

## Orthogonally additive holomorphic maps between $C^*$ -algebras

by

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**Abstract.** Let  $A, B$  be  $C^*$ -algebras,  $B_A(0; r)$  the open ball in  $A$  centered at 0 with radius  $r > 0$ , and  $H : B_A(0; r) \rightarrow B$  an orthogonally additive holomorphic map. If  $H$  is zero product preserving on positive elements in  $B_A(0; r)$ , we show, in the commutative case, i.e.,  $A = C_0(X)$  and  $B = C_0(Y)$ , that there exist weight functions  $h_n$  and a symbol map  $\varphi : Y \rightarrow X$  such that

$$H(f) = \sum_{n \geq 1} h_n (f \circ \varphi)^n, \quad \forall f \in B_{C_0(X)}(0; r).$$

In the general case, we show that if  $H$  is also conformal then there exist central multipliers  $h_n$  of  $B$  and a surjective Jordan isomorphism  $J : A \rightarrow B$  such that

$$H(a) = \sum_{n \geq 1} h_n J(a)^n, \quad \forall a \in B_A(0; r).$$

If, in addition,  $H$  is zero product preserving on the whole  $B_A(0; r)$ , then  $J$  is an algebra isomorphism.

We also study orthogonally additive  $n$ -homogeneous polynomials which are  $n$ -isometries.

**1. Introduction.** Let  $E$  and  $F$  be (complex) Banach spaces, and  $n$  a positive integer. A map  $P : E \rightarrow F$  is called a *bounded  $n$ -homogeneous polynomial* if there is a bounded symmetric  $n$ -linear operator  $L : E \times \cdots \times E \rightarrow F$  such that

$$P(x) = L(x, \dots, x), \quad \forall x \in E.$$

Let  $B_E(a; r)$  denote the open ball of  $E$  centered at  $a$  with radius  $r > 0$ . A map  $H : U \rightarrow F$  is said to be *holomorphic* on a nonempty open subset  $U$

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of  $E$  if for each  $a$  in  $U$  there exist an open ball  $B_E(a; r) \subset U$  and a unique sequence of bounded  $n$ -homogeneous polynomials  $P_n : E \rightarrow F$  such that

$$H(x) = \sum_{n=0}^{\infty} P_n(x - a)$$

uniformly for all  $x$  in  $B_E(a; r)$ . Here,  $P_0$  is the constant function with value  $H(a)$ . After translation, we can assume  $a = 0$ , and a holomorphic function  $H : B_E(0; r) \rightarrow F$  has its Taylor series at zero:

$$H(x) = \sum_{n=0}^{\infty} P_n(x)$$

uniformly for all  $x$  in  $B_E(0; r)$ . In this case,  $P_n$  is given by vector-valued integration as

$$P_n(x) = \frac{1}{2\pi i} \int_{|\lambda|=1} \frac{H(\lambda x)}{\lambda^{n+1}} d\lambda, \quad n = 0, 1, 2, \dots$$

See, for example, [38, pp. 40–47]. For the general theory of homogeneous polynomials and holomorphic functions, we refer to [38, 20].

When  $E, F$  are function spaces or Banach algebras, a map  $\Phi : E \rightarrow F$  is said to be *orthogonally additive* if

$$fg = gf = 0 \text{ implies } \Phi(f + g) = \Phi(f) + \Phi(g), \quad \forall f, g \in E,$$

and *zero product preserving* if

$$fg = 0 \text{ implies } \Phi(f)\Phi(g) = 0, \quad \forall f, g \in E.$$

The notions of orthogonally additive and zero product preserving transformations have been studied by many authors, for example, [5, 27, 28, 24, 14, 3, 31, 42, 6, 12, 39, 32, 33, 44].

The main goal of this paper is to establish Theorems 3.3 and 3.11. Let  $A, B$  be  $C^*$ -algebras and  $H : B_A(0; r) \rightarrow B$  an orthogonally additive and zero product preserving holomorphic map. In the commutative case when  $A = C_0(X)$  and  $B = C_0(Y)$ , the algebras of continuous complex-valued functions vanishing at infinity, we show in Theorem 3.3 that

$$H(f)(y) = \sum_{n \geq 1} h_n(y) f(\varphi(y))^n, \quad \forall y \in Y, \forall f \in B_{C_0(X)}(0; r).$$

Here,  $\varphi : Y \rightarrow X$  is continuous whenever any weight function  $h_n$  is nonvanishing. When  $A, B$  are general  $C^*$ -algebras, we assume further that  $H$  is *conformal*, i.e., the derivative  $P_1$  of  $H$  at 0 is a bounded invertible linear map from  $A$  onto  $B$ . We show in Theorem 3.11 that there is a sequence  $(h_n)$  in the center of the multiplier algebra  $M(B)$  of  $B$  and an algebra isomorphism

$J : A \rightarrow B$  such that

$$H(a) = \sum_{n \geq 1} h_n J(a)^n, \quad \forall a \in B_A(0; r).$$

If we only assume that  $H$  preserves zero products of elements in the positive part,  $A_+$ , of  $A$  intersected with  $B_A(0; r)$ , then  $J$  is still a Jordan isomorphism.

To achieve Theorems 3.3 and 3.11, we need to study homogeneous polynomials first. In [43], Sundaresan characterized the linearization of orthogonally additive  $n$ -homogeneous polynomials on  $L_p$ -spaces. Several authors have extended his results to, e.g.,  $C(K)$ -spaces [42],  $C^*$ -algebras [39], and Banach lattices [6, 9]. Adopting these important linearization tools (see Section 2), in this paper we establish Banach–Stone type theorems for orthogonally additive homogeneous polynomials and holomorphic maps between  $C^*$ -algebras (Sections 3 and 4).

A counterpart of Theorem 3.11 is given in [10] for orthogonally additive and zero product preserving holomorphic functions between matrix algebras. See Theorem 3.8 and Examples 3.9 for details. In a very interesting recent paper [23], the authors consider orthogonally additive holomorphic maps  $H$  between general  $C^*$ -algebras which preserve double orthogonality, i.e.,

$$a^*b = ab^* = 0 \quad \text{implies} \quad H(a)^*H(b) = H(a)H(b)^* = 0.$$

In this case, with the extra assumption that the range of  $H$  contains an invertible element, a corresponding result to Theorem 3.11 is established in [23] through the technique of  $JB^*$ -algebras. Nevertheless, we will see in Examples 3.9 that one cannot directly make use of this new result to study zero product preserving holomorphic maps.

In Theorems 4.2 and 4.3, we establish similar representation results for orthogonally additive  $n$ -homogeneous polynomials,  $P : A \rightarrow B$  between  $C^*$ -algebras, which are  $n$ -isometries of positive elements, i.e.,

$$\|P(a)\| = \|a\|^n, \quad \forall a \in A_+.$$

To end the introduction, we would like to mention [13, 26, 41] for some related contributions to orthogonally additive scalar holomorphic functions of bounded type on  $C(K)$ -spaces and on general  $C^*$ -algebras.

## 2. Preliminaries

**2.1. Orthogonally additive and zero product preserving polynomials.** For a  $C^*$ -algebra  $A$ , we write  $A_{sa}$  and  $A_+$  for the set of all its self-adjoint and positive elements, respectively. The following proposition follows from [10, Lemma 2.1] and its proof (see also [23, Proposition 6]).

PROPOSITION 2.1. *Let  $A, B$  be  $C^*$ -algebras. Let  $H : B_A(0; r) \rightarrow B$  be a holomorphic function with Taylor series  $H = \sum_{n=0}^{\infty} P_n$  at zero. Let  $D = A, A_{sa}$  or  $A_+$ .*

- (a) *If  $H$  is orthogonally additive on  $B_E(0; r) \cap D$  then each  $P_n$  is orthogonally additive on  $D$ .*
- (b) *If  $H$  is zero product preserving on  $B_E(0; r) \cap D$  then each  $P_n$  is zero product preserving on  $D$ . Furthermore,*

$$fg = 0 \Rightarrow P_m(f)P_n(g) = 0, \quad \forall f, g \in D, \forall m, n = 0, 1, 2, \dots$$

It follows from Proposition 2.1 that if  $H$  is orthogonally additive or zero product preserving, then the constant term  $P_0$  is zero.

