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Definability aspects of the Denjoy integral

by

Sean Walsh (Irvine, CA)

Abstract. The Denjoy integral is an integral that extends the Lebesgue integral and can integrate any derivative. In this paper, it is shown that the graph of the indefinite Denjoy integral $f \mapsto \int_a^x f$ is a coanalytic non-Borel relation on the product space $M[a,b] \times C[a,b]$, where $M[a,b]$ is the Polish space of real-valued measurable functions on $[a,b]$, and $C[a,b]$ is the Polish space of real-valued continuous functions on $[a,b]$. Using the same methods, it is also shown that the class of indefinite Denjoy integrals, denoted by $ACG_*[a,b]$, is a coanalytic but not Borel subclass of $C[a,b]$, thus answering a question posed by Dougherty and Kechris. Some basic model theory of the associated spaces of integrable functions is also studied. Here the main result is that, when viewed as an $\mathbb{R}[X]$-module with the indeterminate X being interpreted as the indefinite integral, the space of continuous functions on $[a,b]$ is elementarily equivalent to the Lebesgue integrable and Denjoy integrable functions on this interval, and each is stable but not superstable, and that they all have a common decidable theory when viewed as $\mathbb{Q}[X]$-modules.

Contents

1. Introduction. The Denjoy integral is an integral that extends the integrals of Riemann and Lebesgue and that can integrate any derivative.

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This paper studies the Denjoy integral from two perspectives from mathematical logic, namely that of descriptive set theory and model theory. From the perspective of descriptive set theory, the natural question to ask is: how hard is it to define the Denjoy integral when viewed as a subset of a Polish space? Recall that a Polish space is a separable topological space whose topology can be given by a complete metric, and the measure of complexity of definitions on Polish spaces is induced by the hierarchy of Borel sets: open and closed sets are regarded as minimally complex, Borel sets formed from them by the operations of countable union and intersection are regarded as more complex, and continuous images of Borel sets and their complements are regarded as yet more complex. The continuous images of Borel sets turn out to be the same as the continuous images of closed sets, and these sets are called analytic sets, and their complements are called coanalytic sets. Our results show that certain sets pertaining to the Denjoy integral are coanalytic but not Borel, and thus are comparatively complex under the measure of complexity coming from descriptive set theory.

As with the Riemann and Lebesgue integrals, the indefinite Denjoy integrals $F(x) = \int_a^x f$ of real-valued functions f on $[a, b]$ are themselves continuous, and so it is natural to view them as a subset of the Polish space of real-valued continuous functions defined on $[a, b]$. This space is denoted by $C[a, b]$, and its topological structure is taken to be induced by the supremum metric. One of our main results (Theorem 1.6 below) says that the set of indefinite Denjoy integrals is coanalytic but not Borel when viewed as a subspace of $C[a, b]$. This is important for two reasons. First, this result provides another example of a logically complex object that occurs naturally in analysis. For a survey of other such examples, see Becker [Bec92]. The second reason that this result is important is that it answers a question of Dougherty and Kechris from their earlier study [DK91] of descriptive set theory and Denjoy integration.

Prior to stating Dougherty and Kechris' question, and describing their own results, it is necessary to first present the definition of the Denjoy integral. There are many equivalent definitions, but the one which is most apt for our purposes is a generalization of the fundamental theorem for the Lebesgue integral. This theorem gives an equivalent condition for a measurable function $f : [a, b] \to \mathbb{R}$ and a continuous function $F : [a, b] \to \mathbb{R}$ with $F(a) = 0$ to be such that f is Lebesgue integrable with $F(x) = \int_a^x f$. In particular, the fundamental theorem says that this is equivalent to F being absolutely continuous and for F' to exist almost everywhere with $F' = f$ almost everywhere (Theorem 1.4 below). The Denjoy integral generalizes the Lebesgue integral via a generalization of absolute continuity. Let us then proceed by first recalling the definition of absolute continuity and then specifying Denjoy's generalization.

To this end, it will be convenient to introduce some notation, employed throughout the paper, for describing partitions and related notions. As will become clear, it will often be necessary to indicate that the edges of these partitions lie in some given closed subset of $[a,b]$. Hence, given a closed subset K of $[a,b]$, *a K-edged pre-partition $\mathcal{D}$ of $[a,b]$* is a finite non-empty collection $J_1,\ldots,J_n$ of non-overlapping closed subintervals of $[a,b]$ which have both their endpoints in K. Here, two closed intervals $J=[c,d], J'=[c',d']$ are said to be *non-overlapping* if either $d\leq c'$ or $d'\leq c$, so that sharing an endpoint is allowed, i.e. $J=[c,d]$ and $J''=[d,e]$ count as non-overlapping. A pre-partition $\mathcal{D}$ of $[a,b]$ is called a *partition* if its union is the whole interval $[a,b]$. The length of a closed interval J will be denoted by its Lebesgue measure $\mu(J)$.

