

Diametral dimensions of Fréchet spaces

by

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Abstract. The diametral dimension is an important topological invariant in the category of Fréchet spaces which has been used, e.g., to distinguish types of Stein manifolds. We introduce variants of the classical definition in order to solve an old conjecture of Bessaga, Mityagin, Pełczyński, and Rolewicz at least for nuclear Fréchet spaces. Moreover, we clarify the relation between an invariant recently introduced by Terzioğlu and the by now classical condition $(\bar{\Omega})$ of Vogt and Wagner.

1. Kolmogorov widths and diametral dimensions. Kolmogorov widths (or diameters) are a quantitative measure for compactness in normed spaces: for absolutely convex sets V and U of a vector space X (typically U is the unit ball of a given norm) and $n \in \mathbb{N}_0$ the n th *width* is

$$\delta_n(V, U) = \inf\{\delta > 0 : V \subseteq \delta U + L \text{ for a subspace } L \text{ of } X \text{ with } \dim(L) \leq n\}$$

(the dependence on X is notationally suppressed). If V is bounded with respect to U (i.e., $\delta_0(V, U) < \infty$) then V is precompact with respect to the the Minkowski functional of U (which is a seminorm with unit ball U) if and only if $\delta_n(V, U) \rightarrow 0$. This elementary fact is Proposition 1.2 in Pinkus' book [Pin85] where much more information about n -widths can be found.

For a locally convex space (l.c.s.) X with the system $\mathcal{U}_0(X)$ of absolutely convex 0-neighbourhoods the *diametral dimension* of X is the sequence space

$$\Delta(X) = \{\xi \in \mathbb{R}^{\mathbb{N}_0} : \forall U \in \mathcal{U}_0(X) \exists V \in \mathcal{U}_0(X) : \xi_n \delta_n(V, U) \rightarrow 0\}.$$

This space is a topological invariant, i.e., if X and Y are isomorphic l.c.s. then $\Delta(X) = \Delta(Y)$. Even more, if Y is a quotient of X then we have

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the inclusion $\Delta(X) \subseteq \Delta(Y)$. Moreover, X is a Schwartz space (i.e., every 0-neighbourhood U contains another one which is precompact with respect to U) if and only if $\ell^\infty \subseteq \Delta(X)$.

There are several versions of diametral dimensions of locally convex spaces in the literature, all going back to an idea of Pełczyński [Peł57] from 1957. The formulation above can be found in [Mit61] where Mityagin refers to a joint work with Bessaga, Pełczyński, and Rolewicz which, to our best knowledge, eventually did not appear in print.

Implicitly, Mityagin also considered the following variant:

$$\Delta_b(X) = \{\xi \in \mathbb{R}^{\mathbb{N}_0} : \forall U \in \mathcal{U}_0(X) \forall B \in \mathcal{B}(X) : \xi_n \delta_n(B, U) \rightarrow 0\}.$$

where $\mathcal{B}(X)$ is the system of all bounded subsets of X . The obvious property $\delta_n(B, U) \leq S\delta_n(V, U)$ for $B \subseteq SV$ implies

$$\Delta(X) \subseteq \Delta_b(X)$$

for all l.c.s., and all bounded sets of X are precompact if and only if $\ell^\infty \subseteq \Delta_b(X)$. Referring to the joint work mentioned above, [Mit61, Proposition 9] claims $\Delta_b(X) = \Delta(X)$ for all Fréchet spaces X (actually, $\xi_n \delta_n(B, U) \rightarrow 0$ is only required for compact sets B but then the statement is clearly wrong even for Banach spaces).

It was probably soon realized that $\Delta = \Delta_b$ cannot be true in full generality, since the famous Grothendieck–Köthe example [MV97, 27.21] of a Fréchet–Montel space X which is not Schwartz satisfies $\Delta(X) = c_0$ and $\Delta_b(X) \supseteq \ell^\infty$. Terzioğlu [Ter13] describes this explicitly but without a definite conclusion when the equality is in fact true. The most optimistic conjecture is that $\Delta(X) = \Delta_b(X)$ for all Fréchet–Schwartz spaces (for spaces which are not Montel we trivially have $\Delta(X) = \Delta_b(X) = c_0$).