The following linearization of orthogonally additive  $n$ -homogeneous polynomials is the key tool for us. Note that although the theorem in [42] is stated for compact spaces, the proof there works also for locally compact spaces.

THEOREM 2.2 ([42, Theorem 2.1]; see also [6, 12]). *Let  $X$  be a locally compact Hausdorff space and  $F$  a Banach space. Let  $P : C_0(X) \rightarrow F$  be a bounded orthogonally additive  $n$ -homogeneous polynomial. Then there exists a bounded linear operator  $T : C_0(X) \rightarrow F$  such that*

$$P(f) = T(f^n), \quad \forall f \in C(K).$$

Since commutative  $C^*$ -algebras are algebras of continuous functions, the following can be considered as the noncommutative version of Theorem 2.2.

THEOREM 2.3 ([39, 11]). *Let  $A$  be a  $C^*$ -algebra,  $F$  a complex Banach space, and  $P : A \rightarrow F$  a bounded  $n$ -homogeneous polynomial. The following are equivalent:*

- (1) *There exists a bounded linear operator  $T : A \rightarrow F$  such that*

$$P(a) = T(a^n), \quad \forall a \in A.$$

- (2)  *$P$  is orthogonally additive on  $A$ , i.e.,*

$$ab = ba = 0 \Rightarrow P(a + b) = P(a) + P(b), \quad \forall a, b \in A.$$

- (3)  *$P$  is orthogonally additive on  $A_{sa}$ , i.e.,*

$$ab = 0 \Rightarrow P(a + b) = P(a) + P(b), \quad \forall a, b \in A_{sa}.$$

In view of Theorem 2.3, the assumption in Theorem 2.2 can be weakened to  $P$  being orthogonally additive on  $C_0(X)_{sa}$ . So, due to Proposition 2.1, whenever we say that a holomorphic map or a homogeneous polynomial on a  $C^*$ -algebra is orthogonally additive, it does not matter if it is orthogonally additive on all elements or just on self-adjoint ones.

Similarly, we have

LEMMA 2.4. *Let  $A, B$  be  $C^*$ -algebras. Let  $P : A \rightarrow B$  be a bounded orthogonally additive  $n$ -homogeneous polynomial. Consider the following conditions:*

(1)  *$P$  is zero product preserving on  $A$ , i.e.,*

$$ab = 0 \Rightarrow P(a)P(b) = 0, \quad \forall a, b \in A.$$

(2)  *$P$  is zero product preserving on  $A_{\text{sa}}$ , i.e.,*

$$ab = 0 \Rightarrow P(a)P(b) = 0, \quad \forall a, b \in A_{\text{sa}}.$$

(3)  *$P$  is zero product preserving on  $A_+$ , i.e.,*

$$ab = 0 \Rightarrow P(a)P(b) = 0, \quad \forall a, b \in A_+.$$

*We have (1) $\Rightarrow$ (2) $\Leftrightarrow$ (3) in general; and the three conditions above are all equivalent when  $A$  is commutative.*

*Proof.* It is clear that (1) $\Rightarrow$ (2) $\Rightarrow$ (3). Suppose  $a, b \in A_{\text{sa}}$  with  $ab = 0$ . Write  $a = a_+ - a_-$  and  $b = b_+ - b_-$  as the orthogonal differences of their positive and negative parts. By functional calculus, we see that

$$a_+b_+ = a_+b_- = a_-b_+ = a_-b_- = 0.$$

Now the orthogonal additivity of  $P$  and condition (3) give

$$\begin{aligned} P(a)P(b) &= P(a_+)P(b_+) + P(a_+)P(-b_-) + P(-a_-)P(b_+) + P(-a_-)P(-b_-) = 0. \end{aligned}$$

Hence, we have (3) $\Rightarrow$ (2).

Now suppose  $A = C_0(X)$  is commutative. We verify (3) $\Rightarrow$ (1). Let  $T : A \rightarrow B$  be the bounded linear operator associated to  $P$  as in Theorem 2.3.

First let  $f, g$  in  $C_0(X)$  be real-valued such that  $fg = 0$ . Then

$$f_+g_+ = f_+g_- = f_-g_+ = f_-g_- = 0,$$

and hence

$$\sqrt[n]{f_+} \sqrt[n]{g_+} = \sqrt[n]{f_+} \sqrt[n]{g_-} = \sqrt[n]{f_-} \sqrt[n]{g_+} = \sqrt[n]{f_-} \sqrt[n]{g_-} = 0.$$

It follows from Theorem 2.3(1) and assumption (3) that

$$\begin{aligned} T(f)T(g) &= T(f_+)T(g_+) - T(f_+)T(g_-) - T(f_-)T(g_+) + T(f_-)T(g_-) \\ &= P(\sqrt[n]{f_+})P(\sqrt[n]{g_+}) - P(\sqrt[n]{f_+})P(\sqrt[n]{g_-}) \\ &\quad - P(\sqrt[n]{f_-})P(\sqrt[n]{g_+}) + P(\sqrt[n]{f_-})P(\sqrt[n]{g_-}) = 0. \end{aligned}$$

Let  $f_1, f_2, g_1, g_2$  in  $C_0(X)$  be real-valued such that  $(f_1 + ig_1) \cdot (f_2 + ig_2) = 0$ . Then for each  $t$  in  $X$ , we have  $f_1(t) = g_1(t) = 0$  or  $f_2(t) = g_2(t) = 0$ . Therefore,

$$f_1 \cdot f_2 = g_1 \cdot g_2 = f_1 \cdot g_2 = g_1 \cdot f_2 = 0.$$

It follows from the above conditions that

$$\begin{aligned} T(f_1 + ig_1) \cdot T(f_2 + ig_2) \\ = T(f_1) \cdot T(f_2) + iT(f_1) \cdot T(g_2) + iT(g_1) \cdot T(f_2) - T(g_1) \cdot T(g_2) = 0. \end{aligned}$$

Therefore,  $T$  preserves zero products. Let  $f, g \in C_0(X)$  with  $fg = 0$ . Then  $f^n g^n = 0$  as well. Consequently,

$$P(f)P(g) = T(f^n)T(g^n) = 0,$$

and the assertion follows. ■

We remark that the surjective linear isometry  $a \mapsto a^t$  gives a counter-example for the implication (2) $\Rightarrow$ (1) in Lemma 2.4, when  $A = B = B(\mathcal{H})$ . Here,  $a^t$  denotes the transpose of an operator  $a$  in  $B(\mathcal{H})$  with respect to an arbitrary but fixed orthonormal basis of the Hilbert space  $\mathcal{H}$ .

**2.2. Classical Banach–Stone type theorems.** The classical Banach–Stone Theorem can be stated in the following two ways.

**THEOREM 2.5** (Banach–Stone Theorem for isometries; see, e.g., [19]). *Let  $T : C_0(X) \rightarrow C_0(Y)$  be a linear operator. If  $T$  is a surjective isometry, then there exists a homeomorphism  $\varphi : Y \rightarrow X$  such that*

$$Tf(y) = h(y)f(\varphi(y)), \quad \forall f \in C_0(X), \forall y \in Y.$$

*Here,  $h$  is a continuous unimodular scalar function on  $Y$ , i.e.,  $|h(y)| = 1$  for all  $y \in Y$ .*

**THEOREM 2.6** (Banach–Stone Theorem for zero product preserving maps; see [1, 5], and also [27, 22, 28]). *Let  $T : C_0(X) \rightarrow C_0(Y)$  be a bounded linear operator. If  $T$  is zero product preserving, then there exist a bounded scalar function  $h$  on  $Y$  and a map  $\varphi : Y \rightarrow X$  such that*

$$Tf(y) = h(y)f(\varphi(y)), \quad \forall f \in C_0(X), \forall y \in Y.$$

*Both  $\varphi$  and  $h$  are continuous on the cozero set  $\text{coz}(h) := \{y \in Y : h(y) \neq 0\}$ , which is open in  $Y$ . If  $T$  is bijective, then  $h$  is away from zero and  $\varphi$  is a homeomorphism from  $Y$  onto  $X$ .*

The original form of Theorem 2.6 is established for Riesz isomorphisms (see, e.g., [35, p. 172]). Note that a linear operator on Riesz spaces is a Riesz homomorphism if and only if it is positive and disjointness preserving. So the above form is indeed an improvement. In the following, we will see that the disjointness structure (= zero product structure) alone gives rise to a rather rich theory.

