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Discrete maximal regularity for abstract Cauchy problems

by

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Abstract. Maximal regularity is a fundamental concept in the theory of nonlinear partial differential equations, for example, quasilinear parabolic equations, and the Navier–Stokes equations. It is thus natural to ask whether the discrete analogue of this notion holds when the equation is discretized for numerical computation. In this paper, we introduce the notion of discrete maximal regularity for the finite difference method (θ-method), and show that discrete maximal regularity is roughly equivalent to (continuous) maximal regularity for bounded operators in the case of UMD spaces. The feature of our result is that it includes the conditionally stable case ($0 \leq \theta < 1/2$). We pay close attention to the dependence of the constants appearing in estimates. In addition, we show that this characterization is also true for unbounded operators in the case of the backward Euler method.

1. Introduction. In this paper, we consider the following abstract Cauchy problem in a Banach space X:

$$\begin{cases} u'(t) = Au(t) + f(t), & t > 0, \\ u(0) = 0, \end{cases}$$

where u is an unknown X-valued function, f is a given one, and A is a linear operator on X. The operator A is said to have (continuous) *maximal regularity* if, for some $p \in (1, \infty)$ and every $f \in L^p(0, \infty; X)$, the above problem has a unique solution u (the precise meaning of "solution" is described in Definition 2.4) satisfying

$$\|u'\|_{L^p(0,\infty;X)} + \|Au\|_{L^p(0,\infty;X)} \leq C\|f\|_{L^p(0,\infty;X)},$$

uniformly with respect to f. For example, it is known that the Laplace operator and the Stokes operator have maximal regularity under suitable conditions, and that this property can be applied to quasilinear parabolic equations and the Navier–Stokes equations (see e.g. [2, 26]). We are con-

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cerned about the numerical computation of the Cauchy problem above. As is well-known, the analytic semigroup theory and its discrete counterparts play important roles in construction and study of numerical schemes for parabolic equations (cf. [12, 13, 14, 24, 25]). Hence, it is natural to ask whether a discrete analogue of maximal regularity holds when the above problem is discretized for numerical computations. Moreover, if this is the case, then it is expected that the discrete version of maximal regularity can be applied to the numerical analysis of nonlinear evolution equations, for example, the stability analysis and the error estimate of the finite element approximation for the equations as given above. Indeed, Geissert [15, 16] and Li [20] considered the continuous maximal regularity for the discrete Laplacian, and applied it to the semidiscrete problem for the linear and semilinear heat equations. However, since they only dealt with the semidiscrete problem, the results cannot be applied to the analysis of practical computations. Thus, we need to consider the time-discretized problem and the discrete version of maximal regularity.

In the present paper, we concentrate on the discretization of the time variable, and postpone that of the space variables to further studies (cf. [17]). That is, we discretize the Cauchy problem by the finite difference method:

$$\begin{cases} \dfrac{u^{n+1} - u^n}{\tau} = Au^{n+\theta} + f^{n+\theta}, \quad n \in \mathbb{N} = \{0, 1, 2, \dots\}, \\ u^0 = 0, \end{cases} \tag{1.1}$$

where $\tau > 0$ is the time step, $\theta \in [0,1]$ is a fixed parameter, $u = (u^n)$ is an unknown X-valued sequence, and $f = (f^n)$ is a given one. We say that A has *discrete maximal regularity* if for every $f \in l^p(\mathbb{N}; X)$, there exists a unique solution u of (1.1) with the estimate

$$\|D_\tau u\|_{l^p_\tau(\mathbb{N};X)} + \|Au_\theta\|_{l^p_\tau(\mathbb{N};X)} \le C\|f_\theta\|_{l^p_\tau(\mathbb{N};X)},$$

uniformly with respect to τ. The meaning of symbols in the above inequality will be specified in Section 2. We are interested in (1) when the operator A has discrete maximal regularity, and (2) what the constant C depends on.

We now summarize several previous studies. For more details, one can refer to [1]. The first one is Blunck's result [5]. Here the forward Euler method ($\theta = 0$) was considered, and the notion of discrete maximal regularity was introduced. The main result was the discrete version of the operator-valued Fourier multiplier theorem (cf. [27, 21]) and a characterization of discrete maximal regularity. While these results seem powerful, Blunck only considered the case where the time step is unity (i.e., $\tau = 1$). Therefore, the result, especially the characterization of discrete maximal regularity, cannot be directly applied to numerical analysis. Although there are several works following Blunck such as [9, 23], these results are not applicable to analysis of numerical schemes either. Another study we mention is that of

Ashyralyev et al. [3]. They considered the backward Euler method ($\theta = 1$) and the Crank–Nicolson method ($\theta = 1/2$), with an arbitrary time step $\tau > 0$, and showed that if A has continuous maximal regularity, then it also has discrete maximal regularity for $\theta = 0, 1/2$. Recently, Kovács et al. [18] showed the same results for A-stable time discretizations, such as BDF and implicit Runge–Kutta methods. They also considered fully-discretized problems with abstract space-discretization. However, the constants appearing in those results may depend on the Banach spaces under consideration, due to utilization of Blunck's discrete multiplier theorem. In contrast to those results, we focus on general single-step finite difference methods (1.1) including the conditionally stable case, and we shall pay attention to what the constants depend on, especially whether they depend on Banach spaces.

Our main goal is to show that continuous maximal regularity implies discrete maximal regularity for general θ-methods in the case of UMD spaces. When $\theta \in [1/2, 1]$, our main theorem (Theorem 3.2) includes the results of Ashyralyev et al., and the proof is integrated and simplified. Our result also includes the conditionally stable case ($\theta \in [0, 1/2)$), which is not considered in the literature and is the main feature of the present paper. In this case, a certain stability condition is assumed, which is not mentioned in [5] when $\tau = 1$. This condition is reasonable since it often appears in the context of numerics for parabolic problems. Many operators exist that have continuous maximal regularity. Furthermore, many approaches to continuous maximal regularity have already been developed. Therefore, even if we do not know whether a given operator has maximal regularity, this can be investigated. As a result, our sufficient condition is quite reasonable from both analytical and numerical viewpoints.

As in the previous studies, our method is based on R-boundedness and Blunck's discrete Fourier multiplier theorem (Theorem 2.11). However, as mentioned before, the constant appearing in this theorem depends on the Banach space X. This is troublesome in view of numerical analysis, since X is assumed to be a finite-dimensional space that depends on the space discretization parameter (e.g., E_n in [3] and X_h in [18]) in applications. Therefore, we discuss this problem, and obtain an applicable version of the main result (Corollary 3.3). Although one may think that this discussion is obvious, it is important from the numerical point of view.

We also consider the opposite assertion: discrete maximal regularity implies continuous maximal regularity (Theorem 3.5), of just mathematical interest. This question was considered in [5] when $\theta = 0$ and $\tau = 1$. In [5, Theorem 1.1], the power-boundedness for $T = I + A$ is assumed, which is equivalent to the stability condition mentioned above for $\tau = 1$. However, it does not hold in general settings of numerical analysis. Therefore, we present a numerical-analytic statement, that is, we show that a uniform estimate (for small τ) implies continuous maximal regularity. This is proved for general θ by a limiting argument as $\tau \downarrow 0$. See also Theorem 4.1.

As mentioned above, we restrict our consideration to the (time-)discrete Cauchy problem. We have succeeded in applying the results of this paper to the stability and convergence analysis of the finite element method for linear and semilinear heat equations [17].

The plan of the rest of this paper is as follows. Section 2 is devoted to preliminaries on maximal regularity. In Subsection 2.1, we introduce the notion of R-boundedness. It plays an important role in operator-valued Fourier multiplier theorems, in both the continuous and discrete cases. Subsections 2.2 and 2.3 are devoted to continuous and discrete maximal regularity. In Subsection 2.3, we reduce the problem of discrete maximal regularity to R-boundedness of certain sets of operators. Our main result appears in Section 3. Its significance is in the dependence of the constant. We also demonstrate that the converse of the main theorem holds.

We conclude this paper by dealing with some additional topics in Section 4, where we focus on the backward Euler method. In this case, we can also show other analogous properties. This section consists of two parts. First, we consider the characterization of discrete maximal regularity for unbounded operators (Subsection 4.1). This corresponds to the result given by Blunck, which deals with bounded operators. The key point is Yosida approximation. Next, we obtain an a priori estimate for non-zero initial values (Subsection 4.2). The results in this subsection are important for numerical analysis of nonlinear equations. However, since the arguments are essentially the same as in [4, Chapter 2], which deals with the same problem in the time interval $(0, 1)$, we omit the detailed proofs.