The main result of this part of the paper is a proof for hilbertizable Fréchet–Schwartz spaces, i.e., the seminorms can be given by (semi-) scalar products, in particular, this includes the important case of nuclear Fréchet spaces.

The following variants of the diametral dimensions turn out to be useful for this purpose:

$$\begin{aligned} \Delta^\infty(X) &= \{\xi \in \mathbb{R}^{\mathbb{N}_0} : \forall U \in \mathcal{U}_0(X) \exists V \in \mathcal{U}_0(X) : \xi_n \delta_n(V, U) \text{ bounded}\}, \\ \Delta_b^\infty(X) &= \{\xi \in \mathbb{R}^{\mathbb{N}_0} : \forall U \in \mathcal{U}_0(X) \forall B \in \mathcal{B}(X) : \xi_n \delta_n(B, U) \text{ bounded}\}. \end{aligned}$$

For every locally convex space X we have the obvious inclusions:

$$\begin{array}{ccc} \Delta^\infty(X) & \subseteq & \Delta_b^\infty(X) \\ \cup & & \cup \\ \Delta(X) & \subseteq & \Delta_b(X) \end{array}$$

Although we are interested in whether equality holds in the bottom row, we first consider the top row:

PROPOSITION 1.1. *The equality $\Delta^\infty(X) = \Delta_b^\infty(X)$ holds for every Fréchet–Schwartz space X .*

Proof. Take $\xi \in \Delta_b^\infty(X)$, a convex 0-neighbourhood U in X , and assume that $\xi_n \delta_n(U_\ell, U)$ is unbounded for all ℓ where U_ℓ is a decreasing basis of $\mathcal{U}_0(X)$. Then we find a strictly increasing sequence of integers n_ℓ such that $|\xi_{n_\ell}| \delta_{n_\ell}(U_\ell, U) > \ell$ for all $\ell \in \mathbb{N}$.

Since X is Schwartz we may assume that all U_ℓ are precompact with respect to U . Hence, there are finite sets $F_\ell \subseteq U_\ell$ with

$$U_\ell \subseteq \frac{1}{|\xi_{n_\ell}|}U + F_\ell.$$

Then $B = \bigcup_\ell F_\ell$ is bounded in X because, for each $p \in \mathbb{N}$, all but the finitely many elements of $F_0 \cup \dots \cup F_{p-1}$ belong to U_p . Since $\xi \in \Delta_b^\infty(X)$ there is a constant C such that $|\xi_{n_\ell}| \delta_{n_\ell}(B, U) < C$ for all ℓ .

For fixed $\ell > C + 1$ there is, by definition of $\delta_n(B, U)$, an at most n_ℓ -dimensional subspace L with $B \subseteq \frac{C}{|\xi_{n_\ell}|}U + L$, which implies

$$U_\ell \subseteq \frac{1}{|\xi_{n_\ell}|}U + F_\ell \subseteq \frac{1}{|\xi_{n_\ell}|}U + \frac{C}{|\xi_{n_\ell}|}U + L \subseteq \frac{C+1}{|\xi_{n_\ell}|}U + L.$$

Hence the contradiction $\delta_{n_\ell}(U_\ell, U) \leq (C + 1)/|\xi_{n_\ell}|$. ■

The proposition is trivially true for Fréchet spaces which are not Montel (because then $\Delta^\infty(X) = \Delta_b^\infty(X) = \ell^\infty$). It thus holds for all *quasinormable* Fréchet spaces. Up to changing c_0 to ℓ^∞ in the definition of the diametral dimension this saves [Ter13, Proposition 1].

In view of the proposition and the trivial inclusions mentioned above it is enough to investigate whether $\Delta(X) = \Delta^\infty(X)$ for Fréchet–Schwartz spaces.

If V and U are the unit balls of seminorms $p \geq q$, the Kolmogorov widths describe approximation properties of the inclusion $(X, p) \hookrightarrow (X, q)$ and it is easy to see (and well-known) that we may pass to the Hausdorff completions of these spaces, that is, $\delta_n(V, U) = \delta_n(T(B_p), B_q)$, where B_p is the unit ball of the completion X_V of $(X, p)/\text{Kern}(p)$ and T is the canonical map $X_V \rightarrow X_U$ induced by the inclusion. X_V is called the *local Banach space* corresponding to V . If p is induced by a (semi-) scalar product we call it the *local Hilbert space*.