The classical Banach–Stone Theorems have been generalized in several contexts and appeared in, e.g., the vector-valued version, the lattice version, and the  $C^*$ -algebra version [1, 5, 27, 22, 28, 21, 15]. Going in a different

direction from [7], we will continue this line and obtain polynomial versions for  $C^*$ -algebras in Sections 3 and 4.

**3. Orthogonally additive and zero product preserving holomorphic maps.** Let  $X$  and  $Y$  be locally compact Hausdorff spaces. Let  $P : C_0(X) \rightarrow C_0(Y)$  be a bounded orthogonally additive  $n$ -homogeneous polynomial. By Theorem 2.2, there exists a bounded linear operator  $T : C_0(X) \rightarrow C_0(Y)$  such that

$$(3.1) \quad P(f) = T(f^n), \quad \forall f \in C_0(X).$$

In the following, a subset  $F$  of  $C_0(Y)$  is called

- *strongly* (respectively, *strictly*) *separating points* in  $Y$  if for every pair of distinct points  $y_1, y_2$  in  $Y$  there is an  $f$  in  $F$  such that  $|f(y_1)| \neq |f(y_2)|$  (resp.  $f(y_1) \neq f(y_2) = 0$ );
- *regular* if for any closed subset  $Y_0$  of  $Y$  and any point  $y$  in  $Y \setminus Y_0$  there is an  $f$  in  $F$  such that  $f = 0$  on  $Y_0$  and  $f(y) \neq 0$ .

We say that an  $n$ -homogeneous polynomial  $P : C_0(X) \rightarrow C_0(Y)$  has *trivial positive kernel* if

$$P(f) = 0 \Rightarrow f = 0, \quad \forall f \in C_0(X)_+.$$

**THEOREM 3.1** (Banach–Stone Theorem for zero product preserving polynomials). *Let  $P : C_0(X) \rightarrow C_0(Y)$  be a bounded orthogonally additive  $n$ -homogeneous polynomial. Assume that  $P$  is zero product preserving on positive elements, i.e.,*

$$fg = 0 \Rightarrow P(f)P(g) = 0, \quad \forall f, g \in C_0(X)_+.$$

*Then there exist a bounded scalar function  $h$  on  $Y$  and a map  $\varphi : Y \rightarrow X$  such that*

$$P(f)(y) = h(y)f(\varphi(y))^n, \quad \forall f \in C_0(X), \forall y \in Y.$$

*Here, both  $h$  and  $\varphi$  are continuous wherever  $h$  is nonvanishing.*

*If, in addition,  $P$  has trivial positive kernel and its range separates points in  $Y$  strictly, then  $h$  is nonvanishing on  $Y$  and  $\varphi$  is a homeomorphism from  $Y$  onto a dense subset of  $X$ .*

*Proof.* Let  $T$  be the bounded linear map associated to  $P$  as in (3.1). As seen in the proof of Lemma 2.4,  $T$  is zero product preserving. It follows from Theorem 2.6 that there exists a bounded weight function  $h$  on  $Y$  continuous on its open cozero set  $Y_1 = \{y \in Y : h(y) \neq 0\}$ , and a continuous map  $\varphi : Y_1 \rightarrow X$  such that

$$(3.2) \quad P(f)(y) = \begin{cases} h(y)f(\varphi(y))^n, & \forall f \in C_0(X), \forall y \in Y_1, \\ 0, & \text{on } Y \setminus Y_1. \end{cases}$$

Without further assumptions, we might define  $\varphi$  arbitrarily on  $Y \setminus Y_1$ . This finishes the first assertion of the theorem.

CLAIM 1. *If  $P$  has trivial positive kernel, then  $\varphi(Y_1)$  is dense in  $X$ .*

Otherwise, there would be a nonempty open set  $U$  in  $X$  disjoint from  $\varphi(Y_1)$ . Let  $f$  be nonzero in  $C_0(X)_+$  vanishing outside  $U$ . Then  $P(f) = 0$  implies  $f = 0$ , a contradiction.

CLAIM 2. *If  $P$  separates points in  $Y$  strictly, then  $Y = Y_1$  and  $\varphi$  is a homeomorphism from  $Y$  onto  $\varphi(Y)$ .*

Since for every  $y$  in  $Y$ , there is an  $f$  in  $C_0(X)$  such that  $P(f)(y) \neq 0$ , we see that  $Y = Y_1$  on which  $h$  is nonvanishing. Hence

$$P(f)(y) = h(y)f(\varphi(y))^n, \quad \forall f \in C_0(X), \forall y \in Y.$$

Moreover, for any distinct points  $y_1, y_2$  in  $Y$  the strict separation property of  $P$  ensures again that  $\varphi(y_1) \neq \varphi(y_2)$ . Thus,  $\varphi$  is one-to-one. Finally, it is routine to see that  $\varphi$  is a homeomorphism from  $Y$  onto  $\varphi(X)$ . ■

EXAMPLES 3.2. We remark that  $\varphi(Y)$  can be a proper dense subset of  $X$  in Theorem 3.1. For example, consider the map  $P : C[0, 1] \rightarrow C_0(0, 1]$  defined by  $P(f)(t) = tf(t)$ . On the other hand, the weight function  $h$  might not be continuous on the whole  $Y$ , as shown by the map  $P : C_0(0, 1] \rightarrow C[0, 1]$  defined by  $P(f)(t) = \sin(1/t)f(t)$  on  $Y_1 = (0, 1]$ . Moreover, the strict separation assumption on the range of  $P$  cannot be weakened to the usual one. Consider  $P : C[0, 1] \rightarrow C([0, 1] \cup [2, 3])$  defined by  $P(f)|_{[0,1]}(t) = f(t)$  and  $P(f)|_{[2,3]}(t) = f(t - 2)/2$ . It is obvious that the range of  $P$  separates points in  $Y = [0, 1] \cup [2, 3]$  (but not strictly), and  $\varphi(t) = \varphi(2 + t)$  for all  $t$  in  $[0, 1]$ .