With this in place, we can now define the notion of absolute continuity and the generalization that is operative in the definition of the Denjoy integral:

Definition 1.1. Let $F:[a,b]\to\mathbb{R}$ be continuous and let $K\subseteq[a,b]$. Then F is *absolutely continuous* on K, written $F\in AC(K)$, if for every $\epsilon>0$ there is $\delta>0$ such that for all K-edged pre-partitions $\mathcal{D}$ of $[a,b]$,

$$\sum_{J\in\mathcal{D}}\mu(J)<\delta \;\Rightarrow\; \sum_{J\in\mathcal{D}}|F(\max(J))-F(\min(J))|<\epsilon. \tag{1.1}$$

The generalizations are obtained by relaxing the consequent of this conditional. One does this by introducing the notation

$$\omega(F,J)=\sup\{|F(x)-F(y)|:x,y\in J\}, \tag{1.2}$$

and then by defining:

Definition 1.2. Let $F:[a,b]\to\mathbb{R}$ be continuous and let $K\subseteq[a,b]$. Then F is *absolutely continuous in the restricted sense* on K, written $F\in AC_*(K)$, if for every $\epsilon>0$ there is $\delta>0$ such that for all K-edged pre-partitions $\mathcal{D}$ of $[a,b]$, if $\sum_{J\in\mathcal{D}}\mu(J)<\delta$ then $\sum_{J\in\mathcal{D}}\omega(F,J)<\epsilon$. Finally, F is *generalized absolutely continuous in the restricted sense*, written $F\in ACG_*(K)$, if there is a countable sequence of closed $K_n\subseteq[a,b]$ such that $K=\bigcup_n K_n$ and $F\in AC_*(K_n)$.

Note that on this definition, all $AC_*(K)$ and $ACG_*(K)$ functions are continuous. One could obviously define analogous notions for non-continuous functions. But since the functions which interest us are indefinite integrals which are automatically continuous, we maintain the convention that all $AC_*(K)$ and all $ACG_*(K)$ functions are continuous. The Denjoy integral may then be defined as follows:

Definition 1.3. Suppose that $f:[a,b]\to\mathbb{R}$. Then f is *Denjoy integrable*, written $f\in\mathrm{Den}[a,b]$, if there is $F\in ACG_*([a,b])$ such that F' exists almost everywhere and $F' = f$ almost everywhere. If in addition $F(a) = 0$, then one defines $\int_a^x f = F(x)$.

The motivation for this definition comes from the parallel with the fundamental theorem for the Lebesgue integral, which in virtue of the above definitions we can state as follows:

Theorem 1.4 (Fundamental Theorem of Calculus for Lebesgue Integrals, [Fol99, Theorem 3.35, p. 106]). *Suppose $f : [a, b] \to \mathbb{R}$ is measurable and $F : [a, b] \to \mathbb{R}$ is continuous with $F(a) = 0$. Then*

$$\Big[f \in L^1[a, b] \ \&\ F(x) = \int_a^x f\Big] \quad \textit{iff} \quad [F \in AC([a, b]) \ \&\ F' = f \ \textit{a.e.}].$$

Here we use the standard notation $L^1[a, b]$ for the space of real-valued Lebesgue integrable functions on $[a, b]$, and the standard abbreviation a.e. for almost everywhere equality. It turns out that all Denjoy integrable functions $f : [a, b] \to \mathbb{R}$ are Lebesgue measurable [Gor94, Theorem 7.6, p. 109]. From this and the definition of the Denjoy integral (Definition 1.3) we can immediately deduce the following analogue of Theorem 1.4:

Theorem 1.5. *Suppose $f : [a, b] \to \mathbb{R}$ is measurable and $F : [a, b] \to \mathbb{R}$ is continuous with $F(a) = 0$. Then*

$$\Big[f \in \mathrm{Den}[a, b] \ \&\ F(x) = \int_a^x f\Big] \quad \textit{iff} \quad [F \in ACG_*([a, b]) \ \&\ F' = f \ \textit{a.e.}].$$

Since any function in $ACG_*([a, b])$ is differentiable a.e. (§2), this theorem says that the Denjoy integrable functions are, up to almost everywhere equality, exactly the derivatives of $ACG_*([a, b])$ functions. The analogy between Theorem 1.4 and Theorem 1.5 thus becomes all the more apparent when one observes that $F \in AC(K)$ iff $F \in AC_*(K)$ in the specific case where K is a closed interval. For, Theorem 1.4 then says that the Lebesgue integrable functions are, up to almost everywhere equality, exactly the derivatives of $AC_*([a, b])$ functions.