For an operator $T : X \rightarrow Y$ between Banach spaces with unit balls B_X and B_Y we abbreviate $\delta_n(T) = \delta_n(T(B_X), B_Y)$. The velocity of convergence of this sequence is then a measure for the compactness of T .

For a scalar sequence $(\delta_n)_{n \in \mathbb{N}_0}$ we write, as usual,

$$o(\delta_n) = \{ \xi \in \mathbb{R}^{\mathbb{N}_0} : \forall \varepsilon > 0 \exists N \in \mathbb{N} \forall n \geq N : |\xi_n| \leq \varepsilon |\delta_n| \}.$$

We now consider the plausible statement that the product (composition) of two compact operators is “strictly more compact” than each factor. Unfortunately, we can only prove this for Hilbert spaces.

PROPOSITION 1.2. *Let X, Y, Z be Hilbert spaces and $T : X \rightarrow Y$ and $S : Y \rightarrow Z$ be compact operators. Then*

$$\delta_n(S \circ T) \in o(\delta_n(S)) \quad \text{and} \quad \delta_n(S \circ T) \in o(\delta_n(T)).$$

Proof. The advantage of the Hilbert space setting is that the Kolmogorov widths coincide with the singular numbers of the compact operator (see, e.g., [Pin85, Chapter IV] or [Vog00]). Let thus

$$S = \sum_{k=0}^{\infty} s_k \langle \cdot, e_k \rangle f_k$$

be a Schmidt representation of S with orthonormal systems $(e_k)_k$ and $(f_k)_k$ in Y and Z , respectively, and the decreasing sequence of singular numbers $s_k = \delta_k(S)$. Taking the span of f_0, \dots, f_{n-1} as a candidate for the infimum in the definition of $\delta_n(S \circ T)$ we get

$$\begin{aligned} \delta_n(S \circ T)^2 &\leq \sup \left\{ \left\| \sum_{k=n}^{\infty} s_k \langle T(x), e_k \rangle f_k \right\|_Z^2 : \|x\|_X \leq 1 \right\} \\ &= \sup \left\{ \sum_{k=n}^{\infty} s_k^2 |\langle T(x), e_k \rangle|^2 : \|x\|_X \leq 1 \right\} \\ &\leq s_n^2 \sup \left\{ \sum_{k=n}^{\infty} |\langle y, e_k \rangle|^2 : y \in K \right\} \end{aligned}$$

where $K = \overline{T(B_X)}$ is compact in Y . The sequence of functions

$$r_n(y) = \left(\sum_{k=n}^{\infty} |\langle y, e_k \rangle|^2 \right)^{1/2} = \|\pi_n(y)\|_Y$$

with the orthogonal projection π_n onto the closed span of $\{e_k : k \geq n\}$ is equicontinuous and converges pointwise to 0. It therefore converges uniformly to 0 on the compact set K . This proves $\delta_n(S \circ T) \in o(\delta_n(S))$.

The other assertion then follows by duality since

$$\delta_n(S \circ T) = \delta_n((S \circ T)^*) = \delta_n(T^* \circ S^*) \in o(\delta_n(T^*)) = o(\delta_n(T)). \blacksquare$$

PROPOSITION 1.3. $\Delta(X) = \Delta^\infty(X)$ for every hilbertizable Schwartz space X .

Proof. In the definitions of the diametral dimensions we may replace $\mathcal{U}_0(X)$ by the system of all 0-neighbourhoods which are unit balls of seminorms induced by scalar products. Given $\xi \in \Delta^\infty(X)$ and $U \in \mathcal{U}_0(X)$ we choose $V \in \mathcal{U}_0(X)$ such that the canonical map $T : X_V \rightarrow X_U$ between the local Hilbert spaces is compact and then we choose $W \in \mathcal{U}_0(X)$ such that $\xi_n \delta_n(W, V)$ is bounded and $S : X_W \rightarrow X_V$ is compact. The previous proposition then implies $\xi_n \delta_n(W, U) \in o(\xi_n \delta_n(W, V))$, hence $\xi_n \delta_n(W, U) \rightarrow 0$. ■

We remark that either statement in Proposition 1.2 for Banach instead of Hilbert spaces (i.e., the product of two compact operators between Banach spaces is “strictly more compact” than at least one of the factors) would give 1.3 for all Schwartz spaces.