THEOREM 3.3. *Let  $H : B_{C_0(X)}(0; r) \rightarrow C_0(Y)$  be a bounded orthogonally additive and zero product preserving holomorphic function. Then there exist a sequence  $(h_n)$  of bounded scalar functions on  $Y$  in which each  $h_n$  is continuous on its cozero set, which is open, and a map  $\varphi : Y \rightarrow X$  such that*

$$H(f)(y) = \sum_{n=1}^{\infty} h_n(y)f(\varphi(y))^n, \quad \forall y \in Y,$$

*uniformly for all  $f$  in  $B_{C_0(X)}(0; r)$ . Here,  $\varphi$  is continuous whenever any  $h_n$  is nonvanishing.*

*Proof.* The holomorphic function  $H : B_{C_0(X)}(0; r) \rightarrow C_0(Y)$  has its Taylor series  $\sum_{n \geq 0} P_n$  at zero. By Proposition 2.1,  $P_n : C_0(X) \rightarrow C_0(Y)$  is a bounded orthogonally additive and zero product preserving  $n$ -homogeneous polynomial for every  $n \geq 1$ , and  $P_0 = 0$ . It follows from Theorem 3.1 that, for each  $n = 1, 2, \dots$ , there exist a bounded scalar function  $h_n$  on  $Y$  continuous on its cozero set  $\text{coz}(h_n) := \{y \in Y : h_n(y) \neq 0\}$ , which is open, and a



map  $\varphi_n : Y \rightarrow X$  continuous on  $\text{coz}(h_n)$  such that

$$P_n(f)(y) = h_n(y)f(\varphi_n(y))^n, \quad \forall f \in C_0(X), \forall y \in Y.$$

For any two positive integers  $m \neq n$ , we claim that

$$\varphi_m(y) = \varphi_n(y), \quad \forall y \in \text{coz}(h_m) \cap \text{coz}(h_n).$$

Suppose  $y \in Y$  is such that  $h_m(y)h_n(y) \neq 0$  and  $x_m = \varphi_m(y) \neq x_n = \varphi_n(y)$ . Let  $f, g \in C_0(X)_+$  be such that  $fg = 0$ , and  $f(x_m) = g(x_n) = 1$ . By Proposition 2.1(b), we see that

$$0 = P_m(f)(y)P_n(g)(y) = h_m(y)h_n(y).$$

This contradiction shows that  $\varphi_m, \varphi_n$  agree on  $\text{coz}(h_m) \cap \text{coz}(h_n)$ . Therefore, we can define a map  $\varphi : Y \rightarrow X$  by set-theoretical union, which agrees with  $\varphi_n$  and is continuous on  $\text{coz}(h_n)$  for  $n = 1, 2, \dots$  ■

To obtain the noncommutative version of Theorem 3.3, we need the following counterpart of Theorem 3.1.

**THEOREM 3.4.** *Let  $A, B$  be  $C^*$ -algebras. Let  $P : A \rightarrow B$  be a bounded orthogonally additive  $n$ -homogeneous polynomial. Let  $T : A \rightarrow B$  be the bounded linear operator such that  $P(a) = T(a^n)$  for all  $a \in A$ . Let  $h := T^{**}(1)$ , where  $T^{**}$  is the bidual map of  $T$ . Suppose that*

$$ab = 0 \Rightarrow P(a)P(b) = 0, \quad \forall a, b \in A_+.$$

- (a) *If  $h$  is invertible, then there is a Jordan homomorphism  $J : A \rightarrow B$  such that*

$$P(a) = hJ(a)^n = J(a)^nh, \quad \forall a \in A.$$

- (b) *If  $B = \text{span } P(A)$ , the linear span of the range of  $P$ , then  $h$  is a central invertible multiplier of  $B$  and  $J$  in (a) is surjective.*

*In both cases,  $J$  is injective if and only if  $P$  has trivial positive kernel.*

*Proof.* By Theorem 2.3, we have a bounded linear operator  $T : A \rightarrow B$  such that  $P(a) = T(a^n)$  for all  $a \in A$ . By functional calculus, for every  $x$  in  $A_+$  there is a positive element  $y$  in  $C^*(x)$ , the  $C^*$ -subalgebra of  $A$  generated by  $x$ , such that  $y^n = x$ . As in the proof of Lemma 2.4, we see that  $T$  sends zero products in  $A_{\text{sa}}$  to zero products in  $B$ . It follows from [14, Lemma 4.5] and [45, Lemma 2.3] that

$$T^{**}(1)T(a^2) = T(a^2)T^{**}(1) = T(a)^2, \quad \forall a \in A.$$

Since  $A^2 = A$ , we see that  $h = T^{**}(1)$  commutes with all elements in the range of  $T$ . If  $h$  is invertible, then  $J = h^{-1}T = Th^{-1}$  is a Jordan homomorphism. Since Jordan homomorphisms preserve powers [25, Theorem 1], we have

$$P(a) = T(a^n) = hJ(a^n) = hJ(a)^n = J(a)^nh, \quad \forall a \in A.$$

Now, suppose  $B = \text{span}P(A)$  instead. In this case, every element  $b$  in  $B$  can be written as a linear sum  $b = \sum_j \alpha_j P(b_j) = \sum_j \alpha_j T(b_j^n) = T(\sum_j \alpha_j b_j^n)$ . Therefore,  $T$  is surjective. Moreover,  $T$  sends zero products in  $A_{\text{sa}}$  to zero products in  $B$ . By [45, Theorem 2.4] (where  $A$  can be nonunital; see also [14, Theorem 4.12] for the unital case), we see that  $h$  is an invertible central multiplier of  $B$ , and  $J = h^{-1}T$  is a surjective Jordan homomorphism from  $A$  onto  $B$ .

Suppose now any one of the two cases holds. In particular,  $h$  is invertible. Consequently,

$$P(a) = hJ(a)^n = hJ(a^n) = 0 \iff J(a^n) = 0, \quad \forall a \in A.$$

Therefore,  $P$  has trivial positive kernel whenever  $J$  is injective. Next, we show the converse. Let  $a \in A$  be such that  $J(a) = 0$ . Since the kernel of  $J$  is a closed Jordan ideal, and thus a closed self-adjoint two-sided ideal of  $A$ , [18, Lemma 5.2 and Theorem 5.3], we have  $J(a^*) = 0$ . Replacing with its real or imaginary part, we can assume  $a = a^*$ . Observe that  $P(a^2) = hJ(a^2)^n = hJ(a)^{2n} = 0$  and  $a^2 \geq 0$ . Since  $P$  has trivial positive kernel, we get  $a^2 = 0$  and thus  $a = 0$ . This ensures the injectivity of  $J$ . ■

The following lemma might be known, although we have not found a reference in the literature. We thank Lawrence G. Brown for telling us the following proof.

LEMMA 3.5. *Let  $A$  be a noncommutative  $C^*$ -algebra. Then there exist  $a, b$  in  $A$  such that  $ab = 0$  but  $b^n a^n \neq 0$  for  $n = 1, 2, \dots$*

*Proof.* By Kaplansky’s theorem [30, p. 292] there is a norm one element  $x$  in  $A$  with  $x^2 = 0$ . Let  $h = |x|$  and  $k = |x^*|$ . Consider the left support projection  $p = \lim_n k^{1/n}$  and the right support projection  $q = \lim_n h^{1/n}$  of  $x$ . Then  $p, q$  are open projections of  $A$  such that  $pq = 0$  and  $x = pxq$ . Moreover, 1 is in the spectrum of the positive norm one element  $k$ . Let  $a = k$  and  $b = x^* + h$ . Then  $ab = 0$ .

We verify that  $b^n a^n \neq 0$ . Let  $B$  be the commutative  $C^*$ -subalgebra of  $A$  generated by the orthogonal positive elements  $h$  and  $k$ . Indeed,  $B$  consists of elements  $f(k) + g(h)$ , where  $f$  and  $g$  are continuous functions vanishing at 0. Note that every complex homomorphism of  $B$  extends to a pure state of  $A$ . Thus there is a pure state  $\phi$  such that  $\phi(f(k)) = f(1)$ , and  $\phi(g(h)) = 0$ . Consider the GNS representation  $(\pi, \mathcal{H}, v)$  for  $\phi$ , where  $v$  is the state vector. Thus  $\pi(f(k))v = f(1)v$ , and  $\pi(g(h))v = 0$ . Let  $w = \pi(x)^*v$ . Then  $v$  and  $w$  form an orthonormal basis for a two-dimensional subspace of the Hilbert space  $\mathcal{H}$  which is invariant under both  $\pi(x)$  and  $\pi(x)^*$ . The matrix representations of the restrictions of  $\pi(a)$  and  $\pi(b)$  to this subspace can be

written as two idempotent  $2 \times 2$  matrices

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}.$$

Their product is 0 in one order but non-zero in the other order. In particular,  $\pi(b^n a^n) = \pi(b)^n \pi(a)^n = \pi(b)\pi(a) \neq 0$ . Thus  $b^n a^n \neq 0$  for  $n = 1, 2, \dots$  ■

