1^{\lambda+1},$$

since for successor  $\lambda$  it is exactly what is required, and for limit  $\lambda$  we have

$$\Phi_\lambda = \Phi\left(\bigcup_{\eta < \lambda} \Phi_\eta\right) = \Phi\left(\bigcup_{\eta < \lambda} \mathcal{B}_1^{\eta+1}\right) \subseteq \Phi(\mathcal{B}_1^\lambda).$$

Let  $(f_n)_{n \in \mathbb{N}}$  be a sequence from  $\mathcal{B}_1^\lambda$  converging pointwise to a bounded function  $f$ .

**CLAIM 4.6.** *For every closed set  $F$  and  $\varepsilon > 0$ ,*

$$D_{f,\varepsilon}^{\omega^\lambda}(F) \subseteq D_{(f_n)_{n \in \mathbb{N}},\varepsilon/4}(F).$$

*Proof.* Let  $x \in F \setminus D_{(f_n)_{n \in \mathbb{N}, \varepsilon/4}}^{\omega^\lambda}(F)$ , we need to show that  $x \notin D_{f, \varepsilon}^{\omega^\lambda}(F)$ . By the definition of the derivative, there exists a neighborhood  $x \in U$  and  $N \in \mathbb{N}$  such that for every  $y \in F \cap U$  and  $n, m \geq N$  we have  $|f_n(y) - f_m(y)| < \varepsilon/4$ . As  $f_n(y) \rightarrow f(y)$  for every  $y \in X$ , we have  $|f_N(y) - f(y)| \leq \varepsilon/4$  for every  $y \in F \cap U$ . Applying Lemma 3.8 with  $g = f_N$  and  $\eta = \omega^\lambda$ , we get

$$D_{f, \varepsilon}^{\omega^\lambda}(F) \cap U \subseteq D_{f_N, \varepsilon/4}^{\omega^\lambda}(F) \cap U = \emptyset,$$

since  $f_N \in \mathcal{B}_1^\lambda$ . Hence  $x \notin D_{f, \varepsilon}^{\omega^\lambda}(F)$ , proving the claim. ■

Now suppose moreover that  $\gamma((f_n)_{n \in \mathbb{N}}) \leq \omega$ . We need to show that  $\beta(f) \leq \omega^{\lambda+1}$ . Applying the claim repeatedly with  $F = D_{(f_n)_{n \in \mathbb{N}, \varepsilon/4}}^n(X)$ , by induction we get  $D_{f, \varepsilon}^{\omega^\lambda \cdot n}(X) \subseteq D_{(f_n)_{n \in \mathbb{N}, \varepsilon/4}}^n(X)$  for each  $n \in \mathbb{N}$ . Taking the intersection over all  $n \in \mathbb{N}$ , we get  $D_{f, \varepsilon}^{\omega^{\lambda+1}}(X) \subseteq D_{(f_n)_{n \in \mathbb{N}, \varepsilon/4}}^\omega(X) = \emptyset$ , hence  $f \in \mathcal{B}_1^{\lambda+1}$ , showing (4.1) and thus finishing the proof of  $\Phi_\lambda \subseteq \mathcal{B}_1^{\lambda+1}$ .

Now we show the other direction, i.e.,  $\Phi_\lambda \supseteq \mathcal{B}_1^{\lambda+1}$ . We use transfinite induction on  $\lambda$ . The inclusion is obvious for  $\lambda = 0$ . For  $\lambda > 0$ , using the statement for each  $\eta < \lambda$ , we have  $\Phi_\lambda = \Phi(\bigcup_{\eta < \lambda} \Phi_\eta) = \Phi(\bigcup_{\eta < \lambda} \mathcal{B}_1^{\eta+1})$ , hence it is enough to show that  $\Phi(\bigcup_{\eta < \lambda} \mathcal{B}_1^{\eta+1}) \supseteq \mathcal{B}_1^{\lambda+1}$ .

Let  $f \in \mathcal{B}_1^{\lambda+1}$  be a characteristic function, i.e.,  $f = \chi_A$  for some  $A \subseteq X$ . Using the same argument as in the proof of Theorem 3.4, one can write  $A$  as

$$A = \bigcup_{\substack{\eta < \omega^{\lambda+1} \\ \eta \text{ even}}} F_\eta \setminus F_{\eta+1},$$

where  $(F_\eta)_{\eta < \omega^{\lambda+1}}$  is a decreasing, continuous sequence of closed sets with  $F_0 = X$  and  $\bigcap_{\eta < \omega^{\lambda+1}} F_\eta = \emptyset$ .