Combining 1.1 and 1.3 we can now confirm the claim of Bessaga, Mityagin, Pełczyński, and Rolewicz at least for hilbertizable Fréchet–Schwartz spaces:

THEOREM 1.4. $\Delta(X) = \Delta_b(X) = \Delta^\infty(X) = \Delta_b^\infty(X)$ for every hilbertizable Fréchet–Schwartz space X and, in particular, for every nuclear Fréchet space X .

We do not know if $\Delta_b(X) = \Delta_b^\infty(X)$ for all Fréchet–Schwartz spaces X . However, the same method as in 1.3 gives this for hilbertizable Fréchet–Montel spaces (note that the Grothendieck–Köthe example can be chosen hilbertizable so that the following statement is not contained in 1.4):

PROPOSITION 1.5. $\Delta_b(X) = \Delta_b^\infty(X)$ for every hilbertizable Fréchet–Montel space X .

Proof. We use the fact that a hilbertizable Fréchet space has a fundamental system of bounded sets consisting of unit balls of Hilbert spaces (see, e.g., [MV97, end of proof of 29.16]) and that Fréchet spaces satisfy the *strict Mackey condition* introduced by Grothendieck: for every bounded set B there is an absolutely convex bounded set $D \supseteq B$ whose Minkowski functional induces on B the topology of X [PCB87, Theorem 5.1.27]. In our case we thus find, for each bounded Hilbert ball B , another bounded Hilbert ball D such that $\text{span}(B) \hookrightarrow \text{span}(D)$ is a compact inclusion between Hilbert spaces. For $U \in \mathcal{U}_0(X)$ and the associated local Hilbert space X_U we can thus apply Proposition 1.2 to $\text{span}(B) \hookrightarrow \text{span}(D) \rightarrow X_U$ and obtain $\delta_n(B, U) \in o(\delta_n(D, U))$. ■

2. Prominent sets. In his last publication [Ter13] T. Terzioğlu called a bounded subset B of a l.c.s. X *prominent* if

$$\Delta(X) = \{\xi \in \mathbb{R}^{\mathbb{N}_0} : \xi_n \delta_n(B, U) \rightarrow 0 \text{ for all } U \in \mathcal{U}_0(X)\}.$$

Having a prominent set is obviously a topological invariant, and since the right hand side above contains $\Delta_b(X)$, we have $\Delta(X) = \Delta_b(X)$ for all l.c.s. with a prominent bounded set. By an elegant application of Grothendieck’s factorization theorem Terzioglu [Ter13, Proposition 3] proved that an absolutely convex bounded set B of a Fréchet space X is prominent if and only if, for every $U \in \mathcal{U}_0(X)$, there are $V \in \mathcal{U}_0(X)$ and $C \geq 0$ such that $\delta_n(V, U) \leq C\delta_n(B, V)$ for all $n \in \mathbb{N}_0$.

Moreover, he proved the existence of prominent bounded sets in so-called G_1 spaces, a class of Köthe sequence spaces containing power series spaces of finite type, and he showed that power series spaces of infinite type do not have prominent sets.

There are some more topological invariants distinguishing power series spaces of finite and infinite type, in particular condition $(\overline{\Omega})$ of Vogt and Wagner [MV97, Chapter 29]. We recall that a Fréchet space X with a fundamental sequence of seminorms $\|\cdot\|_k$ and corresponding dual norms $\|\cdot\|_k^*$ satisfies $(\overline{\Omega})$ if

$$\forall k \in \mathbb{N} \exists \ell \geq k \forall m \geq \ell \exists C > 0 \forall \varphi \in X' : (\|\varphi\|_\ell^*)^2 \leq C\|\varphi\|_k^*\|\varphi\|_m^*.$$

THEOREM 2.1. *Every Fréchet space X with $(\overline{\Omega})$ has a prominent set.*