**THEOREM 3.6.** *Let  $A, B$  be  $C^*$ -algebras. Let  $P : A \rightarrow B$  be a bounded orthogonally additive  $n$ -homogeneous polynomial. Suppose that  $B = \text{span } P(A)$ , and*

$$ab = 0 \Rightarrow P(a)P(b) = 0, \quad \forall a, b \in A.$$

*Then there is a central invertible multiplier  $h$  of  $B$  and a bounded surjective algebra homomorphism  $J$  from  $A$  onto  $B$  such that*

$$P(a) = hJ(a)^n, \quad \forall a \in A.$$

*Moreover,  $J$  is an algebra isomorphism if and only if  $P$  has trivial positive kernel.*

*Proof.* In view of Theorem 3.4, it suffices to verify that the surjective Jordan homomorphism  $J : A \rightarrow B$  is multiplicative. By Brešar’s theorem [8, Theorem 2.3], there are closed ideals  $I_1, I_2$  of  $A$  and  $I'_1, I'_2$  of  $B$  satisfying the following properties:

- (i)  $I_1 + I_2$  is an essential ideal of  $A$  with  $I_1 \cap I_2 = \ker J$ .
- (ii)  $I'_1 + I'_2$  is an essential ideal of  $B$  with  $I'_1 \cap I'_2 = \{0\}$ .
- (iii)  $J(I_1) = I'_1$  and  $J(I_2) = I'_2$ .
- (iv)  $J(ux) = J(u)J(x)$  for all  $u \in I_1$  and  $x \in A$ .
- (v)  $J(vx) = J(x)J(v)$  for all  $v \in I_2$  and  $x \in A$ .

Let  $I$  be the kernel of the Jordan homomorphism  $J|_{I_2}$ . Then  $I$  is a closed two-sided ideal of the  $C^*$ -algebra  $I_2$  (see [18]). Therefore,  $J$  induces a Jordan isomorphism  $\tilde{J}$  from the  $C^*$ -algebra  $I_2/I$  onto  $I'_2$ . By (v),  $\tilde{J}$  is anti-multiplicative.

We claim that  $I'_2$  is commutative. Otherwise, by Lemma 3.5 there exist  $a', b'$  in  $I'_2$  such that  $a'b' = 0$  but  $b'^n a'^n \neq 0$ . Let  $\tilde{a}, \tilde{b} \in I_2/I$  be such that  $\tilde{J}(\tilde{a}) = a'$  and  $\tilde{J}(\tilde{b}) = b'$ . Then

$$\tilde{J}(\tilde{b}\tilde{a}) = \tilde{J}(\tilde{a})\tilde{J}(\tilde{b}) = a'b' = 0.$$

Since  $\tilde{J}$  is injective,  $\tilde{b}\tilde{a} = 0$  in  $I_2/I$ . By [2, Proposition 2.3] (see also [14, Lemma 4.14]), there are  $c, d$  in  $I_2$  such that  $\tilde{a} = c + I$ ,  $\tilde{b} = d + I$  and  $dc = 0$ . It follows from the zero product preserving property of  $P$  that

$$P(d)P(c) = h^2 J(d)^n J(c)^n = 0.$$

Since  $h$  is invertible, we have  $J(d)^n J(c)^n = 0$ . This in turn provides a contradiction;

$$0 = \tilde{J}(\tilde{b})^n \tilde{J}(\tilde{a})^n = b'^n a'^n \neq 0,$$

which proves the commutativity of  $I'_2$ .

Denote by  $W = I_1 + I_2$  the essential ideal of  $A$ . It follows from (iv) and (v) that  $J(wx) = J(w)J(x)$  for all  $w$  in  $W$  and  $x$  in  $A$ . Consequently, for all  $w$  in  $W$  and  $x, y$  in  $A$ ,

$$J(w)J(xy) = J(wxy) = J(wx)J(y) = J(w)J(x)J(y).$$

It turns out that

$$J(W)(J(xy) - J(x)J(y)) = 0.$$

As  $J(W) = I'_1 + I'_2$  is an essential ideal of  $B$  by (ii), we establish the desired conclusion that  $J(xy) = J(x)J(y)$ , and so  $J$  is an algebra homomorphism. ■

Recall that a *standard*  $C^*$ -algebra  $A$  on a Hilbert space  $\mathcal{H}$  is a  $C^*$ -subalgebra of  $B(\mathcal{H})$  containing all compact operators. In particular,  $B(\mathcal{H})$  and  $K(\mathcal{H})$ , the  $C^*$ -algebra of compact operators, are standard. However, the readers are referred to [17, 4] for other usages of the term “standard  $C^*$ -algebras” and “standard operator algebras”.

**COROLLARY 3.7.** *Let  $\mathcal{H}$  be a complex Hilbert space of arbitrary dimension. Let  $A$  be a standard  $C^*$ -algebra on  $\mathcal{H}$ . Let  $P : A \rightarrow A$  be a bounded orthogonally additive  $n$ -homogeneous polynomial such that  $A = \text{span } P(A)$ .*

- (a) *If  $P(a)P(b) = 0$  whenever  $a, b \in A_+$  with  $ab = 0$ , then there exist a nonzero scalar  $\lambda$  and an invertible operator  $S$  in  $B(\mathcal{H})$  such that either*

$$P(a) = \lambda Sa^n S^{-1}, \quad \forall a \in A, \quad \text{or} \quad P(a) = \lambda S(a^\dagger)^n S^{-1}, \quad \forall a \in A.$$

- (b) *If  $P(a)P(b) = 0$  whenever  $a, b \in A$  with  $ab = 0$ , then there exist a nonzero scalar  $\lambda$  and an invertible operator  $S$  in  $B(\mathcal{H})$  such that*

$$P(a) = \lambda Sa^n S^{-1}, \quad \forall a \in A.$$

*Proof.* By Theorems 3.4 and 3.6, we obtain a surjective Jordan or algebra homomorphism  $J : A \rightarrow A$ . Note that the kernel of  $J$  is a two-sided ideal of  $A$  (see [18]). From [31, Lemma 2] (and its proof) we deduce that  $J$  is indeed bijective. The assertions then follow from the known facts about Jordan and algebra automorphisms of standard  $C^*$ -algebras and the triviality of the center of  $A$  (see, e.g., [36], [37], [16, Corollary 3.2] and [40, §6]). ■

For holomorphic maps of matrices, we have a counterpart to Theorem 3.3.

**THEOREM 3.8** ([10]). *Let  $m$  and  $s$  be positive integers with  $m \geq 2$  and  $m \geq s$ . Let  $H : B_{M_m}(0; r) \rightarrow M_s$  be a holomorphic function between com-*

plex matrix algebras. Assume  $H$  is orthogonally additive and zero product preserving on self-adjoint elements. Then either

- (a) the range of  $H$  consists of zero trace elements (this case occurs whenever  $s < m$ ), or
- (b)  $s = m$ , and there exist a scalar sequence  $(\lambda_n)$  (some  $\lambda_n$  can be zero) and an invertible  $m \times m$  matrix  $S$  such that

$$(3.3) \quad H(x) = \sum_{n \geq 1} \lambda_n S^{-1} x^n S, \quad \forall x \in B_{M_m}(0; r),$$

or

$$H(x) = \sum_{n \geq 1} \lambda_n S^{-1} (x^t)^n S, \quad \forall x \in B_{M_m}(0; r).$$

In case (b), we always have the representation (3.3) when  $H$  preserves all zero products, i.e.,

$$ab = 0 \Rightarrow H(a)H(b) = 0, \quad \forall a, b \in B_{M_m}(0; r).$$

The following examples borrowed from [14, 10] tell us that one cannot get a complete analog to Theorem 3.3 for the noncommutative case. We also remark that Example 3.9(c) below tells us that a similar conclusion of [23, Theorem 18] for orthogonally additive and double orthogonality preserving holomorphic functions does not hold for zero product preserving ones.