2. Preliminaries

2.1. R-boundedness. In this subsection, we introduce the definition of R-boundedness and its basic property to be used later. For standard facts on R-boundedness, we refer to [19, Section 2]. R-boundedness is a fundamental concept in this paper since it plays a crucial role in Weis's operator-valued Fourier multiplier theorem on UMD spaces [27, Theorem 3.4], as well as in its discrete version [5, Theorem 1.3].

DEFINITION 2.1. A Banach space X is an *UMD space* if for some $p \in (1, \infty)$ and $C > 0$, we have

$$\Big\| u_0 + \sum_{j=1}^{n} \varepsilon_j (u_j - u_{j-1}) \Big\|_{L^p(\Omega, \mathcal{F}, P; X)} \le C \|u_n\|_{L^p(\Omega, \mathcal{F}, P; X)} \tag{2.1}$$

for all $n \in \mathbb{N}$, $\varepsilon_j \in \{\pm 1\}$, and X-valued martingales (u_j) on a probability space $(\Omega, \mathcal{F}, P)$.

We refer to [7, 8, 6] for characterizations of UMD spaces by ζ-convexity and the Hilbert transform.

Let X and Y be Banach spaces and let $\mathcal{L}(X,Y)$ denote the space of bounded operators from X to Y. Let $\{r_j\}_{j\in\mathbb{N}}$ be a sequence of independent and symmetric $\{\pm 1\}$-valued random variables on $[0,1]$, for example, the Rademacher functions $r_j(t) = \mathrm{sign}[\sin(2^{j+1}\pi t)]$.

DEFINITION 2.2. A set $\mathcal{T} \subset \mathcal{L}(X,Y)$ is said to be *R-bounded* if there exists a constant $C > 0$ such that

$$\int_0^1 \Big\| \sum_{j=0}^n r_j(t) T_j x_j \Big\|_Y \, dt \le C \int_0^1 \Big\| \sum_{j=0}^n r_j(t) x_j \Big\|_X \, dt \tag{2.2}$$

for all $n \in \mathbb{N}$, $x_0, \ldots, x_n \in X$, and $T_0, \ldots, T_n \in \mathcal{T}$. The infimum of the C's satisfying (2.2) is called the *R-bound* of $\mathcal{T}$ and is denoted by $R(\mathcal{T})$.

We now introduce a property of sectorial operators. The following lemma is a modification of [5, Corollary 3.5] and is obtained by the same argument as in [5]. Here $\Sigma_\delta = \{z \in \mathbb{C} \setminus \{0\} \mid |\arg z| < \delta\}$ is a sector domain in $\mathbb{C}$ for $\delta \in (0,\pi)$.

LEMMA 2.3. *Let A be a closed and densely defined linear operator on X. Assume that there exists $\delta \in (0,\pi/2)$ such that $\Sigma_{\pi/2+\delta} \subset \rho(A)$, and set $\mathcal{T}_\theta = \{\lambda R(\lambda;A) \mid \lambda \in \Sigma_\theta\}$ for $\theta \in (0, \pi/2+\delta)$. If $\mathcal{T}_{\pi/2}$ is R-bounded, then so is $\mathcal{T}_{\pi/2+\delta_0}$ for each δ_0 satisfying*

$$0 < \delta_0 < \min\left\{\delta, \arctan \frac{1}{R(\mathcal{T}_{\pi/2})}\right\}.$$

Moreover, $R(\mathcal{T}_{\pi/2+\delta_0}) \le P_1(R(\mathcal{T}_{\pi/2}))$, where

$$P_1(X) = \frac{2}{1-\alpha}\left(1 + \frac{X}{\alpha}\right) + X \tag{2.3}$$

is a polynomial of degree one with $\alpha = R(\mathcal{T}_{\pi/2}) \tan \delta_0$.

2.2. Maximal regularity. We consider the following abstract Cauchy problem in a Banach space X:

$$\begin{cases} u'(t) = Au(t) + f(t), & t > 0, \\ u(0) = 0, \end{cases} \tag{2.4}$$

where $f \colon \mathbb{R}^+ = (0,\infty) \to X$ is a given function, $u \colon \mathbb{R}^+ \to X$ is the unknown, and A is a linear operator on X with domain $D(A)$.

DEFINITION 2.4. Let $p \in (1,\infty)$. A linear operator A has *maximal L^p-regularity* with a constant C if, for every $f \in L^p(\mathbb{R}^+;X)$, (2.4) has a unique solution u with the following properties:

(1) $u(t) \in D(A)$ for almost every $t > 0$,
(2) u is strongly differentiable in X for almost every $t > 0$,
(3) there exists a constant $C > 0$, independent of f, such that

$$\|u'\|_{L^p(\mathbb{R}^+;X)} + \|Au\|_{L^p(\mathbb{R}^+;X)} \le C\|f\|_{L^p(\mathbb{R}^+;X)}. \tag{2.5}$$

In Definition 2.4, we do not require that $u \in L^p(\mathbb{R}^+;X)$. However, if $0 \in \rho(A)$, maximal L^p-regularity implies that $u \in L^p(\mathbb{R}^+;X)$, since $\|A\cdot\|$ is a norm in $D(A)$ equivalent to the graph norm. Since maximal L^p-regularity is p-independent (see e.g. [10, Theorem 4.2]), we say that A has *maximal regularity* if A has maximal L^p-regularity for some $p \in (1,\infty)$. To distinguish this from the discrete case below, we occasionally call this *continuous* maximal regularity.

It is known that an operator of maximal regularity generates a bounded analytic semigroup (see e.g. [10, Theorem 2.1]). Thus maximal regularity is stronger than analyticity of the semigroup. The Laplace operator and the Stokes operator have maximal regularity under suitable conditions, which can be applied to the analysis of quasilinear parabolic equations and the Navier–Stokes equations.

As a sufficient condition for maximal regularity, the result of Dore and Venni [11] is well-known. On the other hand, in [27, Corollary 4.4], Weis characterized maximal regularity by the R-boundedness of some sets of operators.

THEOREM 2.5 (Weis). *Let X be a UMD space and $T(t)$ a bounded analytic semigroup on X with generator A. Then the following conditions are equivalent:*

(a) *A has maximal regularity.*
(b) *$\{\lambda R(\lambda;A) \mid \lambda \in i\mathbb{R}\setminus\{0\}\}$ is R-bounded.*
(c) *$\{T(t) \mid t > 0\}$ and $\{tAT(t) \mid t > 0\}$ are R-bounded.*

REMARK 2.6. Let $p \in (1,\infty)$, and let X and A be as in Theorem 2.5. Assume that A has maximal L^p-regularity with a constant C_{MR}. Then, tracing the constants in [5, Proposition 1.4], one can observe that

$$R(\{\lambda R(\lambda;A) \mid \lambda \in i\mathbb{R}\setminus\{0\}\}) \le c_p C_{\mathrm{MR}},$$

where $c_p > 0$ is a constant depending only on p.

2.3. Discrete maximal regularity. We next discretize the notion of maximal regularity. First, we need to consider the discrete problem for (2.4). In this paper, we use the single-step finite difference method to discretize the time variable. That is, we consider the discrete Cauchy problem in X:

$$\begin{cases} \dfrac{u^{n+1}-u^n}{\tau} = Au^{n+\theta} + f^{n+\theta}, \quad n \in \mathbb{N}, \\ u^0 = 0, \end{cases} \tag{2.6}$$

where $\tau > 0$ is the time step, $\theta \in [0,1]$ is a fixed parameter, $f = (f^n)_n \in X^{\mathbb{N}}$ is a given sequence, $u = (u^n)_n \in X^{\mathbb{N}}$ is an unknown sequence, and

$$v^{n+\theta} = (1-\theta)v^n + \theta v^{n+1}$$

for $v = (v^n) \in X^{\mathbb{N}}$. Hereafter, we assume that A is bounded, since the problem (2.6) is ill-posed for general f when A is unbounded and $\theta \neq 1$.

In general, the discretization (2.6) is called the *θ-method*. It is known as the *forward Euler method* when $\theta = 0$, the *backward Euler method* when $\theta = 1$, and the *Crank–Nicolson method* when $\theta = 1/2$. Note that the solvability of (2.6) is equivalent to the invertibility of $I - \theta\tau A$, since (2.6) can be rewritten as

$$(I - \theta\tau A)u^{n+1} = (I + (1-\theta)\tau A)u^n + \tau f^{n+\theta}. \tag{2.7}$$

In particular, if (2.6) is solvable, then the solution must be unique.

For the space $X^{\mathbb{N}}$, we introduce some notation.

Definition 2.7. Let $p \in (1,\infty)$.