Let  $\lambda_k \rightarrow \omega^\lambda$ ,  $\lambda_k < \omega^\lambda$  be an increasing sequence of even ordinals with  $\lambda_k > 0$  and let

$$B_k = \bigcup_{n \in \mathbb{N}} \bigcup_{\substack{\omega^\lambda \cdot n \leq \eta < \omega^\lambda \cdot n + \lambda_k \\ \eta \text{ even}}} F_\eta \setminus F_{\eta+1}.$$

Let  $f_k = \chi_{B_k}$ . It is easy to see that  $f_k \rightarrow f$  pointwise. We need to show that this convergence is pseudouniform, and  $f_k \in \bigcup_{\eta < \lambda} \mathcal{B}_1^{\eta+1}$  for every  $k \in \mathbb{N}$ .

The proof of the former statement is based on the following claim.

CLAIM 4.7. *For every  $n \in \mathbb{N}$  and  $\varepsilon > 0$  we have  $D_{(f_k)_{k \in \mathbb{N}, \varepsilon}}^n(X) \subseteq F_{\omega^\lambda \cdot n}$ .*

*Proof.* For  $n = 0$  this is a consequence of the definitions, so we need to show that it holds for  $n + 1$  if it holds for  $n$ . For this, it is enough to show that  $D_{(f_k)_{k \in \mathbb{N}, \varepsilon}}(F_{\omega^\lambda \cdot n}) \subseteq F_{\omega^\lambda \cdot (n+1)}$ . Let  $x \in F_{\omega^\lambda \cdot n} \setminus F_{\omega^\lambda \cdot (n+1)}$ . We need to show that  $x \notin D_{(f_k)_{k \in \mathbb{N}, \varepsilon}}(F_{\omega^\lambda \cdot n})$ . The sequence  $(F_\eta)_{\eta < \omega^{\lambda+1}}$  is decreasing and

continuous, hence  $F_{\omega^{\lambda \cdot (n+1)}} = \bigcap_{\eta < \omega^{\lambda \cdot (n+1)}} F_{\eta} = \bigcap_{k \in \mathbb{N}} F_{\omega^{\lambda \cdot n + \lambda_k}}$ , so there is a  $k \in \mathbb{N}$  such that  $x \notin F_{\omega^{\lambda \cdot n + \lambda_k}}$ .

Since  $F_{\omega^{\lambda \cdot n + \lambda_k}}$  is closed, there is a neighborhood  $U \ni x$  such that  $U \cap F_{\omega^{\lambda \cdot n + \lambda_k}} = \emptyset$ . If  $i, j \geq k$  then  $f_i(y) = f_j(y)$  for all  $y \in U \cap F_{\omega^{\lambda \cdot n}}$ , hence  $x \notin D_{(f_k)_{k \in \mathbb{N}, \varepsilon}}(F_{\omega^{\lambda \cdot n}})$ , proving the claim. ■

Now

$$D_{(f_k)_{k \in \mathbb{N}, \varepsilon}}^{\omega}(X) = \bigcap_{n \in \mathbb{N}} D_{(f_k)_{k \in \mathbb{N}, \varepsilon}}^n(X) \subseteq \bigcap_{n \in \mathbb{N}} F_{\omega^{\lambda \cdot n}} = \emptyset,$$

hence the convergence  $f_k \rightarrow f$  is pseudouniform.

It remains to prove that  $f_k \in \bigcup_{\eta < \lambda} \mathcal{B}_1^{\eta+1}$  for each  $k$ .