EXAMPLES 3.9. Let  $\{e_n : n = 1, 2, \dots\}$  be an orthonormal basis of a separable Hilbert space  $\mathcal{H}$ . Let  $E_{ij} = e_i \otimes e_j$  be the matrix unit in  $B(\mathcal{H})$  given by  $E_{ij}(h) = \langle h, e_j \rangle e_i$ .

(a) Consider the linear map  $\theta : B(\mathcal{H}) \rightarrow B(\mathcal{H})$  defined by  $\theta(T) := E_{11} T E_{12}$ . Then  $\theta$  is a bounded linear (and thus holomorphic) map. Since the range of  $\theta$  has trivial multiplication,  $\theta$  is zero product preserving. However,  $\theta$  cannot be written in the standard form as stated in Corollary 3.7 or Theorem 3.8.

(b) Consider  $\theta : M_k \rightarrow M_{k+2}$  defined by

$$(a_{ij}) \mapsto \begin{pmatrix} 0 & a_{11} & a_{12} & \dots & a_{1k} & 0 \\ 0 & 0 & 0 & \dots & 0 & a_{11} \\ 0 & 0 & 0 & \dots & 0 & a_{21} \\ \vdots & & & \ddots & & \vdots \\ 0 & 0 & 0 & \dots & 0 & a_{k1} \\ 0 & 0 & 0 & \dots & 0 & 0 \end{pmatrix}.$$

Then  $\theta$  is linear (and thus holomorphic), and zero product preserving on self-adjoint elements. Note that the range of  $\theta$  does not have trivial multiplication, since  $\theta(E_{11})^2 = E_{1,k+2}$ . However, the range of  $\theta$  consists of elements

of zero trace. We verify that  $\theta$  cannot be written in the form  $c\varphi$  for any fixed element  $c$  in  $M_{k+2}$  and any homomorphism or anti-homomorphism  $\varphi : M_k \rightarrow M_{k+2}$ . Assume, for example, that  $\theta = c\varphi$  and  $\varphi$  is a homomorphism. Then we arrive at the contradiction

$$\begin{aligned} E_{1,k+2} &= \theta(E_{11})^2 = \theta(E_{11})c\varphi(E_{11}) = \theta(E_{11})c(\varphi(E_{12})\varphi(E_{21})) \\ &= \theta(E_{11})\theta(E_{12})\varphi(E_{21}) = 0\varphi(E_{21}) = 0. \end{aligned}$$

(c) Let  $E$  and  $F$  be the isometries in  $B(\mathcal{H})$  such that  $E(e_n) = e_{2n}$  and  $F(e_n) = e_{2n-1}$  for  $n = 1, 2, \dots$ , respectively. Define a holomorphic function  $\theta : B(\mathcal{H}) \rightarrow B(\mathcal{H})$  by

$$\theta(a) = EaE^* + Fa^2F^*, \quad \forall a \in B(\mathcal{H}).$$

Then  $\theta$  is orthogonally additive and zero product preserving. The range of  $\theta$  contains the identity  $\theta(1) = 1$ . However, it cannot be written in any form as stated in Theorem 3.8(b).

To get an analogous result (Theorem 3.11 below) to Theorem 3.3 for holomorphic maps between general  $C^*$ -algebras, we need the following lemma.

LEMMA 3.10. *Let  $A, B$  be  $C^*$ -algebras,  $r > 0$ , and  $H = \sum_{n \geq 1} P_n : B_A(0; r) \rightarrow B$  be an orthogonally additive holomorphic map. Suppose  $H$  is zero product preserving on positive (resp. all) elements in  $B_A(0; r)$ . Assume there is a polynomial term  $P_k(x) = h_k J(x)^k$  providing a central invertible multiplier  $h_k$  in  $M(B)$  and a Jordan (resp. algebra) isomorphism  $J : A \rightarrow B$ . Then there are central multipliers  $h_n$  in  $M(B)$  for  $n \geq 1$  such that*

$$H(a) = \sum_{n \geq 1} h_n J(a)^n, \quad \forall a \in B_A(0; r).$$

*Proof.* Replacing  $H$  with the map  $x \mapsto J^{-1}(h_k^{-1}H(x))$ , we can assume that  $P_k(x) = x^k$  for all  $x$  in  $A = B$ . Let  $T_n$  be the bounded linear map associated to  $P_n$  such that  $P_n(x) = T_n(x^n)$ . For any positive  $x, y$  in  $A_+$  with  $xy = 0$ , by Proposition 2.1(b) we have  $P_k(x)P_n(y) = P_n(y)P_k(x) = 0$ . This gives  $x^k T_n(y^n) = T_n(y^n)x^k = 0$ . It follows that

$$(3.4) \quad aT_n(b) = T_n(b)a = 0 \quad \text{whenever } ab = 0 \text{ and } a, b \in A_+.$$

Let  $x, y \in M(A)_+$  with  $xy = 0$ . Choose  $a_\lambda, b_\lambda \in A_+$  such that  $a_\lambda \uparrow x$  and  $b_\lambda \uparrow y$ . Since  $a_\lambda b_\lambda = 0$  for all  $\lambda$ , (3.4) and the  $\sigma(A^{**}, A^*)$  continuity of  $T_n^{**}$  give

$$(3.5) \quad xy = 0 \Rightarrow xT_n^{**}(y) = T_n^{**}(y)x = 0, \quad \forall x, y \in M(A)_+.$$

Let  $a \in A_+$  with  $\|a\| = 1$ . Identify the  $C^*$ -subalgebra of  $M(A)$  generated by 1 and  $a$  with  $C(X)$ , where  $X \subseteq [0, 1]$  is the spectrum of  $a$ . Under this convention,  $C(X) \subseteq M(A)$ , and (3.5) applies.

Denote by  $\theta : C(X)^{**} \rightarrow B^{**}$  the map induced from  $T_n^{**}$ . For each positive integer  $N$  and each integer  $k = -1, 0, 1, \dots, N$ , let

$$X_{N,k} = (k/N, (k + 1)/N] \cap X.$$

Pick an arbitrary point  $x_{N,k}$  from each nonempty  $X_{N,k}$ . For any  $f$  in  $C(X)$ , we have

$$(3.6) \quad f = \lim_{N \rightarrow \infty} \sum_{X_{N,k} \neq \emptyset} f(x_{N,k}) 1_{X_{N,k}},$$

where  $1_{X_{N,k}}$  is the characteristic function of the Borel set  $X_{N,k}$ , and the limit of the finite sums converges uniformly on  $X$ . In particular, for every fixed positive integer  $N$  we have

$$1 = \sum_{X_{N,k} \neq \emptyset} 1_{X_{N,k}}.$$

For two disjoint nonempty sets  $X_{N,j}$  and  $X_{N,k}$ , we can find two sequences  $(f_m)$  and  $(g_m)$  in  $C(X)$  such that  $f_{m+p}g_m = 0$  for  $m, p = 0, 1, \dots$ , and  $f_m \rightarrow 1_{X_{N,j}}$  and  $g_m \rightarrow 1_{X_{N,k}}$  pointwise on  $X$ . By the weak\*-continuity of  $\theta$ , for all  $m = 1, 2, \dots$ , we have

$$1_{X_{N,j}}\theta(g_m) = \lim_{p \rightarrow \infty} f_{m+p}\theta(g_m) = 0, \quad \theta(1_{X_{N,j}})g_m = \lim_{p \rightarrow \infty} \theta(f_{m+p})g_m = 0.$$

Thus

$$1_{X_{N,j}}\theta(1_{X_{N,k}}) = \lim_{m \rightarrow \infty} 1_{X_{N,j}}\theta(g_m) = 0$$

and

$$\theta(1_{X_{N,j}})1_{X_{N,k}} = \lim_{m \rightarrow \infty} \theta(1_{X_{N,j}})g_m = 0.$$

Hence, for each positive integer  $N$  and  $j = -1, 0, 1, \dots, N$ ,

$$(3.7) \quad \begin{aligned} \theta(1_{X_{N,j}}) &= \sum_{X_{N,k} \neq \emptyset} 1_{X_{N,k}}\theta(1_{X_{N,j}}) = 1_{X_{N,j}}\theta(1_{X_{N,j}}) \\ &= 1_{X_{N,j}}\theta(1) = \theta(1)1_{X_{N,j}}. \end{aligned}$$

It follows from (3.6) and (3.7) that

$$\theta(f) = f\theta(1) = \theta(1)f, \quad \forall f \in C(X)_+.$$

In particular,

$$T_n(a) = aT_n^{**}(1) = T_n^{**}(1)a$$

for all positive norm one—and thus all—elements  $a$  in  $A$ .