(1) We define the discrete L^p-norm $\|\cdot\|_{l^p_\tau(\mathbb{N};X)}$ as

$$\|v\|_{l^p_\tau(\mathbb{N};X)} = \Big(\sum_{n=0}^{\infty} \|v^n\|_X^p \tau\Big)^{1/p} \qquad \text{for } v = (v^n) \in l^p(\mathbb{N};X).$$

(2) For $v = (v^n) \in X^{\mathbb{N}}$, $\tau > 0$, and $\theta \in [0,1]$, we define the sequences $D_\tau v$, Av, and v_θ as

$$(D_\tau v)^n = \frac{v^{n+1} - v^n}{\tau}, \quad (Av)^n = Av^n, \quad (v_\theta)^n = v^{n+\theta}.$$

Now, we can define the discrete version of maximal L^p-regularity.

Definition 2.8. Suppose that $p \in (1,\infty)$ and $\theta \in [0,1]$. A linear operator A has *maximal l^p-regularity* with a constant C if, for every $\tau > 0$ small enough and $f \in l^p(\mathbb{N};X)$, problem (2.6) has a unique solution $u = (u^n) \in X^{\mathbb{N}}$ satisfying

$$\|D_\tau u\|_{l^p_\tau(\mathbb{N};X)} + \|Au_\theta\|_{l^p_\tau(\mathbb{N};X)} \leq C\|f_\theta\|_{l^p_\tau(\mathbb{N};X)}, \tag{2.8}$$

where $C > 0$ is independent of $\tau > 0$ and f. We say that A has *discrete maximal regularity* if A has maximal l^p-regularity for some $p \in (1,\infty)$.

We characterize maximal l^p-regularity by the boundedness of the Fourier multiplier. Hereafter, we write $\mathbb{T} = \{z \in \mathbb{C} \mid |z| = 1\}$, and we assume that A is the infinitesimal generator of a bounded analytic semigroup on X, so that (2.6) is solvable. When $\mathbb{T} \setminus \{1\} \subset \rho(T_\tau)$, we set

$$M_\tau(z) = (I - \theta\tau A)^{-1}(z-1)R(z;T_\tau), \qquad z \in \mathbb{T}, \tag{2.9}$$

$$(T_{M_\tau} f)^n = [\mathcal{F}^{-1}(M_\tau \mathcal{F} f)]^n, \quad f \in c_{00}(\mathbb{Z};X),\ n \in \mathbb{Z}, \tag{2.10}$$

where

$$T_\tau = (I - \theta\tau A)^{-1}(I + (1-\theta)\tau A). \tag{2.11}$$

Here and hereafter, $\mathcal{F}$ and $\mathcal{F}^{-1}$ are the Fourier transforms on $\mathbb{Z}$ and on $\mathbb{T}$, respectively, and $c_{00}(\mathbb{Z};X) \subset X^{\mathbb{Z}}$ is the space of X-valued sequences with compact supports.

LEMMA 2.9. *Let $p \in (1,\infty)$, and let $A \in \mathcal{L}(X)$ be the infinitesimal generator of a bounded analytic semigroup on X. Suppose that $\mathbb{T} \setminus \{1\} \subset \rho(T_\tau)$. Then the following assertions are equivalent:*

(a) *A has maximal l^p-regularity.*
(b) *T_{M_τ} can be extended to a bounded operator on $l^p(\mathbb{Z};X)$, and its operator norm is bounded by a constant independent of $\tau > 0$.*

Proof. From (2.7), $D_\tau u$ is written as

$$(D_\tau u)^n = (I-\theta\tau A)^{-1}\Big[(T_\tau - I)\sum_{j=0}^{n-1} T_\tau^{n-j-1} f^{j+\theta} + f^{n+\theta}\Big]$$

for $n \in \mathbb{N}$. Therefore, by a basic computation, we can obtain

$$(D_\tau u)^n = (T_{M_\tau}\tilde{f}_\theta)^n, \quad n \in \mathbb{N},$$

where $\tilde{f}_\theta$ is the zero extension of f_θ to $\mathbb{Z}$. Hence, we obtain the desired equivalence. ∎

Now, we check when $\mathbb{T} \setminus \{1\} \subset \rho(T_\tau)$. We define

$$g_{\theta,\tau}(\zeta) = \frac{\zeta - 1}{\theta\tau\zeta + (1-\theta)\tau}, \quad \zeta \in \mathbb{C}. \tag{2.12}$$

Assume that $g_{\theta,\tau}(\mathbb{T}\setminus\{1\}) \subset \rho(A)$, and let $\lambda \in \mathbb{T}\setminus\{1\}$. If $\theta = 1/2$ and $\lambda = -1$, then $\lambda I - T_\tau = -4(2-\tau A)^{-1}$ is invertible. Otherwise, since $\theta\tau\lambda + (1-\theta)\tau \neq 0$, we have

$$\lambda I - T_\tau = [\theta\tau\lambda + (1-\theta)\tau](g_{\theta,\tau}(\lambda)I - A)(I-\theta\tau A)^{-1}, \tag{2.13}$$

which implies that $\lambda \in \rho(T_\tau)$. It remains to determine the set $g_{\theta,\tau}(\mathbb{T} \setminus \{1\})$. By a simple calculation,

$$\begin{cases} g_{\theta,\tau}(\mathbb{T}\setminus\{1\}) = \begin{cases} C\Big(\dfrac{-1}{(1-2\theta)\tau}; \dfrac{1}{(1-2\theta)\tau}\Big) \setminus \{0\}, & 0 \le \theta < 1/2, \\ C\Big(\dfrac{1}{(2\theta-1)\tau}; \dfrac{1}{(2\theta-1)\tau}\Big) \setminus \{0\}, & 1/2 < \theta \le 1, \end{cases} \\ g_{\theta,\tau}(\mathbb{T}\setminus\{\pm 1\}) = i\mathbb{R}\setminus\{0\}, \quad \theta = 1/2, \end{cases} \tag{2.14}$$

where $C(a;r) = \{z \in \mathbb{C} \mid |z-a| = r\}$ for $a \in \mathbb{C}$ and $r > 0$. Since A generates a bounded analytic semigroup, we have $g_{\theta,\tau}(\mathbb{T}\setminus\{1\}) \subset \overline{\Sigma_{\pi/2}} \setminus \{0\} \subset \rho(A)$ when $1/2 < \theta \le 1$ ($\overline{\Sigma_{\pi/2}}$ is the closure in $\mathbb{C}$). Therefore, we need no condition for $\mathbb{T} \setminus \{1\} \subset \rho(T_\tau)$ in this case. Much the same is true when $\theta = 1/2$. However, an additional condition is necessary when $0 \le \theta < 1/2$. We then give the following condition (S) (cf. Fig. 1).

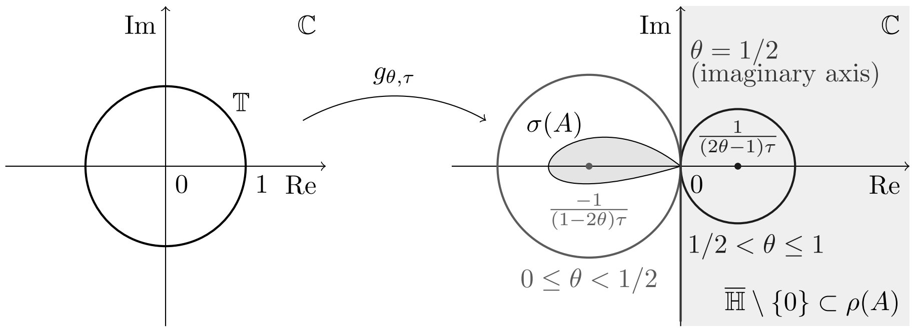

Fig. 1. The set $g_{\theta,\tau}(\mathbb{T} \setminus \{1\})$ and condition (S)

(S) The operator A satisfies

$$\sigma(A) \subset \mathbb{D}\left(\frac{-1}{(1-2\theta)\tau}; \frac{1}{(1-2\theta)\tau}\right) \cup \{0\}.$$

Here, $\mathbb{D}(a; r) = \{z \in \mathbb{C} \mid |z - a| < r\}$ is an open disk for $a \in \mathbb{C}$ and $r > 0$. Note that (S) is satisfied if τ is sufficiently small, since the spectrum of $A \in \mathcal{L}(X)$ is a bounded set. Now, we have a sufficient condition for $\mathbb{T} \setminus \{1\} \subset \rho(T_\tau)$.