At the end of their study of the Denjoy integral, Dougherty and Kechris ([DK91, p. 166], cf. [Kec87, p. 312]) posed the following question about the generalization of absolute continuity which features in Theorem 1.5:

> A second problem is related to the definability aspects of the so-called "descriptive definitions of integrals" (see [Sak37, Chapts. VII, VIII]). These are essentially implicit definitions like the original one of the primitive. For example, the Lebesgue integral F of an integrable function f can be defined as the unique (up to a constant) F such that (i) F is absolutely continuous, and (ii) $F'(x) = f(x)$ for almost all x. By replacing in (i) absolute continuity by more general conditions, one can obtain descriptive definitions of integrals involving any derivative. The question is whether these conditions can possibly be Borel.

In this quotation, Dougherty and Kechris refer to Saks' book [Sak37], and §VIII.1 of that book is called "The descriptive definition of the Denjoy integral" and contains Definition 1.3 [Sak37, p. 241]. It seems then that Dougherty and Kechris are asking about the descriptive set theory complexity of the set $ACG_*([a,b])$. The present paper answers this question by showing that $ACG_*([a,b])$ is not Borel:

THEOREM 1.6. *The set $ACG_*([a,b])$ is coanalytic but not Borel in $C[a,b]$.*

The proof of this theorem occurs at the close of §5. After posing the above question about $ACG_*([a,b])$, Dougherty and Kechris pose another question about whether there is a uniform Borel method of recovering the integral of a function which one knows to be integrable.

Before stating this other question precisely (cf. Question 1.8 below), let us briefly summarize Dougherty and Kechris' own results. These are reported in [DK91] as well as the paper associated to Kechris' 1986 ICM talk [Kec87]. A distinctive feature of Dougherty and Kechris' work is that it restricts attention to the action of the Denjoy integral on the derivatives of everywhere differentiable functions. For, if $F:[a,b]\to\mathbb{R}$ with $F(a)=0$ is everywhere differentiable, then F is $ACG^*([a,b])$ [Gor94, Theorem 7.2, p. 108]. Then Theorem 1.5 implies that $f=F'$ is Denjoy integrable with indefinite integral $\int_a^x f = F(x)$. Now, it is a classical result, due to Mazurkiewicz, that $\mathrm{Diff}[a,b]$, the set of everywhere differentiable real-valued functions on $[a,b]$, is a coanalytic complete subset of $C[a,b]$ [Kec95, §33.D, Theorem 33.9, p. 248]. The topic which Dougherty and Kechris pursued was the complexity of the associated set of derivatives $\Delta=\{F' : F\in \mathrm{Diff}[a,b]\}$.

To make this question precise, one must find some Polish space in which Δ or a related set can naturally be viewed as a subspace. Dougherty and Kechris opted to work in the Polish space $(C[a,b])^\omega$, the countable product space of $C[a,b]$. Inside this space, they focused attention on the sets

$$CN=\Big\{\{f_n\}_{n=1}^\infty\in (C[a,b])^\omega : \forall x \lim_n f_n(x) \text{ exists}\Big\}, \tag{1.3}$$

$$\overline{\Delta}=\Big\{\{f_n\}_{n=1}^\infty\in CN : \lim_n f_n\in\Delta\Big\}. \tag{1.4}$$