CLAIM 4.8. *For every  $\varepsilon > 0$  and  $m \in \mathbb{N}$  we have  $D_{f_k, \varepsilon}^{(\lambda_k+4) \cdot m}(X) \subseteq F_{\omega^{\lambda \cdot m}}$ .*

First we show that it is enough to prove the claim. Since  $\lambda_k > 0$ , we have  $(\lambda_k + 4) \cdot \omega = \lambda_k \cdot \omega$ , hence using the fact that  $\bigcap_{\eta < \omega^{\lambda+1}} F_{\eta} = \emptyset$  we obtain

$$D_{f_k, \varepsilon}^{\lambda_k \cdot \omega}(X) = \bigcap_{m \in \mathbb{N}} D_{f_k, \varepsilon}^{(\lambda_k+4) \cdot m}(X) \subseteq \bigcap_{m \in \mathbb{N}} F_{\omega^{\lambda \cdot m}} = \emptyset,$$

showing that  $\beta(f_k) \leq \lambda_k \cdot \omega$ . If  $\lambda$  is limit then  $\lambda_k \leq \omega^{\theta}$  for some  $\theta < \lambda$ , hence  $\beta(f_k) \leq \lambda_k \cdot \omega \leq \omega^{\theta+1}$ , showing that  $f_k \in \bigcup_{\eta < \lambda} \mathcal{B}_1^{\eta+1}$  in this case. If  $\lambda$  is successor then let  $\lambda = \theta + 1$ . Now  $\lambda_k < \omega^{\theta} \cdot l$  for some  $l \in \mathbb{N}$ , hence  $\lambda_k \cdot \omega \leq \omega^{\theta+1}$ , showing that  $f_k \in \mathcal{B}_1^{\theta+1} \subseteq \bigcup_{\eta < \lambda} \mathcal{B}_1^{\eta+1}$ . Now it only remains to prove the claim.

*Proof of Claim 4.8.* We use induction on  $m$ . For  $m = 0$  the assertion is a consequence of the definitions. Suppose it holds for  $m$ ; to prove it for  $m+1$  we need to show that if  $x \in F_{\omega^{\lambda \cdot m}} \setminus F_{\omega^{\lambda \cdot (m+1)}}$  then  $x \notin D_{f_k, \varepsilon}^{\lambda_k+4}(F_{\omega^{\lambda \cdot m}})$ .

There exists a neighborhood  $U$  of  $x$  with  $U \cap F_{\omega^{\lambda \cdot (m+1)}} = \emptyset$ . Let

$$H = \bigcup_{\substack{\omega^{\lambda \cdot m} \leq \eta < \omega^{\lambda \cdot m + \lambda_k} \\ \eta \text{ even}}} F_{\eta} \setminus F_{\eta+1}.$$

It is easy to see that  $\alpha_1(\chi_H) \leq \lambda_k + 4$ , since  $H$  can be written as a transfinite difference of closed sets of length  $\lambda_k + 4$  by taking the sequence  $(P_{\eta})_{\eta < \lambda_k + 4}$ , where

$$P_{\eta} = \begin{cases} X & \text{if } \eta = 0 \text{ or } 1, \\ F_{\omega^{\lambda \cdot m + \eta - 2}} & \text{if } 2 \leq \eta < \omega, \\ F_{\omega^{\lambda \cdot m + \eta}} & \text{if } \omega \leq \eta < \lambda_k, \\ F_{\omega^{\lambda \cdot m + \lambda_k}} & \text{if } \lambda_k \leq \eta \leq \lambda_k + 1, \\ \emptyset & \text{if } \lambda_k + 2 \leq \eta \leq \lambda_k + 3. \end{cases}$$

Note that  $\lambda_k$  is even, hence  $H$  is indeed the transfinite difference of this sequence. Using Lemma 2.2 and [2, 3.14] we have  $\beta(\chi_H) = \alpha(\chi_H) \leq \alpha_1(\chi_H) \leq \lambda_k + 4$ . But as  $f_k(y) = \chi_{B_k}(y) = \chi_H(y)$  for every  $y \in F_{\omega^{\lambda \cdot m}} \cap U$ , we have  $D_{f_k, \varepsilon}^{\lambda_k+4}(F_{\omega^{\lambda \cdot m}}) \cap U = D_{\chi_H, \varepsilon}^{\lambda_k+4}(F_{\omega^{\lambda \cdot m}}) \cap U = \emptyset$ , hence  $x \notin D_{f_k, \varepsilon}^{\lambda_k+4}(F_{\omega^{\lambda \cdot m}})$ , proving the claim. ■

This finishes the proof that  $f \in \Phi_\lambda$  for a characteristic function  $f \in \mathcal{B}_1^{\lambda+1}$ .