LEMMA 2.10. *Let $\theta \in [0, 1]$. Assume that $A \in \mathcal{L}(X)$ is the infinitesimal generator of a bounded analytic semigroup on X. Suppose that condition* (S) *is fulfilled when $0 \le \theta < 1/2$. Then $\mathbb{T} \setminus \{1\} \subset \rho(T_\tau)$. In particular, the two assertions in the previous lemma are equivalent.*

From the viewpoint of Lemma 2.9, we need to examine the boundedness of the multiplier operator. For this purpose, Blunck proved the following multiplier theorem [5, Theorem 1.3]. This is the discrete version of Weis's operator-valued Fourier multiplier theorem [27, Theorem 3.4]. See also [21] for the scalar-valued discrete Fourier multiplier theorem.

THEOREM 2.11 (Blunck). *Let X be a UMD space, $J = (-\pi, \pi) \setminus \{0\}$, and $M : J \to \mathcal{L}(X)$. Set*

$$T_M f = \mathcal{F}^{-1}[\tilde{M}\mathcal{F}f], \quad f \in c_{00}(\mathbb{Z}; X), \tag{2.15}$$

where $\tilde{M}(z) = M(\arg z)$ for $z \in \mathbb{T} \setminus \{1\}$. Assume that M is differentiable and the set

$$\mathcal{T} = \{M(t) \mid t \in J\} \cup \{(e^{it} - 1)(e^{it} + 1)M'(t) \mid t \in J\} \tag{2.16}$$

is R-bounded. Then T_M can be extended to a bounded operator on $l^p(\mathbb{Z};X)$ for all $p \in (1,\infty)$. Moreover,

$$\|T_M\|_{\mathcal{L}(l^p(\mathbb{Z};X))} \leq C_{\mathrm{mul}} R(\mathcal{T}), \tag{2.17}$$

where $C_{\mathrm{mul}} > 0$ depends only on p and X.

In view of numerical analysis, it is troublesome that the constant C_{mul} depends on the Banach space X, since X is supposed to be the finite element space, which depends on the discretization parameter. Tracing the constants in the proof of [5, Theorem 1.3], we find that the dependence on X is caused by the constant of the UMD property (2.1). Let $C_{\mathrm{UMD}}(p,X)$ be the infimum of the constant C in (2.1). Then it is obvious that $Y \subset X \Rightarrow C_{\mathrm{UMD}}(p,Y) \leq C_{\mathrm{UMD}}(p,X)$. Therefore, we can state the following.

Corollary 2.12. *Let X be a UMD space, and $X_0 \subset X$ a closed subspace. Furthermore, let $J = (-\pi,\pi) \setminus \{0\}$ and $M \colon J \to \mathcal{L}(X_0)$. Define T_M as in* (2.15) *for $f \in c_{00}(\mathbb{Z};X_0)$, and $\mathcal{T}$ as in* (2.16)*. Assume that M is differentiable and $\mathcal{T}$ is R-bounded. Then T_M can be extended to a bounded operator on $l^p(\mathbb{Z};X_0)$ for all $p \in (1,\infty)$. Moreover,*

$$\|T_M\|_{\mathcal{L}(l^p(\mathbb{Z};X_0))} \leq C R(\mathcal{T}), \tag{2.18}$$

where $C > 0$ depends only on p and X, but is independent of X_0.

3. Main result. Our main result is based on the characterization given in Lemma 2.9, with condition (S) assumed when $\theta \in [0,1/2)$. However, to obtain a uniform estimate for τ, condition (S) is not sufficient. Therefore, we consider the stronger condition given below (cf. Fig. 2).

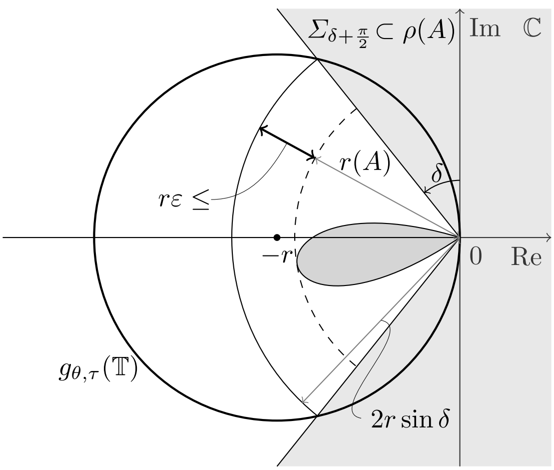

Fig. 2. Condition $(\mathrm{NR})_{\delta,\varepsilon}$. Here, we set $r = 1/[(1-2\theta)\tau]$.

$(\mathrm{NR})_{\delta,\varepsilon}$ The following two conditions are fulfilled:

(NR1) $S(A) \subset \mathbb{C} \setminus \Sigma_{\delta+\pi/2}$,
(NR2) $(1-2\theta)\tau r(A) + \varepsilon \leq 2\sin\delta$.

Here, the set

$$S(A) = \{\langle x^*, Ax\rangle \mid x \in X,\ x^* \in X^*,\ \|x\| = \|x^*\| = \langle x^*, x\rangle = 1\}$$

is the numerical range of A, and $r(A) = \sup_{z\in S(A)} |z|$ (*not* the spectral radius of A).

REMARK 3.1. (1) One can observe that condition $(\mathrm{NR})_{\delta,\varepsilon}$ is stronger than (S), since $\sigma(A) \subset \overline{S(A)}$, where the overline indicates closure in $\mathbb{C}$. See, for example, [22, Theorem 3.9 in Chapter 1].

(2) If A is the infinitesimal generator of a bounded analytic semigroup on X, then condition (NR1) is fulfilled for some $\delta \in (0, \pi/2)$. Therefore, if one wants to achieve $(\mathrm{NR})_{\delta,\varepsilon}$, it suffices to fix ε small enough, and consider τ satisfying

$$\tau \leq \frac{2\sin\delta - \varepsilon}{(1-2\theta)r(A)}.$$

Now, we are in a position to state our main theorem.

THEOREM 3.2 (Discrete maximal regularity for the θ-method). *Let X be a UMD space, and let $p \in (1,\infty)$ and $\theta \in [0,1]$. Assume that $A \in \mathcal{L}(X)$ has maximal L^p-regularity with a constant C_A. When $\theta \in [0, 1/2)$, suppose that A satisfies condition $(\mathrm{NR})_{\delta,\varepsilon}$ for some $\delta \in (0, \pi/2)$ and $\varepsilon > 0$. Then A has maximal l^p-regularity with a constant C_{DMR} which depends only on p, θ, C_A, δ, ε, and the Banach space X. Moreover,*

$$C_{\mathrm{DMR}} = C_{\mathrm{mul}} C_0, \tag{3.1}$$

where C_{mul} is the constant appearing in (2.17), *and $C_0 > 0$ depends only on p, θ, C_A, δ, ε, but is independent of τ, X and the operator A.*

Proof. The proof will be divided into four steps.

STEP 1. Let M_τ, T_{M_τ}, and T_τ be defined by (2.9)–(2.11), respectively. In view of Lemma 2.9, it suffices to show that T_{M_τ} is bounded in $l^p(\mathbb{Z}; X)$. Let $J = (-\pi,\pi) \setminus \{0\}$, and define $\check{M}_\tau \colon J \to \mathcal{L}(X)$ as

$$\check{M}_\tau(t) = M_\tau(e^{it}) = (I - \theta\tau A)^{-1}(e^{it}-1)R(e^{it}; T_\tau), \quad t \in J.$$

By Theorem 2.11, we only need to show that the set

$$\mathcal{T}_\tau = \{\check{M}_\tau(t) \mid t \in J\} \cup \{(e^{it}-1)(e^{it}+1)\check{M}'_\tau(t) \mid t \in J\}$$

is R-bounded uniformly in $\tau > 0$. We will calculate the sets

$$\mathcal{T}_1 = \{\check{M}_\tau(t) \mid t \in J\}, \quad \mathcal{T}_2 = \{(e^{it}-1)(e^{it}+1)\check{M}'_\tau(t) \mid t \in J\}.$$

Let $\lambda = e^{it}$ for $t \in J$. Then, from (2.13), we have

$$R(\lambda; T_\tau) = \frac{1}{\theta\tau\lambda + (1-\theta)\tau}(I - \theta\tau A)R(g_{\theta,\tau}(\lambda); A),$$

where $g_{\theta,\tau}$ is defined in (2.12). Therefore, setting $\mu = g_{\theta,\tau}(\lambda)$, we have

$$\check{M}_\tau(t) = \mu R(\mu; A), \tag{3.2}$$

which implies that

$$\mathcal{T}_1 = \{\mu R(\mu; A) \mid \mu \in g_{\theta,\tau}(\mathbb{T} \setminus \{1\})\}. \tag{3.3}$$