Proof. A combination of Lemmas 29.13 and 29.16 in [MV97] shows that there exists a bounded Banach disk B of X such that, for every $U \in \mathcal{U}_0(X)$, there exist $V \in \mathcal{U}_0(X)$ and $C > 0$ with

$$V \subseteq rU + \frac{C}{r}B \quad \text{for all } r > 0.$$

Fix $U \in \mathcal{U}_0(X)$ and take V and C as above. Let $n \in \mathbb{N}_0$ and $\delta > \delta_n(B, V)$. Then there exists an at most n -dimensional subspace L with $B \subseteq \delta V + L$. For $r = 2C\delta$, we obtain

$$V \subseteq rU + \frac{C}{r}B \subseteq rU + \frac{C\delta}{r}V + L = 2C\delta U + \frac{1}{2}V + L.$$

Inserting this inclusion into its right hand side, we get

$$V \subseteq 2C\delta U + \frac{1}{2}\left(2C\delta U + \frac{1}{2}V + L\right) + L \subseteq 2C\delta\left(1 + \frac{1}{2}\right)U + \frac{1}{4}V + L.$$

By iteration, for every $j \in \mathbb{N}$, this implies

$$V \subseteq 2C\delta\left(1 + \frac{1}{2} + \dots + \frac{1}{2^{j-1}}\right)U + \frac{1}{2^j}V + L \subseteq 4C\delta U + \frac{1}{2^j}V + L.$$

Choosing j such that $\frac{1}{2^j}V \subseteq C\delta U$ we get

$$V \subseteq 5C\delta U + L.$$

We have shown $\delta_n(V, U) \leq 5C\delta$ and thus

$$\delta_n(V, U) \leq 5C\delta_n(B, V).$$

Hence B is a prominent set of E . ■

We will next show that the implication in this theorem is in fact a characterization for *regular* Köthe spaces. We recall that for a matrix $A = (a_k(n))_{(k,n) \in \mathbb{N}_0^2}$ of positive weights with $a_k(n) \leq a_{k+1}(n)$ the corresponding Köthe space (of order 1—but everything below holds for all other orders) is

$$\lambda_1(A) = \left\{ (x_n)_{n \in \mathbb{N}_0} \in \mathbb{C}^{\mathbb{N}_0} : \|x\|_k = \sum_{n=0}^{\infty} a_k(n)|x_n| < \infty \text{ for all } k \in \mathbb{N}_0 \right\}.$$

The space and the matrix are called *regular* if $n \mapsto a_k(n)/a_{k+1}(n)$ is decreasing for every $k \in \mathbb{N}_0$. The advantage of regularity is that the Kolmogorov widths are then very easy to calculate: for the unit balls U_k of the seminorms $\|\cdot\|_k$ and $\ell \geq k$ we have, e.g., by [Ter08],

$$\delta_n(U_\ell, U_k) = \frac{a_k(n)}{a_\ell(n)} \quad \text{for all } n \in \mathbb{N}_0.$$

More information about diametral dimensions of Köthe spaces can be found in [Ter08, BD16].

A characterization of $(\overline{\Omega})$ for Köthe spaces is well-known (see, e.g., [Wag80, Satz 1.10]), and in fact easily obtained from $\|\pi_n\|_\ell^* = 1/a_\ell(n)$ (where $\pi_n(x) = x_n$) and the definition of $(\overline{\Omega})$:

$\lambda_1(A)$ has $(\overline{\Omega})$ if and only if, for every $k \in \mathbb{N}_0$, there exists $\ell \geq k$ such that for every $m \geq \ell$ there is $c > 0$ with

$$a_\ell^2(n) \geq ca_m(n)a_k(n) \quad \text{for all } n \in \mathbb{N}_0.$$

PROPOSITION 2.2. *A regular Köthe space $\lambda_1(A)$ has a prominent set if and only if it satisfies $(\overline{\Omega})$.*

Proof. We still have to show necessity of $(\overline{\Omega})$ given a prominent bounded set B . As supersets of prominent sets are prominent we may assume that $B = \bigcap_{k \in \mathbb{N}_0} r_k U_k$ for a sequence of scalars $r_k > 0$. For $k \in \mathbb{N}_0$, Terzioğlu’s characterization mentioned above for $U = U_k$ yields $V = U_\ell$ for some $\ell \geq k$ and $C > 0$ such that

$$\delta_n(U_\ell, U_k) \leq C\delta_n(B, U_\ell).$$

Given $m \geq \ell$, the inclusion $B \subseteq r_m U_m$ then yields

$$\delta_n(U_\ell, U_k) \leq C\delta_n(B, U_\ell) \leq Cr_m\delta_n(U_m, U_\ell).$$