Now let  $f \in \mathcal{B}_1^{\lambda+1}$  be a step function, i.e.,  $f = \sum_{i=1}^n c_i \chi_{A_i}$ , where the  $c_i$ 's are distinct real numbers and the  $A_i$ 's form a partition of  $X$ . For each  $i$ ,  $\chi_{A_i} \in \mathcal{B}_1^{\lambda+1}$  by [2, 3.38], hence for each  $i$  there exists a sequence  $(f_i^k)_{k \in \mathbb{N}}$  such that  $(f_i^k)_{k \in \mathbb{N}} \rightarrow \chi_{A_i}$  pseudouniformly, and  $f_i^k \in \bigcup_{\eta < \lambda} \mathcal{B}_1^{\eta+1}$ . Let  $f^k = \sum_{i=1}^n c_i \cdot f_i^k$ . Lemma 2.3 yields  $\gamma((f^k)_{k \in \mathbb{N}}) \leq \omega$ , and it can be easily seen that  $f^k \rightarrow f$  pointwise. It remains to prove that  $f^k \in \bigcup_{\eta < \lambda} \mathcal{B}_1^{\eta+1}$  for each  $k$ . Fix  $k \in \mathbb{N}$ . Then  $f_i^k \in \mathcal{B}_1^{\lambda_i+1}$  for some  $\lambda_i < \lambda$ . Hence with  $\lambda' = \max\{\lambda_i : 1 \leq i \leq n\} < \lambda$  we have  $f_i^k \in \mathcal{B}_1^{\lambda'+1}$  for every  $i$ . Now [2, 3.29] yields  $f^k \in \mathcal{B}_1^{\lambda'+1} \subseteq \bigcup_{\eta < \lambda} \mathcal{B}_1^{\eta+1}$ , proving that  $f \in \Phi_\lambda$ .

To finish the proof of the theorem, it remains to show that  $f \in \Phi_\lambda$  for every  $f \in \mathcal{B}_1^{\lambda+1}$ .

Let  $f \in \mathcal{B}_1^{\lambda+1}$ . By Lemma 3.6 there exists a sequence  $(g^k)_{k \in \mathbb{N}}$  of non-negative step functions such that  $g^k \in \mathcal{B}_1^{\lambda+1}$ ,  $\inf f + \sum_k g^k = f$  and  $\|g^k\| \leq 1/2^k$  for  $k \geq 1$ . We can replace  $g^0$  with  $g^0 + \inf f$ , so now we have  $\sum_k g^k = f$ . Since  $g^k$  is a step function,  $g^k \in \Phi_\lambda$ , hence for each  $k$  we have a sequence  $(g_n^k)_{n \in \mathbb{N}}$  tending pseudouniformly to  $g^k$  with  $g_n^k \in \bigcup_{\eta < \lambda} \Phi_\eta = \bigcup_{\eta < \lambda} \mathcal{B}_1^{\eta+1}$  for each  $n, k \in \mathbb{N}$ .

We first show that we can suppose that  $\|g_n^k\| \leq \|g^k\|$ . For every  $k \in \mathbb{N}$  define  $h^k : \mathbb{R} \rightarrow \mathbb{R}$  by

$$h^k(x) = \begin{cases} 0 & \text{if } x < 0, \\ x & \text{if } 0 \leq x \leq 1/2^k, \\ 1/2^k & \text{if } 1/2^k < x. \end{cases}$$

Then  $h^k$  is a Lipschitz function, hence  $\beta(h^k \circ g_n^k) \leq \beta(g_n^k)$  by Lemma 2.4, thus  $h^k \circ g_n^k \in \bigcup_{\eta < \lambda} \mathcal{B}_1^{\eta+1}$ . Using the same arguments as in the proof of Lemma 2.4, it is easy to see that  $\gamma((h^k \circ g_n^k)_{n \in \mathbb{N}}) \leq \gamma((g_n^k)_{n \in \mathbb{N}}) \leq \omega$ , hence the sequence  $(h^k \circ g_n^k)_{n \in \mathbb{N}}$  is pseudouniformly convergent for every  $k$ . Using the continuity of  $h^k$  we have  $(h^k \circ g_n^k)_{n \in \mathbb{N}} \rightarrow h^k \circ g^k = g^k$ . This shows that by replacing  $g_n^k$  with  $h^k \circ g_n^k$ , we can indeed assume that  $\|g_n^k\| \leq \|g^k\|$ .