Let us now calculate $\mathcal{T}_2$. Since

$$\check{M}_\tau'(t) = (I - \theta\tau A)^{-1} i e^{it} R(e^{it}; T_\tau)[I - (e^{it} - 1)R(e^{it}; T_\tau)],$$

we have

$$(e^{it} - 1)(e^{it} + 1)\check{M}_\tau'(t) = i e^{it}(e^{it} + 1)\check{M}_\tau(t)[I - (I - \theta\tau A)\check{M}_\tau(t)].$$

Moreover, by (3.2),

$$I - (I - \theta\tau A)\check{M}_\tau(t) = I - (I - \theta\tau A)\mu R(\mu; A) = (1 - \theta\tau\mu)[I - \mu R(\mu; A)],$$

where $\mu = g_{\theta,\tau}(e^{it})$. Therefore,

$$(e^{it} - 1)(e^{it} + 1)\check{M}_\tau'(t) = i e^{it}(e^{it} + 1)(1 - \theta\tau\mu)\check{M}_\tau(t)[I - \check{M}_\tau(t)].$$

Noting that

$$(z + 1)(1 - \theta\tau g_{\theta,\tau}(z)) \in C(1; 1)$$

for $z \in \mathbb{T}$, regardless of θ and τ, we can obtain

$$R(\mathcal{T}_2) \le 4R(\mathcal{T}_1)(1 + R(\mathcal{T}_1)),$$

and thus

$$R(\mathcal{T}_\tau) \le 5R(\mathcal{T}_1)(1 + R(\mathcal{T}_1)), \tag{3.4}$$

provided that $\mathcal{T}_1$ is R-bounded. Here, we have used the standard facts $R(\mathcal{S} \cup \mathcal{T}) \le R(\mathcal{S}) + R(\mathcal{T})$ and $R(\mathcal{S}\mathcal{T}) \le R(\mathcal{S})R(\mathcal{T})$ (cf. [19, Fact 2.8]). Hence, it suffices to prove the R-boundedness of $\mathcal{T}_1$. Set

$$\mathcal{T}_0 = \{isR(is; A) \mid s \in \mathbb{R} \setminus \{0\}\},$$

which is R-bounded with

$$R(\mathcal{T}_0) \le c_p C_A \tag{3.5}$$

by the maximal L^p-regularity of A, where $c_p > 0$ depends only on p (Remark 2.6).

Step 2. We prove the assertion when $1/2 \le \theta \le 1$. In this case, from (3.3), (2.14), Theorem 2.5, and the convexity property of R-boundedness [19, Example 2.15], $\mathcal{T}_1$ is R-bounded with

$$R(\mathcal{T}_1) \le R(\mathcal{T}_0), \tag{3.6}$$

which is a uniform estimate in τ. Thus A has maximal l^p-regularity and

$$C_{\mathrm{DMR}} \leq C_{\mathrm{mul}} R(\mathcal{T}_\tau) \leq C_{\mathrm{mul}} \cdot 5c_p C_A(1 + c_p C_A),$$

from (3.4)–(3.6), which implies (3.1) with

$$C_0 = 5c_p C_A(1 + c_p C_A).$$

STEP 3. We assume that $0 \leq \theta < 1/2$. In this step, we show that $R(\mathcal{T}_1)$ is bounded from above uniformly in τ. This case is not as simple as Step 2. We set

$$\gamma = \gamma_{\theta,\tau} = g_{\theta,\tau}(\mathbb{T}) = C\left(\frac{-1}{(1-2\theta)\tau}; \frac{1}{(1-2\theta)\tau}\right),$$

and $\dot{\gamma} = \gamma \setminus \{0\}$. Then $\mathcal{T}_1 = \{\mu R(\mu; A) \mid \mu \in \dot{\gamma}\}$. Take $\delta_0 \in (0, \delta)$ satisfying

$$0 < \delta_0 < \arctan \frac{1}{R(\mathcal{T}_0)}.$$

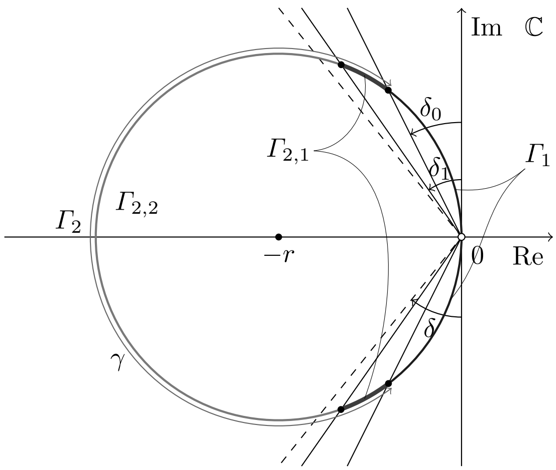

Fig. 3. The arcs in the proof of the main theorem

We will choose an appropriate δ_0 later. We decompose $\dot{\gamma}$ into two parts, Γ_1 and Γ_2 (cf. Fig. 3), as

$$\Gamma_1 = \{\mu \in \dot{\gamma} \mid |\arg \mu| < \delta_0 + \pi/2\}, \quad \Gamma_2 = \{\mu \in \dot{\gamma} \mid |\arg \mu| \geq \delta_0 + \pi/2\},$$

and we set $\mathcal{S}_j = \{\mu R(\mu; A) \mid \mu \in \Gamma_j\}$ for $j = 1, 2$. By Lemma 2.3, $\mathcal{S}_1$ is R-bounded with

$$R(\mathcal{S}_1) \leq P_1(R(\mathcal{T}_0)), \tag{3.7}$$

where P_1 is a polynomial of degree one, defined by (2.3). Note that the set $\mathcal{T}_{\pi/2}$ in Lemma 2.3 is R-bounded, and its R-bound is equal to that of $\mathcal{T}_0$ here. It remains to show that $\mathcal{S}_2$ is R-bounded.

We first prove that there exists $\eta > 0$ independent of τ such that

$$\|R(\mu; A)\| \leq (1 - 2\theta)\tau\eta \tag{3.8}$$

for $\mu \in \Gamma_2$. Take $\delta_1 \in (\delta_0, \delta)$ sufficiently close to δ so that

$$2(\sin\delta - \sin\delta_1) = \varepsilon/2. \tag{3.9}$$

We additionally decompose Γ_2 into $\Gamma_{2,1}$ and $\Gamma_{2,2}$ (cf. Fig. 3), where

$$\Gamma_{2,1} = \{\mu \in \Gamma_2 \mid |\arg\mu| < \delta_1 + \pi/2\}, \quad \Gamma_{2,2} = \Gamma_2 \setminus \Gamma_{2,1}.$$

It is well-known that

$$\|R(\mu; A)\| \le \frac{1}{\operatorname{dist}(\mu; S(A))}, \quad \mu \in \mathbb{C} \setminus \overline{S(A)},$$

where $\overline{S(A)}$ is the closure in $\mathbb{C}$ (cf. [22, Theorem 3.9 in Chapter 1]). To compute $\operatorname{dist}(\mu; S(A))$, we set $r = 1/[(1-2\theta)\tau]$, which is the radius of the circle γ. Assume that $\mu \in \Gamma_{2,1}$. Then, since $\mu \in \Sigma_{\delta+\pi/2}$ and $S(A) \subset \mathbb{C} \setminus \Sigma_{\delta+\pi/2}$, by $(\mathrm{NR})_{\delta,\varepsilon}$ we have

$$\operatorname{dist}(\mu; S(A)) \ge \operatorname{dist}(\mu; \partial\Sigma_{\delta+\pi/2}) = |\mu| \sin(\delta + \pi/2 - |\arg\mu|).$$

Noting that $|\mu| = 2r\sin(|\arg\mu| - \pi/2)$, we have

$$\begin{aligned} |\mu| \sin(\delta + \pi/2 - |\arg\mu|) &= 2r\sin(|\arg\mu| - \pi/2)\sin(\delta + \pi/2 - |\arg\mu|) \\ &\ge 2r\sin\delta_0 \sin(\delta - \delta_1). \end{aligned}$$

Therefore,

$$\|R(\mu; A)\| \le \frac{1}{2r\sin\delta_0\sin(\delta-\delta_1)} = \frac{(1-2\theta)\tau}{2\sin\delta_0\sin(\delta-\delta_1)}, \quad \mu \in \Gamma_{2,1}. \tag{3.10}$$