Set  $h'_n = T_n^{**}(1)$  for all  $n = 1, 2, \dots$ . We have thus obtained a sequence  $(h'_n)$  of central multipliers in  $M(A)$  such that

$$H(a) = \sum_{n \geq 1} P_n(a) = \sum_{n \geq 1} T_n(a^n) = \sum_{n \geq 1} h'_n a^n, \quad \forall a \in A.$$

Note that  $h'_k = 1$ .

Going back to the original setting, we set  $h_n = h_k J(h'_n)$ . Since the surjective Jordan isomorphism  $J$  sends central multipliers to central multipliers [29, p. 330], all  $h_n$  are central multipliers in  $M(B)$ . Moreover, we have

$$J(h'_n a^n) = J(h'_n a^n + a^n h'_n)/2 = J(h'_n)J(a^n), \quad \forall a \in A, n = 1, 2, \dots$$

Consequently,

$$H(a) = h_k J\left(\sum_{n \geq 1} h'_n a^n\right) = \sum_{n \geq 1} h_n J(a)^n, \quad \forall a \in A. \blacksquare$$

Recall that a holomorphic map  $H$  is *conformal (at 0)* if its derivative  $P_1$  (at 0) is a bounded invertible linear operator. Combining Theorems 3.4 and 3.6, and Lemma 3.10, we get

**THEOREM 3.11.** *Let  $A, B$  be  $C^*$ -algebras,  $r > 0$ , and  $H : B_A(0; r) \rightarrow B$  be an orthogonally additive conformal holomorphic map. Suppose  $H$  is zero product preserving on positive (resp. all) elements in  $B_A(0; r)$ . Then there exist a sequence  $(h_n)$  of central multipliers in  $M(B)$  and a Jordan (resp. algebra) isomorphism  $J : A \rightarrow B$  such that*

$$(3.8) \quad H(a) = \sum_{n \geq 1} h_n J(a)^n, \quad \forall a \in B_A(0; r).$$

The following result supplements Theorem 3.8.

**COROLLARY 3.12.** *Let  $A$  and  $B$  be standard  $C^*$ -algebras on Hilbert spaces  $\mathcal{H}_1$  and  $\mathcal{H}_2$ , respectively. Let  $H : B_A(0; r) \rightarrow B$  be an orthogonally additive conformal holomorphic map. Suppose  $H$  is zero product preserving on positive elements. Then there exist a sequence  $(\lambda_n)$  of scalars and an invertible operator  $S : \mathcal{H}_2 \rightarrow \mathcal{H}_1$  such that either*

$$(3.9) \quad H(x) = \sum_{n \geq 1} \lambda_n S^{-1} x^n S, \quad \forall x \in B_A(0; r),$$

or

$$H(x) = \sum_{n \geq 1} \lambda_n S^{-1} (x^t)^n S, \quad \forall x \in B_A(0; r).$$

If  $H$  is zero product preserving on all elements in  $B_A(0; r)$ , then case (3.9) holds.

*Proof.* It follows from Theorem 3.11 that  $H$  has the form as in (3.8). Since  $M(B)$  has trivial center, all  $\lambda_n := h_n$  are scalars. It is well-known that the Jordan isomorphism  $J : A \rightarrow B$  has the form either

$$Jx = S^{-1}xS, \quad \forall x \in A, \quad \text{or} \quad Jx = S^{-1}x^tS, \quad \forall x \in A,$$

where  $S$  is a bounded invertible operator from  $\mathcal{H}_2$  onto  $\mathcal{H}_1$  (see for example [36, 37]). When  $H$  preserves zero products on the whole of  $B_A(0; r)$ , the map  $J$  is an algebra isomorphism. Hence, case (3.9) happens.  $\blacksquare$



### 4. Orthogonally additive and isometric polynomials

LEMMA 4.1. *Let  $X$  and  $Y$  be locally compact Hausdorff spaces. Let  $S : C_0(X) \rightarrow C_0(Y)$  be a linear map preserving norms of positive functions, that is,*

$$\|Sf\| = \|f\|, \quad \forall f \in C_0(X)_+.$$

*Suppose that the range of  $S$  strongly separates points in  $Y$ . Then there exist a homeomorphism  $\psi$  from  $X$  onto  $\psi(X) \subseteq Y$  and a continuous unimodular scalar function  $k$  on  $X$  such that*

$$Sf(\psi(x)) = k(x)f(x), \quad \forall f \in C_0(X), \forall x \in X.$$

*If the range of  $S$  is regular, then  $S$  is a surjective linear isometry and  $\psi(X) = Y$ .*

*Proof.* Let  $X_\infty = X \cup \{\infty\}$  be the one-point compactification of  $X$ . In case  $X$  is compact, the point  $\infty$  at infinity will be isolated in  $X$ . We identify

$$C_0(X) = \{f \in C(X_\infty) : f(\infty) = 0\}.$$

The same also applies to  $Y$  and  $C_0(Y)$ .

For every point  $x$  in  $X$ , set

$$S_x := \{y \in Y_\infty : |Sf(y)| = 1 \text{ for all } f \text{ in } C_0(X)_+ \text{ with } f(x) = \|f\| = 1\}.$$

Clearly,  $S_x$  is a compact subset of  $Y$ .

First, we verify that  $S_x$  is nonempty. Otherwise, for every point  $y$  in  $Y_\infty$  there is an  $f_y$  in  $C_0(X)_+$  with  $f_y(x) = \|f_y\| = 1$ , but  $|Sf_y(y)| < 1$ . Let

$$V_y = \{z \in Y_\infty : |Sf_y(z)| < 1\}.$$

Then  $V_y$  is an open neighborhood of  $y$  containing the point  $\infty$ . The open covering  $Y_\infty = \bigcup_y V_y$  of the compact space  $Y_\infty$  has a finite subcover

$$Y_\infty = V_{y_1} \cup \dots \cup V_{y_n}.$$

Let

$$f = \frac{1}{n}(f_{y_1} + \dots + f_{y_n}).$$

Clearly,  $f \in C_0(X)_+$  with  $f(x) = \|f\| = 1$ , and

$$|Sf(y)| \leq \frac{1}{n}(|Sf_{y_1}(y)| + \dots + |Sf_{y_n}(y)|) < 1, \quad \forall y \in Y_\infty.$$

This forces  $1 = \|f\| = \|Sf\| < 1$ , a contradiction.

Next, we verify that  $S_x$  contains exactly a single point in  $Y$ . Otherwise, let  $y_1, y_2$  be two distinct points in  $S_x$ . In other words,

$$(4.1) \quad |Sf(y_1)| = |Sf(y_2)| = 1$$

whenever  $f \in C_0(X)_+$  with  $f(x) = \|f\| = 1$ .

Let  $g$  be in  $C_0(X)_+$  with norm one, vanishing in a neighborhood of  $x$ . Let  $f$  be any function in  $C_0(X)_+$  with  $fg = 0$  and  $f(x) = \|f\| = 1$ . It follows from (4.1) that

$$|S(f + tg)(y_1)| = |S(f + tg)(y_2)| = 1, \quad \forall t \in [0, 1].$$

This forces  $Sg(y_1) = Sg(y_2) = 0$ . Consequently, dealing separately with the positive and negative parts of the real and imaginary parts of a continuous function we find that

$$Sg(y_1) = Sg(y_2) = 0$$

whenever  $g$  in  $C_0(X)$  vanishes in a neighborhood of  $x$ .