Now we prove the following claim.

CLAIM 4.9. *Let  $f_n = \sum_{k \leq n} g_n^k$ . Then the sequence  $(f_n)_{n \in \mathbb{N}}$  tends pseudouniformly to  $f$ .*

*Proof.* First we show that  $f_n \rightarrow f$  pointwise. Fix  $\varepsilon > 0$  and  $x \in X$ , and let  $K \in \mathbb{N}$  be large enough that  $1/2^{K-2} < \varepsilon/2$ . Then there exists a common  $N \geq K \in \mathbb{N}$  such that for all  $k < K$  and  $n > N$  we have  $|g_n^k(x) - g^k(x)| \leq \varepsilon/(2K)$ . Thus, for  $n > N$ ,

$$\begin{aligned} |f_n(x) - f(x)| &= \left| \sum_{k \leq n} g_n^k(x) - \sum_{k \in \mathbb{N}} g^k(x) \right| \\ &\leq \sum_{k < K} |g_n^k(x) - g^k(x)| + \sum_{K \leq k \leq n} |g_n^k(x)| + \sum_{k \geq K} |g^k(x)| \\ &\leq \frac{\varepsilon}{2K} \cdot K + 2 \cdot \frac{1}{2^{K-1}} \leq \varepsilon, \end{aligned}$$

proving the pointwise convergence.

Let  $\varepsilon > 0$ . It remains to show that  $D_{(f_n)_{n \in \mathbb{N}, \varepsilon}}^\omega(X) = \emptyset$ . Let  $K \in \mathbb{N}$  be large enough that  $2 \cdot 1/2^K < \varepsilon/2$ . Then for  $n, m \geq K$  we have

$$\|f_n - f_m\| = \left\| \sum_{k \leq n} g_n^k - \sum_{k \leq m} g_m^k \right\| \leq \left\| \sum_{k \leq K} g_n^k - \sum_{k \leq K} g_m^k \right\| + 2 \frac{1}{2^K},$$

hence if  $|f_n(y) - f_m(y)| \geq \varepsilon$  then  $|\sum_{k \leq K} g_n^k(y) - \sum_{k \leq K} g_m^k(y)| \geq \varepsilon/2$ . From this, using transfinite induction, one can easily deduce for all  $\eta < \omega_1$  that

$$D_{(f_n)_{n \in \mathbb{N}, \varepsilon}}^\eta(X) \subseteq D_{(\sum_{k \leq K} g_n^k)_{n \in \mathbb{N}, \varepsilon/2}}^\eta(X).$$

By Lemma 2.3 the sequence  $(\sum_{k \leq K} g_n^k)_{n \in \mathbb{N}}$  converges pseudouniformly to  $\sum_{k \leq K} g^k$ , hence  $D_{(\sum_{k \leq K} g_n^k)_{n \in \mathbb{N}, \varepsilon/2}}^\omega(X) = \emptyset$ , proving  $D_{(f_n)_{n \in \mathbb{N}, \varepsilon}}^\omega(X) = \emptyset$ . ■

Using this claim it remains to prove that  $f_n \in \bigcup_{\eta < \lambda} \mathcal{B}_1^{\eta+1}$  for each  $n$ . Using the same idea as above, we have a  $\lambda' < \lambda$  with  $g_n^k \in \mathcal{B}_1^{\lambda'+1}$  for every  $k \leq n$ , hence by [2, 3.29] we have  $f_n \in \mathcal{B}_1^{\lambda'+1} \subseteq \bigcup_{\eta < \lambda} \mathcal{B}_1^{\eta+1}$ . This shows that  $\Phi_\lambda \supseteq \mathcal{B}_1^{\lambda+1}$ , finishing the proof of the theorem. ■

Now we give a generalized version of the above theorem for Baire class  $\xi$  functions. From now on, let  $1 < \xi < \omega_1$  be a fixed ordinal.