Next, we assume that $\mu \in \Gamma_{2,2}$. In this case,

$$\operatorname{dist}(\mu; S(A)) \ge |\mu| - r(A) \ge 2r\sin\delta_1 - r(A).$$

By the condition $(\mathrm{NR})_{\delta,\varepsilon}$ and (3.9),

$$2r\sin\delta_1 - r(A) = [2r\sin\delta - r(A)] - 2r(\sin\delta - \sin\delta_1) \ge \varepsilon r - \varepsilon r/2 = \varepsilon r/2,$$

which implies

$$\|R(\mu; A)\| \le \frac{(1-2\theta)\tau}{\varepsilon/2}, \quad \mu \in \Gamma_{2,2}. \tag{3.11}$$

From (3.10) and (3.11), we obtain (3.8) with

$$\eta = \max\left\{\frac{1}{2\sin\delta_0\sin(\delta-\delta_1)}, \frac{2}{\varepsilon}\right\}.$$

We are now ready to demonstrate the R-boundedness of $\mathcal{S}_2$. Fix $\mu_0 \in \Gamma_2$. Then $R(\mu_0; A)$ can be expanded in a Taylor series as

$$R(\mu; A) = \sum_{n=0}^{\infty} (\mu_0 - \mu)^n R(\mu_0; A)^{n+1},$$

provided that $\mu \in \rho(A)$ and $|\mu - \mu_0| < \|R(\mu_0; A)\|^{-1}$. Set

$$r_0 = \frac{1}{(1-2\theta)\tau\eta} = \frac{r}{\eta}, \quad \mathcal{S}(\mu) = \{R(\zeta; A) \mid \zeta \in \Gamma_2,\ |\zeta - \mu| < r_0/4\},$$

for $\mu \in \Gamma_2$. Then, noting that $r_0 \leq \|R(\mu_0; A)\|^{-1}$ by (3.8), we have

$$\begin{aligned} R(\mathcal{S}(\mu_0)) &= R\Big(\Big\{ \sum_{n=0}^{\infty} (\mu_0 - \mu)^n R(\mu_0; A)^{n+1} \;\Big|\; \mu \in \Gamma_2,\, |\mu - \mu_0| < r_0/4 \Big\}\Big) \\ &\leq \sum_{n=0}^{\infty} R(\{(\mu_0 - \mu)^n R(\mu_0; A)^{n+1} \mid \mu \in \Gamma_2,\, |\mu - \mu_0| < r_0/4\}) \\ &\leq \sum_{n=0}^{\infty} \left(\frac{1}{2} r_0\right)^n \|R(\mu_0; A)^{n+1}\| \leq \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n \|R(\mu_0; A)\| \leq \frac{2}{r_0}. \end{aligned}$$

Here, we have applied [27, Lemma 2.4] in the second step. That is, $\mathcal{S}(\mu)$ is R-bounded, and

$$R(\mathcal{S}(\mu)) \leq 2(1 - 2\theta)\tau\eta \tag{3.12}$$

for every $\mu \in \Gamma_2$. Now, we set $B(\mu) = \{\zeta \in \gamma \mid |\zeta - \mu| < r_0/4\}$ for $\mu \in \gamma$, so that

$$\gamma = \bigcup_{\mu \in \gamma} B(\mu).$$

Since γ is compact, there exist $N_0 \in \mathbb{N}$ and $\mu_0, \ldots, \mu_{N_0} \in \gamma$ satisfying

$$\gamma = \bigcup_{j=0}^{N_0} B(\mu_j).$$

Moreover, since the ratio of the radii of γ and $B(\mu)$ is independent of τ, we can take N_0 independent of τ. Thus,

$$\{R(\mu; A) \mid \mu \in \Gamma_2\} = \bigcup_{\substack{0 \leq j \leq N_0 \\ \mu_j \in \Gamma_2}} \mathcal{S}(\mu_j),$$

which implies that $\{R(\mu; A) \mid \mu \in \Gamma_2\}$ is R-bounded, and

$$R(\{R(\mu; A) \mid \mu \in \Gamma_2\}) \leq \sum_{\substack{0 \leq j \leq N_0 \\ \mu_j \in \Gamma_2}} R(\mathcal{S}(\mu_j)) \leq 2(N_0 + 1)(1 - 2\theta)\tau\eta$$

by (3.12). Noting that $|\mu| \leq 2/[(1 - 2\theta)\tau]$ for $\mu \in \gamma$, we can obtain the R-boundedness of $\mathcal{S}_2$ with the uniform bound

$$R(\mathcal{S}_2) \leq 8(N_0 + 1)\eta, \tag{3.13}$$

which implies the uniform R-boundedness of $\mathcal{T}_1$.

STEP 4. Finally, we show (3.1) for $\theta \in [0, 1/2)$. We obtain

$$R(\mathcal{T}_1) \leq R(\mathcal{S}_1) + R(\mathcal{S}_2) \leq P_1(R(\mathcal{T}_0)) + 8(N_0 + 1)\eta \tag{3.14}$$

from (3.7) and (3.13), where $P_1(X)$ is defined by (2.3) with $\alpha = R(\mathcal{T}_0) \tan \delta_0$. By the definition of N_0, it can be seen that

$$N_0 + 1 \leq \frac{2\pi}{1/(4\eta)}. \tag{3.15}$$

Thus, we only need to estimate η. By simple computations, one can obtain

$$\delta - \delta_1 \geq \sin\delta - \sin\delta_1 = \varepsilon/4$$

and

$$\sin(\delta - \delta_1) \geq \sin\frac{\varepsilon}{4} \geq \frac{\varepsilon}{2\pi}.$$

Therefore,

$$\eta \leq \frac{\pi}{\varepsilon}\max\left\{\frac{2}{\pi}, \frac{1}{\sin\delta_0}\right\} = \frac{\pi}{\varepsilon}\frac{1}{\sin\delta_0}. \tag{3.16}$$

Now, taking

$$\delta_0 = \min\left\{\frac{\delta}{2}, \arctan\frac{1}{2R(\mathcal{T}_0)}\right\},$$

we obtain

$$\alpha = \min\left\{R(\mathcal{T}_0)\tan\frac{\delta}{2}, \frac{1}{2}\right\},$$

which yields

$$P_1(R(\mathcal{T}_0)) \leq 4\left(1 + \frac{1}{\tan\frac{\delta}{2}}\right) + R(\mathcal{T}_0).$$

Equations (3.14)–(3.16) imply

$$R(\mathcal{T}_1) \leq P_2(R(\mathcal{T}_0)), \tag{3.17}$$

where P_2 is a polynomial of degree two, and depends only on δ and ε. Hence we can establish (3.1) from (3.4), (3.5), (3.17), and Theorem 2.11. ∎

From Corollary 2.12 and Theorem 3.2, we can deduce the following assertion. This is a version of our main theorem applicable to the finite element method.

COROLLARY 3.3. *Let X be a UMD space, and $X_0 \subset X$ a closed subspace. Suppose that $p \in (1,\infty)$ and $\theta \in [0,1]$. Assume that $A \in \mathcal{L}(X_0)$ has maximal L^p-regularity with a constant C_A, and satisfies condition* $(\mathrm{NR})_{\delta,\varepsilon}$ *for some $\delta \in (0,\pi/2)$ and $\varepsilon > 0$. Then A has maximal l^p-regularity with a constant C_{DMR}, which depends only on p, θ, C_A, δ, ε, and X, but is independent of τ, X_0 and the operator A.*

In view of this corollary, we can assume that X is a Lebesgue space and X_0 is a finite element space.

Roughly speaking, Theorem 3.2 says that continuous maximal regularity implies discrete maximal regularity. This is also true in the opposite direction, which is a generalization of [5, Theorem 1.1] in a sense. To show this assertion, we apply Blunck's result. We refer the reader to [5, Proposition 1.4] for the proof. Although Blunck did not mention the dependence of C below, one can obtain it by tracing the proof carefully.

Proposition 3.4. *Let X be a Banach space and $M \in L^\infty(\mathbb{T};\mathcal{L}(X))$. Suppose that the operator T_M defined by* (2.10) *can be extended to a bounded operator on $l^p(\mathbb{Z};X)$ for some $p \in (1,\infty)$. Then the set*

$$\mathcal{T}_M = \{M(z) \mid z \textit{ is a Lebesgue point of } M\}$$

is R-bounded with

$$R(\mathcal{T}_M) \le C\|T_M\|_{\mathcal{L}(l^p(\mathbb{Z};X))},$$

where $C > 0$ depends only on p. Here, we denote the extension of T_M by the same symbol.