If $F\in\mathrm{Diff}[a,b]$, then of course $F'=\lim_n f_n$ where $f_n(x)=[n\cdot(F(x)-F(x+1/n))]$, so that each derivative F' might naturally be viewed as coded by the sequence f_n. That is, we map $\mathrm{Diff}[a,b]$ into $\overline{\Delta}$ by $F\mapsto\delta(F)=\{f_n\}_{n=1}^\infty$ where $f_n(x)=[n\cdot(F(x)-F(x+1/n))]$. Now, it turns out that the set CN is complete coanalytic [Kec95, §33.E, pp. 251 ff], so in looking at the complexity of subsets of CN such as $\overline{\Delta}$, one should ask about how complex it is to be in $\overline{\Delta}$ *given* that one is in CN. One of Dougherty and Kechris's results states that $\overline{\Delta}$ is coanalytic but there is no analytic set S in $(C[a,b])^\omega$ such that for all $\{f_n\}_{n=1}^\infty\in CN$, one has $\{f_n\}_{n=1}^\infty\in S$ iff $\{f_n\}_{n=1}^\infty\in\overline{\Delta}$ [DK91, Theorem 2,

p. 147]. This result tells us that $\{f_n\}_{n=1}^{\infty}$ being in $\overline{\Delta}$ given that it is already in CN is a coanalytic but not Borel notion.

This suggests another question closely related to Dougherty and Kechris's question about the complexity of $ACG_*([a,b])$. For there are many functions F in $ACG_*([a,b])$ which are not in $\mathrm{Diff}[a,b]$, and since Dougherty and Kechris focused on the image of $\mathrm{Diff}[a,b]$ under the differentiation operation, their results in general would not have any implications for the larger set $ACG_*([a,b])$. However, it is natural to study the complexity of the image of $ACG_*([a,b])$ under the operation of *almost everywhere* differentiation. To make this question precise, one must specify a Polish space which naturally contains this image. Here we look at the Polish space $M[a,b]$ of real-valued measurable functions on $[a,b]$, modulo almost everywhere equality. The topology is defined so that $f_n \to f$ in this space iff $f_n \to f$ in measure. At the outset of §5 we review the Polish space structure of $M[a,b]$ in more detail. But having specified this Polish space, we can now state our second main result, which is proven at the close of §5:

THEOREM 1.7. *The set* $\mathrm{Den}[a,b]$ *of Denjoy integrable functions is a* $\mathbf{\Sigma}_2^1$*-subset of the Polish space* $M[a,b]$ *and is not analytic.*

Recall that a $\mathbf{\Sigma}_2^1$-*subset* is the continuous (or Borel) image of a coanalytic set, and it is a basic part of the classical theory that all analytic and coanalytic sets are $\mathbf{\Sigma}_2^1$, but not vice versa. In the statement of this theorem, we regard elements of $\mathrm{Den}[a,b]$ as real-valued functions on $[a,b]$ modulo almost everywhere equality. Dougherty and Kechris' result described at the end of the previous paragraph essentially said that the image of $\mathrm{Diff}[a,b]$ under the operation of differentiation was coanalytic but not Borel in the Polish space $(C[a,b])^{\omega}$. Similarly, Theorem 1.7 implies that the image of $ACG_*[a,b]$ under the operation of almost everywhere differentiation is $\mathbf{\Sigma}_2^1$ but not Borel.

After posing their question about $ACG_*([a,b])$, Dougherty and Kechris ask how hard it is to recover the integral of a function which one antecedently knows to be integrable. In their setting, the form this question took was the following:

QUESTION 1.8 ([DK91, p. 166], [Kec87, p. 312]). Is there a Borel set $B \subseteq (C[a,b])^{\omega} \times C[a,b]$ such that for all $\{f_n\}_{n=1}^{\infty} \in \overline{\Delta}$ and all $F \in C[a,b]$, one has $(\{f_n\}_{n=1}^{\infty}, F) \in B$ iff $F'(x) = \lim_n f_n(x)$ for all $x \in [a,b]$?

A negative resolution of this question would generalize Dougherty and Kechris' result that there is no Borel set $B \subseteq (C[a,b])^{\omega}$ such that for all $\{f_n\}_{n=1}^{\infty} \in \overline{\Delta}$, one has $\{f_n\}_{n=1}^{\infty} \in B$ iff $\int_a^b \lim_n f_n(x)\, dx > 0$ [DK91, Theorem 4, p. 147]. The analogous question in our setting would be the following, or perhaps variations on it with $\mathrm{Den}[a,b]$ replaced by various of its subsets:

QUESTION 1.9. Is there a Borel set $B \subseteq M[a,b] \times C[a,b]$ such that for all $f \in \mathrm{Den}[a,b]$ and all $F \in C[a,b]$ with $F(a) = 0$, one has $(f,F) \in B$ iff $\int_a^x f = F(x)$ for all $x \in [a,b]$?

We have been unable to answer Questions 1.8–1.9.