DEFINITION 4.10. Let  $\mathcal{F}$  be a class of bounded Baire class  $\xi$  functions and let

$$\begin{aligned} \Phi(\mathcal{F}) &= \{f \in \mathcal{B}_\xi : f \text{ is bounded, } \exists f_n \in \mathcal{F}, \tau' \supseteq \tau \text{ Polish} \\ &\quad (\tau' \subseteq \Sigma_\xi^0(\tau), f_n, f \in \mathcal{B}_1(\tau'), f_n \rightarrow f \text{ pseudouniformly with respect to } \tau')\}. \end{aligned}$$

As in the Baire class 1 case, we define the families  $\Phi_\lambda$  as follows. Let  $\Phi_0 = \mathcal{B}_\xi^1$  and for  $0 < \lambda < \omega_1$  let

$$\Phi_\lambda = \Phi\left(\bigcup_{\eta < \lambda} \Phi_\eta\right).$$

**THEOREM 4.11.** *For every ordinal  $\lambda < \omega_1$ , we have  $\Phi_\lambda = \mathcal{B}_\xi^{\lambda+1}$ .*

*Proof.* For  $\lambda = 0$  the statement is obvious. We first prove the inclusion  $\Phi_\lambda \supseteq \mathcal{B}_\xi^{\lambda+1}$  by transfinite induction on  $\lambda$ . Let  $f \in \mathcal{B}_\xi^{\lambda+1}$ . By Remark 2.1 there exists a Polish topology  $\tau' \supseteq \tau$  such that  $f \in \mathcal{B}_1^{\lambda+1}(\tau')$ . Thus, by Theorem 4.3 there exists a sequence  $(f_n)_{n \in \mathbb{N}}$  of functions such that  $f_n \rightarrow f$  pseudouniformly in the topology  $\tau'$ , and  $f_n \in \bigcup_{\eta < \lambda} \mathcal{B}_1^{\eta+1}(\tau')$  for each  $n$ .

It is easy to check from the definition that  $\tau' \in T_{f_n, \xi}$  for each  $n$ , hence Remark 2.1 now yields  $f_n \in \bigcup_{\eta < \lambda} \mathcal{B}_\xi^{\eta+1}(\tau)$ . The sequence  $(f_n)_{n \in \mathbb{N}}$  and the topology  $\tau'$  is exactly what is required by the above definition, showing that  $f \in \Phi(\bigcup_{\eta < \lambda} \mathcal{B}_\xi^{\eta+1})$ , so  $f \in \Phi_\lambda$ . This proves that  $\Phi_\lambda \supseteq \mathcal{B}_\xi^{\lambda+1}$ .

We prove the other inclusion by transfinite induction on  $\lambda$ . Let  $f \in \Phi_\lambda$ , i.e., there is a sequence  $(f_n)_{n \in \mathbb{N}}$  and a topology  $\tau' \supseteq \tau$  with  $\tau' \subseteq \Sigma_\xi^0(\tau)$ ,  $f, f_n \in \mathcal{B}_1(\tau')$ ,  $f_n \rightarrow f$  pseudouniformly with respect to the topology  $\tau'$  and finally  $f_n \in \bigcup_{\eta < \lambda} \Phi_\eta = \bigcup_{\eta < \lambda} \mathcal{B}_\xi^{\eta+1}$ , using the induction hypothesis for each  $\eta < \lambda$ . Consequently, for each  $n$  there exists an ordinal  $\lambda_n < \lambda$  such that  $f_n \in \mathcal{B}_\xi^{\lambda_n+1}$ .

Using Remark 2.1 again, we conclude that there exists a Polish topology  $\tau_n \in T_{f_n, \xi}$  such that  $f_n \in \mathcal{B}_1^{\lambda_n+1}(\tau')$ .

By [2, 5.12] there exists a common Polish refinement  $\tau''$  of  $\tau'$  and each  $\tau_n$  with  $\tau'' \subseteq \Sigma_\xi^0(\tau)$ . Then by [2, 5.13],  $f_n, f \in \mathcal{B}_1(\tau'')$ ; moreover,  $f_n \in \mathcal{B}_1^{\lambda_n+1}(\tau')$  for each  $n$  and  $\gamma_{\tau''}((f_n)_{n \in \mathbb{N}}) \leq \gamma_{\tau'}((f_n)_{n \in \mathbb{N}}) \leq \omega$  from the definition. Theorem 4.3 yields  $f \in \mathcal{B}_1^{\lambda+1}(\tau'')$ , but since one can easily check that  $\tau'' \in T_{f, \xi}$ , we have  $f \in \mathcal{B}_\xi^{\lambda+1}(\tau)$  again by Remark 2.1, finishing the proof of the theorem. ■

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