Theorem 3.5. *Let X be a UMD space, $p \in (1,\infty)$ and $\theta \in [0,1]$. Assume that $A \in \mathcal{L}(X)$ has maximal l^p-regularity with a constant C'_A, and condition* (S) *is fulfilled when $0 \le \theta < 1/2$. Then A has maximal L^p-regularity with a constant C_{MR} depending only on p, C'_A and X.*

Proof. Since A has maximal l^p-regularity, the operator T_{M_τ} defined by (2.10) is bounded in $l^p(\mathbb{Z};X)$ uniformly in $\tau > 0$. Combining this with Proposition 3.4, we obtain

$$R(\mathcal{T}_{M_\tau}) \le c_p C'_A, \quad \forall \tau > 0,$$

where $c_p > 0$ depends only on p and

$$\mathcal{T}_{M_\tau} = \{M_\tau(\lambda) \mid \lambda \in \mathbb{T} \setminus \{1\}\}$$

and M_τ is defined by (2.9).

Now, we show that the set $\mathcal{T}_0 = \{\mu R(\mu;A) \mid \mu \in i\mathbb{R} \setminus \{0\}\}$ is R-bounded. Recall that (3.3) and (2.14) hold. Therefore, no further argument is needed when $\theta = 1/2$. Now assume that $1/2 < \theta \le 1$. Set $h_{\theta,\tau}(\zeta) = (1 - e^{(2\theta-1)\tau z})/[(2\theta-1)\tau]$ for $\zeta \in \mathbb{C}$. Then it is easy to see that $h_{\theta,\tau}(\mu) \in g_{\theta,\tau}(\mathbb{T} \setminus \{1\})$ for $\mu \in i\mathbb{R} \setminus \{0\}$ and $h_{\theta,\tau}(\mu) \to \mu$ as $\tau \downarrow 0$. Thus, for $n \in \mathbb{N}$, $x_0,\ldots,x_n \in X$, and $\mu_0,\ldots,\mu_n \in i\mathbb{R} \setminus \{0\}$, we obtain

$$\begin{aligned}&\int_0^1 \Big\| \sum_{j=0}^n r_j(t)\mu_j R(\mu_j;A)x_j \Big\|\,dt \\ &\quad = \lim_{\tau\downarrow 0}\int_0^1 \Big\| \sum_{j=0}^n r_j(t) h_{\theta,\tau}(\mu_j) R(h_{\theta,\tau}(\mu_j);A)x_j\Big\|\,dt \le c_p C'_A \int_0^1 \Big\|\sum_{j=0}^n r_j(t)x_j\Big\|\,dt,\end{aligned}$$

which implies that $\mathcal{T}_0$ is R-bounded. Here, we have applied Lebesgue's convergence theorem in the first equality. The dependence of C_{MR} on X comes from the continuous version of the vector-valued Fourier multiplier theorem [27, Theorem 3.4]. The proof when $0 \le \theta < 1/2$ is almost the same. ∎

Recall that maximal L^p-regularity is independent of $p \in (1,\infty)$. That is, if an operator A has maximal L^{p_0}-regularity for some $p_0 \in (1,\infty)$, then it has maximal L^p-regularity for all $p \in (1, \infty)$. Combining this with Theorems 3.2 and 3.5, we can derive p-independence for maximal l^p-regularity.

COROLLARY 3.6. *Let X be a UMD space, $A \in \mathcal{L}(X)$ the infinitesimal generator of a bounded analytic semigroup on X, and $\theta \in [0, 1]$. Assume that condition $(\mathrm{NR})_{\delta,\varepsilon}$ is fulfilled when $0 \leq \theta < 1/2$, and A has maximal l^{p_0}-regularity for some $p_0 \in (1, \infty)$. Then A has maximal l^p-regularity for all $p \in (1, \infty)$.*

4. The backward Euler method. In the case of the backward Euler method, we can establish several more properties analogous to those investigated in the previous section. Moreover, we can remove a restriction on operators. In our main theorem we have assumed that the operator A is bounded, since the Cauchy problem (2.6) is ill-posed in general. However, (2.6) is well-posed for unbounded operators when $\theta = 1$, whenever $I - \tau A$ is invertible. Therefore, we need not assume that A is bounded.

This section concerns two independent topics. We first characterize discrete maximal regularity for unbounded operators. The result is, in a sense, an extension of Blunck's characterization of discrete maximal regularity for power-bounded operators, and includes the result of [3]. The next topic is the derivation of an a priori estimate for non-zero initial values. In the continuous case, it is well-known that an a priori estimate (2.5) is valid for non-zero initial values with some modifications. With this in mind, we establish an estimate similar to (2.8) with appropriate initial values. We remark that our results on this topic are minor modifications of results in [4, Chapter 2].

4.1. Characterization of discrete maximal regularity. In [5], Blunck considered discrete maximal regularity for the forward Euler method, and characterized it as continuous maximal regularity. However, his proof is valid only in the case where the operator A is bounded, as long as the forward Euler method is considered.

Our aim is to characterize discrete maximal regularity for unbounded operators. Therefore, we consider the backward Euler method:

$$\begin{cases} \dfrac{u^{n+1} - u^n}{\tau} = Au^{n+1} + f^{n+1}, & n \in \mathbb{N}, \\ u^0 = 0, \end{cases}$$

so that the iteration operator T_τ in (2.11) is just the a resolvent of A, i.e.,

$$T_\tau = (I - \tau A)^{-1} = \tau^{-1} R(\tau^{-1}; A)$$

if the operator $I - \tau A$ is invertible. This scheme was considered by Ashyralyev et al. [3] and it was shown that continuous maximal regularity implies discrete maximal regularity. However, the converse was not considered.

Now, we are in a position to characterize discrete maximal regularity in similar terms to Blunck. The following theorem corresponds to Blunck's [5, Theorem 1.1]. Note that $I - \tau A$ is invertible for each $\tau > 0$ if A generates a bounded semigroup.

THEOREM 4.1. *Let X be a UMD space, and let A be a linear operator on X that generates a bounded analytic semigroup $T(t)$ on X. Set $T_\tau = (I - \tau A)^{-1}$ for $\tau > 0$. Then the following statements are equivalent:*

(a) *A has discrete maximal regularity for $\theta = 1$.*
(b) *$\{(\lambda - 1)T_\tau R(\lambda; T_\tau) \mid \lambda \in \mathbb{T} \setminus \{1\}\}$ is R-bounded uniformly for $\tau > 0$.*
(c) *$\{T_\tau^n,\ n(T_\tau - I)T_\tau^n \mid n \in \mathbb{N}\}$ is R-bounded uniformly for $\tau > 0$.*
(d) *A has continuous maximal regularity.*
(e) *$\{\mu R(\mu; A) \mid \mu \in i\mathbb{R} \setminus \{0\}\}$ is R-bounded.*
(f) *$\{T(t),\ tAT(t) \mid t > 0\}$ is R-bounded.*

Proof. The equivalences (d)⇔(e)⇔(f) are due to Weis (Theorem 2.5). The implications (e)⇒(b)⇒(a) are due to Ashyralyev et al. [3], and (a)⇒(b) ⇒(e) are shown in Theorem 3.5. Hence, it suffices to show that (f)⇒(c) and (c)⇒(f).

(f)⇒(c). It suffices to show that the set $\{T_\tau^n,\ n(T_\tau - I)T_\tau^n \mid n \in \mathbb{N}, n \geq 1\}$ is R-bounded uniformly in $\tau > 0$. It is well-known that

$$R(\lambda; A)^n = \frac{1}{(n-1)!} \int_0^\infty t^{n-1} e^{-\lambda t} T(t)\, dt$$

in $\mathcal{L}(X)$ for $\lambda \in \rho(A)$ with $\operatorname{Re} \lambda > 0$ and $n \geq 1$. Therefore,

$$T_\tau^n = \frac{1}{\tau^n (n-1)!} \int_0^\infty t^{n-1} e^{-t/\tau} T(t)\, dt$$

and, noting that $T_\tau - I = \tau A T_\tau$,

$$n(T_\tau - I)T_\tau^n = \frac{1}{\tau^n (n-1)!} \int_0^\infty t^{n-1} e^{-t/\tau} tAT(t)\, dt$$

for $\tau > 0$ and $n \geq 1$. These equations imply (c), by the formula $\int_0^\infty t^n e^{-\alpha t}\, dt = n!\alpha^{-n-1}$ for $\alpha > 0$ and $n \in \mathbb{N}$, and by the convexity property of R-boundedness [19, Theorem 2.13].