Applying Urysohn’s Lemma and the boundedness of  $S$ , we have indeed

$$(4.2) \quad Sg(y_1) = Sg(y_2) = 0 \quad \text{whenever } g \text{ in } C_0(X) \text{ vanishes at } x.$$

Therefore, there are scalars  $\lambda_1, \lambda_2$  such that

$$Sg(y_i) = \lambda_i g(x), \quad \forall g \in C_0(X), i = 1, 2.$$

By (4.1), we have  $|\lambda_1| = |\lambda_2| = 1$ , and thus  $|Sg(y_1)| = |Sg(y_2)|$  for all  $g$  in  $C_0(X)$ . This is absurd, since the range of  $S$  strongly separates points in  $Y$ .

Now, we can define a function  $\psi : X \rightarrow Y$  such that  $S_x = \{\psi(x)\}$ . As in deriving (4.2), we have

$$f(x) = 0 \Rightarrow Sf(\psi(x)) = 0, \quad \forall f \in C_0(X).$$

This provides a scalar  $k(x)$  such that

$$(4.3) \quad Sf(\psi(x)) = k(x)f(x), \quad \forall f \in C_0(X), \forall x \in X.$$

It follows from the definition of  $S_x$  that  $|k(x)| = 1$  for all  $x$  in  $X$ . Consequently,  $\psi$  is one-to-one on  $X$ .

We claim that  $\psi$  is a homeomorphism from  $X$  onto  $\psi(X)$ . To see this, suppose  $x_\lambda \rightarrow x$  in  $X$  and  $y$  is any cluster point of  $\psi(x_\lambda)$  in  $Y_\infty$ . It follows from (4.3) that  $|Sf(y)| = |f(x)| = |Sf(\psi(x))|$  for all  $f$  in  $C_0(X)$ . By strong separability,  $y = \psi(x)$ , and thus  $\lim_\lambda \psi(x_\lambda) = \psi(x)$ . Conversely, assume that  $\psi(x_\lambda) \rightarrow \psi(x)$  and  $\{x_\lambda\}$  has a cluster point  $z$  in  $X_\infty$ . By (4.3) again,  $|f(x)| = |f(z)|$  for all  $f$  in  $C_0(X)$ . This forces  $z = x$ , and  $x = \lim_\lambda x_\lambda$  in  $X$ .

It is now plain that  $k$  is continuous on  $X$ .

Finally, if the range of  $S$  is regular then (4.3) ensures that  $\psi(X)$  is dense in  $Y$ . Consequently,  $S$  is a surjective linear isometry. Applying what we have obtained to the inverse  $S^{-1} : C_0(Y) \rightarrow C_0(X)$ , we can verify that  $\psi$  is invertible and especially  $\psi(X) = Y$ . ■

**THEOREM 4.2** (Banach–Stone Theorem for  $n$ -isometries). *Let  $P : C_0(X) \rightarrow C_0(Y)$  be an orthogonally additive  $n$ -homogeneous polynomial. Assume that  $P$  is an  $n$ -isometry on positive elements, and its range is regular. Then*

there exist a continuous unimodular scalar function  $h$  and a homeomorphism  $\varphi : Y \rightarrow X$  such that

$$P(f)(y) = h(y)f(\varphi(y))^n, \quad \forall f \in C_0(X), \forall y \in Y.$$

*Proof.* Let  $T : C_0(X) \rightarrow C_0(Y)$  be the bounded linear map associated to  $P$  such that  $P(f) = T(f^n)$  as in (3.1). As  $P : C_0(X) \rightarrow C_0(Y)$  is an  $n$ -isometry on positive elements, for every nonnegative function  $f$  in  $C_0(X)_+$  we have

$$\|T(f)\| = \|P(\sqrt[n]{f})\| = \|\sqrt[n]{f}\|^n = \|f\|.$$

Containing the regular subset  $P(C_0(X))$  of  $C_0(Y)$ , the range of  $T$  is also regular. Lemma 4.1 applies and yields a homeomorphism  $\psi$  from  $X$  onto  $Y$ , and a continuous unimodular scalar function  $k$  on  $X$  such that

$$Tf(\psi(x)) = k(x)f(x), \quad \forall f \in C_0(X), \forall x \in X.$$

Letting  $\varphi := \psi^{-1}$  and  $h := k \circ \varphi$ , we arrive at the desired assertions. ■

Without a tool similar to Lemma 4.1, we need extra assumptions in developing the following counterpart of Theorem 4.2.

**THEOREM 4.3.** *Let  $A, B$  be unital  $C^*$ -algebras. Let  $P : A \rightarrow B$  be an orthogonally additive  $n$ -homogeneous polynomial. Suppose that  $h := P(1)$  is a unitary,  $B = \text{span } P(A)$ , and  $\|P(x)\| = \|x\|^n$  for every normal element  $x$  in  $A$ . Then there is a Jordan  $*$ -isomorphism  $J : A \rightarrow B$  such that*

$$P(a) = hJ(a)^n, \quad \forall a \in A.$$

*Proof.* As in the proof of Theorem 3.4, we have a bounded surjective linear operator  $T : A \rightarrow B$  such that  $P(a) = T(a^n)$ ,  $\forall a \in A$ . Replacing  $P$  with  $h^*P$ , we can assume  $T(1) = 1$ . We are going to show that  $T$  is a Jordan  $*$ -isomorphism.

We compute the norm of  $T$  with the following formula (see, e.g., [34, Theorem 2.14.5]):

$$\|T\| = \sup\{\|T(e^{ih})\| : h^* = h \in A\} = \sup\{\|P(e^{ih/n})\| : h^* = h \in A\} = 1.$$

Let  $x$  be a normal element in  $A$ . Let  $\alpha$  be a point in the spectrum of  $x$  such that  $\|x\| = |\alpha|$ . By functional calculus, we can find a normal element  $y$  in  $A$  such that  $y^n = 2\alpha + x$ . Observe that

$$\|2\alpha + T(x)\| = \|P(y)\| = \|y\|^n = \|2\alpha + x\| = 3\|x\|.$$

Hence,  $\|x\| \leq \|Tx\| \leq \|x\|$ . In other words,  $T$  preserves the norm of every normal element in  $A$ . By [29, Lemma 8],  $T(a^*) = T(a)^*$  for every  $a$  in  $A$ . Consequently,  $T$  induces a surjective unital linear isometry  $T_{\text{sa}}$  between the JB-algebras  $A_{\text{sa}}$  and  $B_{\text{sa}}$ . It follows from [46, Theorem 4] that  $T_{\text{sa}}$  is a Jordan isomorphism from  $A_{\text{sa}}$  onto  $B_{\text{sa}}$ , and thus  $T$  is a Jordan  $*$ -isomorphism from  $A$  onto  $B$ . ■

The following is a consequence of Theorem 4.3, and the well-known facts about the structure of Jordan  $*$ -isomorphisms between standard  $C^*$ -algebras (see, e.g., [36, 37]).

**COROLLARY 4.4.** *Let  $\mathcal{H}$  be a complex Hilbert space of arbitrary dimension. Let  $A$  be a unital standard  $C^*$ -algebra on  $\mathcal{H}$ . Let  $P : A \rightarrow A$  be a bounded orthogonally additive  $n$ -homogeneous polynomial such that  $A = \text{span } P(A)$ . Suppose  $h := P(1)$  is a unitary and  $\|P(a)\| = \|a\|^n$  for every normal operator  $a$  in  $A$ . Then there are unitary operators  $U, V$  in  $B(\mathcal{H})$  such that either*

$$P(a) = Ua^nV, \quad \forall a \in A, \quad \text{or} \quad P(a) = U(a^t)^nV, \quad \forall a \in A.$$

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