The formula for $\delta_n(U_\ell, U_k)$ above thus gives

$$\frac{a_k(n)}{a_\ell(n)} \leq Cr_m \frac{a_\ell(n)}{a_m(n)} \quad \text{for all } n \in \mathbb{N}_0,$$

which implies $(\overline{\Omega})$. ■

We will finally show that products of two power series spaces of different type like $H(\mathbb{D}) \times H(\mathbb{C})$ may have prominent bounded sets. This shows that having prominent bounded sets is not inherited by complemented subspaces.

We recall that for an increasing sequence $0 < \alpha_n \rightarrow \infty$ the power series spaces of finite type $\Lambda_0(\alpha) = \lambda_1(A)$ and infinite type $\Lambda_\infty(\alpha) = \lambda_1(B)$ are regular Köthe spaces corresponding to the matrices

$$A = (\exp(-\alpha_n/k))_{(k,n) \in \mathbb{N}_0^2} \quad \text{and} \quad B = (\exp(k\alpha_n))_{(k,n) \in \mathbb{N}_0^2}.$$

The sequence α is called *stable* if α_{2n}/α_n is bounded (this is equivalent to $\Lambda_r(\alpha)$ being isomorphic to its Cartesian square).

PROPOSITION 2.3. *For every stable sequence $\alpha_n \rightarrow \infty$ the space $\Lambda_0(\alpha) \times \Lambda_\infty(\alpha)$ has a prominent bounded set but does not satisfy $(\bar{\Omega})$.*

Proof. As $\Lambda_0(\alpha)$ satisfies $(\bar{\Omega})$ it has a prominent bounded set B , and we will show that $B \times \{0\}$ is prominent in $\Lambda_0(\alpha) \times \Lambda_\infty(\alpha)$.

We write U_k^0 and U_k^∞ for the canonical 0-neighbourhoods in $\Lambda_0(\alpha)$ and $\Lambda_\infty(\alpha)$, respectively, as well as $\delta_n^0, \delta_n^\infty$, and δ_n^\times for the Kolmogorov diameters in $\Lambda_0(\alpha), \Lambda_\infty(\alpha)$, and their product.

The proof is based on the simple observation that

$$\delta_{2n}^\times(V^0 \times V^\infty, U^0 \times U^\infty) \leq \max\{\delta_n^0(V^0, U^0), \delta_n^\infty(V^\infty, U^\infty)\}.$$

Indeed, if $V^r \subseteq \delta U^r + L^r$ for $r \in \{0, \infty\}$ with at most n -dimensional subspaces L^r of $\Lambda_r(\alpha)$ then $L^0 \times L^\infty$ is an at most $2n$ -dimensional subspace of $\Lambda_0(\alpha) \times \Lambda_\infty(\alpha)$ with

$$V^0 \times V^\infty \subseteq \delta(U^0 \times U^\infty) + L^0 \times L^\infty.$$

Since power series spaces are regular the Kolmogorov diameters δ_n^r are easily calculated, in particular $\delta_n^\infty(V^\infty, U^\infty)$ is much smaller than $\delta_n^0(V^0, U^0)$. We will finally need stability to compare δ_n^0 and δ_{2n}^0 . The details of the proof need some care and we first give precise estimates for δ_{2n}^0 which are quite easily obtained from $\delta_n^0(U_k^0, U_m^0) = a_m(n)/a_k(n) = \exp((1/k - 1/m)\alpha_n)$.