(c)⇒(f). Set $\mathcal{S}_\tau = \{T_\tau^n,\ n(T_\tau - I)T_\tau^n \mid n \in \mathbb{N}\}$, and assume that there exists $C > 0$ independent of $\tau > 0$ such that $R(\mathcal{S}_\tau) \leq C$ for each τ. Define $A_\tau = (T_\tau - I)/\tau$, which is the Yosida approximation of A. Then, as is well-known,

$$\lim_{\tau \downarrow 0} e^{tA_\tau} x = T(t)x \quad \text{in } X \tag{4.1}$$

for every $t > 0$ and every $x \in X$. Moreover, the convergence is uniform on each bounded interval. Here, for $B \in \mathcal{L}(X)$, $e^B = \sum_{n \in \mathbb{N}} B^n/n!$ is the usual exponential of B. Now, we show that the set $\{T(t)\}_t$ is R-bounded. Since

$$e^{tA_\tau} = e^{-t/\tau}\sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{t}{\tau}\right)^n T_\tau^n \in \overline{\mathrm{CH}(\mathcal{S}_\tau)},$$

the set $\{e^{tA_\tau} \mid t > 0\}$ is R-bounded, and its R-bound does not exceed C, where $\mathrm{CH}(\cdot)$ is the convex hull and the overline indicates closure in the strong topology of $\mathcal{L}(X)$. Thus, for $N \in \mathbb{N}$, $s_j > 0$, and $x_j \in X$ $(j = 0, \ldots, N)$, we have

$$\int_0^1 \Big\|\sum_{j=0}^{N} r_j(t)T(s_j)x_j\Big\|\,dt = \lim_{\tau\downarrow 0}\int_0^1 \Big\|\sum_{j=0}^{N} r_j(t)e^{s_jA_\tau}x_j\Big\|\,dt \le C\int_0^1 \Big\|\sum_{j=0}^{N} r_j(t)x_j\Big\|\,dt,$$

which implies that $\{T(t)\}_t$ is R-bounded.

We next establish the R-boundedness of $\{tAT(t)\}_t$. Note that for every $t > 0$ and $x \in X$, one can obtain

$$\lim_{\tau\downarrow 0} tA_\tau e^{tA_\tau}x = tAT(t)x \quad \text{in } X$$

in a way similar to the proof of (4.1), and the convergence is uniform on each bounded interval. We claim that the set $\mathcal{S}'_\tau = \{(n+1)(T_\tau - I)T_\tau^n \mid n \in \mathbb{N}\}$ is R-bounded. Indeed, since

$$\mathcal{S}'_\tau = \{T_\tau - I\} \cup \{(1 + n^{-1})nT_\tau^n \mid n \in \mathbb{N},\, n \ge 1\},$$

the set $\mathcal{S}'_\tau$ is R-bounded with

$$R(\mathcal{S}'_\tau) \le \|T_\tau\| + 1 + 2R(\mathcal{S}_\tau) \le 1 + 3C.$$

Therefore, since

$$\begin{aligned} tA_\tau e^{tA_\tau} &= e^{-t/\tau}\sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{t}{\tau}\right)^{n+1}(T_\tau - I)T_\tau^n \\ &= e^{-t/\tau}\sum_{n=0}^{\infty}\frac{1}{(n+1)!}\left(\frac{t}{\tau}\right)^{n+1}(n+1)(T_\tau - I)T_\tau^n \in \overline{\mathrm{CH}(\mathcal{S}'_\tau)}, \end{aligned}$$

the set $\{tA_\tau e^{tA_\tau}\}_t$ is R-bounded, and its R-bound does not exceed $1 + 3C$. This implies the R-boundedness of $\{tAT(t)\}_t$ in the same way as above. ∎

4.2. A priori estimate with non-zero initial values. Let X be a Banach space, and let A be a linear operator on X. In the theory of nonlinear evolution equations, the choice of initial value is important. Therefore, we need to obtain an a priori estimate of maximal regularity (2.5) with non-zero initial values. It is known that the desired estimate is valid for $u(0) \in (X, D(A))_{1-1/p,p}$, which is the real interpolation space provided $0 \in \rho(A)$. The estimate is as follows:

$$\|u'\|_{L^p(\mathbb{R}^+;X)} + \|Au\|_{L^p(\mathbb{R}^+;X)} \le C(\|f\|_{L^p(\mathbb{R}^+;X)} + \|u_0\|_{1-1/p,p}), \tag{4.2}$$

where u is the solution of

$$\begin{cases} u'(t) = Au(t) + f(t), & t > 0, \\ u(0) = u_0, \end{cases}$$

with $f \in L^p(\mathbb{R}^+; X)$ and $u_0 \in (X, D(A))_{1-1/p,p}$. Here, $\|\cdot\|_{1-1/p,p}$ is the usual norm of $(X, D(A))_{1-1/p,p}$.

We present the discrete version of (4.2) in the case of the backward Euler method. This problem in a bounded interval has already been considered by Ashyralyev and Sobolevskiĭ [4, Chapter 2]. We obtain the same estimate in the case where the interval is unbounded. We consider the following problem:

$$\begin{cases} \dfrac{u^{n+1} - u^n}{\tau} = Au^{n+1} + f^{n+1}, \quad n \in \mathbb{N}, \\ u^0 = u_0 \end{cases} \tag{4.3}$$

for $f = (f^n) \in l^p(\mathbb{N}; X)$ and $u_0 \in (X, D(A))_{1-1/p,p}$, with $p \in (1, \infty)$. Recall that $v_1 = (v^{n+1})_n$ for $v = (v^n) \in X^{\mathbb{N}}$.

THEOREM 4.2. *Let X be a Banach space, and let A be a linear operator on X. Suppose that $p \in (1, \infty)$ and $\tau > 0$. Assume that $0 \in \rho(A)$ and A has discrete maximal regularity for $\theta = 1$. Then, for each $f = (f^n) \in l^p(\mathbb{N}; X)$ and $u_0 \in (X, D(A))_{1-1/p,p}$, there exists a unique solution $u = (u^n)$ of (4.3) satisfying*

$$\|D_\tau u\|_{l^p_\tau(\mathbb{N};X)} + \|Au_1\|_{l^p_\tau(\mathbb{N};X)} \le C(\|f_1\|_{l^p_\tau(\mathbb{N};X)} + \|u_0\|_{1-1/p,p}),$$

where $C > 0$ is independent of τ, f, and u_0.

This theorem can be deduced from the following embedding result. The proof is essentially the same as in [4, Theorem 3.1 in Chapter 2]. Thus we omit it. Recall that $T_\tau = (I - \tau A)^{-1}$.

LEMMA 4.3. *Let X be a Banach space, and let A be a linear operator on X. Suppose that $p \in (1, \infty)$ and $\tau > 0$. Assume that $0 \in \rho(A)$ and A generates a bounded analytic semigroup on X. Define a Banach space E^p_τ as*

$$E^p_\tau = \Big\{ x \in X \;\Big|\; \sum_{n=0}^{\infty} \|AT_\tau^{n+1} x\|_X^p < \infty \Big\}$$

with the norm

$$\|x\|_{E^p_\tau} = \|x\|_X + \Big(\sum_{n=0}^{\infty} \|AT_\tau^{n+1} x\|_X^p \Big)^{1/p}.$$

Then the embedding

$$(X, D(A))_{1-1/p,p} \hookrightarrow E^p_\tau$$

holds uniformly for $\tau > 0$ and X.

Proof of Theorem 4.2. We can establish the desired estimate by dividing the problem (4.3) into two problems

$$\begin{cases} (D_\tau v)^n = Av^{n+1} + f^{n+1}, \quad n \in \mathbb{N}, \\ v^0 = 0, \end{cases} \quad \text{and} \quad \begin{cases} (D_\tau w)^n = Aw^{n+1}, \quad n \in \mathbb{N}, \\ w^0 = u_0, \end{cases}$$

and utilizing Theorem 3.2 and Lemma 4.3. ∎

We conclude this paper by stating an applicable version of Theorem 4.2. The corollary below can be established in the same way as Corollary 3.3 since the embedding constant of Lemma 4.3 is independent of X.

Corollary 4.4. *Let X be a Banach space, $X_0 \subset X$ a closed subspace, and A a linear operator on X_0. Suppose that $p \in (1,\infty)$ and $\tau > 0$. Assume that $0 \in \rho(A)$ and A has discrete maximal regularity for $\theta = 1$. Then, for each $f = (f^n) \in l^p(\mathbb{N}; X_0)$ and $u_0 \in (X_0, D(A))_{1-1/p,p}$, there exists a unique solution $u = (u^n)$ of (4.3) satisfying*

$$\|D_\tau u\|_{l^p_\tau(\mathbb{N};X)} + \|Au_1\|_{l^p_\tau(\mathbb{N};X)} \le C(\|f_1\|_{l^p_\tau(\mathbb{N};X)} + \|u_0\|_{1-1/p,p}),$$

where $C > 0$ is independent of τ, f, u_0, and the Banach space X_0. Here, the norm of u_0 is that of $(X, D(A))_{1-1/p,p}$.

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