For all $m \in \mathbb{N}_0$, there exists $s \geq m$ such that, for all $k \geq s$, we have

$$\delta_n^0(U_k^0, U_m^0) \leq \delta_{2n}^0(U_k^0, U_s^0) \quad \text{for every } n \in \mathbb{N}_0.$$

Indeed, we take $c > 0$ such that $\alpha_{2n}/\alpha_n \leq c$ for all $n \in \mathbb{N}$. If $m \in \mathbb{N}$ is given, we choose $s \geq (c + 1)m$. Then, if $k > s$, we have

$$\frac{\alpha_{2n}}{\alpha_n} \leq c \leq \frac{s}{m} - 1 \leq \frac{1/m - 1/k}{1/s} \leq \frac{1/m - 1/k}{1/s - 1/k}$$

for every $n \in \mathbb{N}_0$. Therefore, $(1/s - 1/k)\alpha_{2n} \leq (1/m - 1/k)\alpha_n$ for all $k \geq s$, which implies

$$\delta_n^0(U_k^0, U_m^0) = \exp((1/k - 1/m)\alpha_n) \leq \exp((1/k - 1/s)\alpha_{2n}) = \delta_{2n}^0(U_k^0, U_s^0).$$

Let now U be a 0-neighbourhood in $\Lambda_0(\alpha) \times \Lambda_\infty(\alpha)$ which we may assume to be of the form $U = U_m^0 \times U_m^\infty$ for some $m \in \mathbb{N}$ (because the sets $m^{-1}U_m^r$ are bases of the 0-neighbourhood filters). We choose s as above and then,

according to Terzioğlu’s characterization of prominence, $k \geq s$ and $C \geq 0$ such that for all $n \in \mathbb{N}_0$,

$$\delta_n^0(U_k^0, U_s^0) \leq C\delta_n^0(B, U_k^0).$$

We then get, for all $n \in \mathbb{N}_0$,

$$\delta_n^0(U_k^0, U_m^0) \leq \delta_{2n}^0(U_k^0, U_s^0) \leq C\delta_{2n}^0(B, U_k^0).$$

Moreover, since $\alpha_{n+1} \leq \alpha_{2n} \leq c\alpha_n$ we find $\ell \in \mathbb{N}$ such that

$$\begin{aligned} \delta_n^\infty(U_\ell^\infty, U_m^\infty) &= \exp((m - \ell)\alpha_n) \leq \exp((1/k - 1/m)\alpha_{n+1}) \\ &= \delta_{n+1}^0(U_k^0, U_m^0) \leq \delta_n^0(U_k^0, U_m^0) \end{aligned}$$

for all $n \in \mathbb{N}_0$. With these choices we get

$$\begin{aligned} \delta_{2n}^\times(U_k^0 \times U_\ell^\infty, U_m^0 \times U_m^\infty) &\leq \max\{\delta_n^0(U_k^0, U_m^0), \delta_n^\infty(U_\ell^\infty, U_m^\infty)\} \\ &= \delta_n^0(U_k^0, U_m^0) \leq C\delta_{2n}^0(B, U_k^0) = C\delta_{2n}^\times(B \times \{0\}, U_k^0 \times U_\ell^\infty) \end{aligned}$$

for every $n \in \mathbb{N}_0$ (the last equality is immediate from the definition of Kolmogorov widths). Similarly, for odd dimensions $2n + 1$ we have

$$\begin{aligned} \delta_{2n+1}^\times(U_k^0 \times U_\ell^\infty, U_m^0 \times U_m^\infty) &\leq \max\{\delta_{n+1}^0(U_k^0, U_m^0), \delta_n^\infty(U_\ell^\infty, U_m^\infty)\} \\ &= \delta_{n+1}^0(U_k^0, U_m^0) \leq C\delta_{2n+2}^0(B, U_k^0) \leq C\delta_{2n+1}^0(B, U_k^0) \\ &= C\delta_{2n+1}^\times(B \times \{0\}, U_k^0 \times U_\ell^\infty). \end{aligned}$$

We have thus shown that for all $n \in \mathbb{N}_0$,

$$\delta_n^\times(U_k^0 \times U_\ell^\infty, U_m^0 \times U_m^\infty) \leq C\delta_n^\times(B \times \{0\}, U_k^0 \times U_\ell^\infty),$$

which implies that $B \times \{0\}$ is a prominent bounded set in $\Lambda_0(\alpha) \times \Lambda_\infty(\alpha)$. ■

Combining this with Proposition 2.2 we get the following application of Terzioğlu’s invariant of having a bounded prominent set:

COROLLARY 2.4. *For every stable sequence $\alpha_n \rightarrow \infty$ the space $\Lambda_0(\alpha) \times \Lambda_\infty(\alpha)$ is not isomorphic to a regular Köthe space.*

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