However, a crucial part of our proofs of Theorems 1.6 and 1.7 revolves around the related issue of identifying the complexity of the graph of the indefinite Denjoy integral. Here we establish the following result, whose proof occurs at the close of §5:

THEOREM 1.10. *The graph of the indefinite Denjoy integral $f \mapsto \int_a^x f$, viewed as a subset of the product space $M[a,b] \times C[a,b]$, is coanalytic but not Borel.*

In all three of our theorems, there is a positive claim about a certain set being coanalytic (resp. $\mathbf{\Sigma}^1_2$), and a negative claim that the sets are not Borel (resp. not analytic). The positive part of Theorem 1.7 follows directly from the positive part of Theorem 1.6 and the observation that being differentiable almost everywhere is a Borel property of an element $F \in C[a,b]$, and the relation $F' = f$ a.e. is a Borel property of a pair $(F,f) \in C[a,b] \times M[a,b]$ (Proposition 5.6). So $\mathrm{Den}[a,b]$ is $\mathbf{\Sigma}^1_2$ because it is the image of a coanalytic set $ACG_*([a,b])$ under the Borel operation $F \mapsto \gamma(F)$, where $\gamma(F) = f$ if F' is differentiable a.e. and $F' = f$ a.e., and $\gamma(F) = 0$ otherwise. It remains to prove the positive parts of Theorems 1.6 and 1.10, as well as the negative parts of all three theorems.

The basic idea of these proofs is to look at coanalytic ranks associated to maps on the Polish space $K[a,b]$ of closed subsets of $[a,b]$ [Kec95, §34.D, pp. 270 ff]. This space has the topology generated by the "miss" sets $\{K \in K[a,b] : K \cap U^c = \emptyset\}$ and the "hit" sets $\{K \in K[a,b] : K \cap U \neq \emptyset\}$, where $U \subseteq [a,b]$ is open [Kec95, §4.F, pp. 24 ff]). The proofs proceed by defining, for each $f \in M[a,b]$, each $F \in C[a,b]$ and each pair $(f,F) \in M[a,b] \times C[a,b]$, three Borel functions from $K[a,b]$ to $K[a,b]$:

$$K \mapsto D_f(K), \quad K \mapsto D_F(K), \quad K \mapsto D_{f,F}(K). \tag{1.5}$$

These functions are called "derivatives" since they resemble the Cantor–Bendixson derivative in certain of their formal properties. The intuitive idea is that $D_f(K)$ consists of those points of K at which f is not locally Lebesgue integrable, $D_F(K)$ consists of those points of K at which F is not locally absolutely continuous in the restricted sense, and $D_{f,F}(K) = D_f(K) \cup D_F(K)$. For the formal definitions, see (3.8)–(3.10) below.

These derivatives can then be iterated countably many times by defining $D^{\alpha+1}(K) = D(D^\alpha(K))$ and taking intersections at limit stages. Since these maps are Borel (§5), it follows from the general theory of such derivatives that the set of elements f, F whose derivatives $D^\alpha_f([a,b])$, $D^\alpha_F([a,b])$,

$D^{\alpha}_{f,F}([a,b])$ are eventually empty, are themselves coanalytic sets. One can then show that $f \in \mathrm{Den}[a,b]$ with $F(x) = \int_a^x f$ iff there is a countable ordinal α such that $D^{\alpha}_{f,F}([a,b])$ is empty and $F' = f$ a.e. (Corollary 4.4). Putting these various results together at the end of §5 immediately gives the positive parts of Theorems 1.6 and 1.10. The general theory of these derivatives also implies that subsets of these spaces whose derivatives vanish below some given countable ordinal are Borel. Hence, by showing that there are elements whose derivative only vanishes at arbitrarily high countable ordinals (Theorem 3.2), we are able to argue for the negative parts of all three theorems at the close of §5.

Before outlining the content of the different sections of this paper, let us briefly describe our results on the model theory of the Denjoy integral. Dougherty and Kechris' question was essentially a question of how difficult it is to define the Denjoy integral. One can also ask about the complexity of the sets which are *defined by* this integral. Here the appropriate setting seems to be that of model theory, where one asks what can be defined in a first-order way from the Denjoy integral, and a natural language for this is the language of $\mathbb{R}[X]$-modules, where the indeterminate X is interpreted as the indefinite Denjoy integral, so that the atomic formulas are a very elementary type of integral equation. One of the basic questions to ask here is whether there is any first-order difference between the Denjoy integrable functions, the Lebesgue integrable functions, and the continuous functions with the Riemann integral. This question is answered here in the negative by the following theorem:

THEOREM 1.11. *As $\mathbb{R}[X]$-modules with the indeterminate X interpreted as the indefinite integral $Xf \mapsto \int_a^x f$, the continuous functions $C[a,b]$, the Lebesgue integrable functions $L^1[a,b]$, and the Denjoy integrable functions* $\mathrm{Den}[a,b]$ *are elementarily equivalent. Further, as $\mathbb{Q}[X]$-modules, they have the same computable complete theory.*

This final theorem is proven at the end of §7. Thus the conclusion of this part of the paper is that from an admittedly elementary model-theoretic standpoint, these integrals are indistinguishable. For suggestions as to less elementary perspectives, see §8.

This paper is organized as follows. In §2, some basic facts related to the Denjoy integral are recalled, and it is noted how one can define a series of subsets of the Denjoy integrable functions which relate to how long it takes to define the Denjoy integral in terms of the Lebesgue integral and improper integrals. In §3, the three Cantor–Bendixson-like derivatives mentioned above in (1.5) are formally defined, and it is shown that there are Denjoy integrable functions whose derivatives vanish only at arbitrarily high countable ordinals (Theorem 3.2). In §4, it is shown that these two measures defined in the previous two subsections correspond exactly, and in particular that the vanishing of the derivatives in §3 is correlated exactly with membership in the sequence of subsets from §2: this is the content of Theorem 4.3 and Corollary 4.4. In §5, it is shown that the derivatives are Borel, which then permits us to deduce the main Theorems 1.6, 1.7, and 1.10. In §6, we turn to the development of the model-theoretic perspective pursued here and use calculations of indices of subgroups to show that these modules are stable but not superstable and hence are model-theoretically more complex than the underlying vector spaces. Finally, in §7 we use the Riesz theorems from the theory of integral equations, in conjunction with the pp-elimination of quantifiers from the model theory of modules, to deduce Theorem 1.11.

2. Basic lemmas and the subspaces. The aim of this section is to briefly review some key facts about the Denjoy integral which we shall employ throughout this paper. As mentioned in the introduction, the indefinite integrals of the Denjoy integrable functions on $[a, b]$ are precisely the functions F in $ACG_*([a, b])$ with $F(a) = 0$. In what follows, we shall repeatedly appeal to the fact that every function in $ACG_*([a, b])$ is differentiable almost everywhere on $[a, b]$ [Gor94, Corollary 6.19, p. 100].

Now, it is worth mentioning that there is a partial converse to this result. In particular, let us say that a function $F : [a, b] \to \mathbb{R}$ is *differential nearly everywhere* if F is differentiable except on a countable set. Then it turns out that if $F \in C([a, b])$ is differentiable nearly everywhere then $F \in ACG_*([a, b])$ ([Gor94, p. 103] or [PY89, p. 29]). However, the full converse to this result is in general false. For examples of real-valued continuous functions F on $[a, b]$ that are differentiable almost everywhere but $F \notin ACG_*([a, b])$, see [Gor94, p. 119].

Frequently, we shall also appeal to certain elementary facts pertaining to the class $AC_*(E)$ (Definition 1.2). First, if E is itself a closed interval, then any continuous function in $AC(E)$ is in $AC_*(E)$ and vice versa; so it is only for more complicated sets E that the two notions diverge. Second, if $Q \subseteq E$ is dense, then any continuous function in $AC_*(Q)$ is in $AC_*(E)$ and vice versa; hence without loss of generality, we may restrict attention to evaluating absolute continuity in the restricted sense on dense subsets [Gor94, Theorem 6.2(d), pp. 90–91]. Third, if F is a continuous function on $[a, b]$ and $E \subseteq [a, b]$ is closed with $(a, b) - E = \bigsqcup_n (c_n, d_n)$, then one has the following [Gor94, Theorem 6.2, pp. 90–91]:

$$F \in AC_*(E) \;\Rightarrow\; \sum_n \omega(F, [c_n, d_n]) < \infty. \tag{2.1}$$

Finally, if $f \in \mathrm{Den}[a, b]$ then there are $K_n \in K[a, b]$ with $[a, b] = \bigcup_n K_n$

and $f\chi_{K_n} \in L^1[a,b]$ [Gor94, Theorem 9.18, pp. 148–149]. Recall that $K[a,b]$ denotes the Polish space of closed subsets of $[a,b]$.

Two basic lemmas on the Denjoy integral are important for what follows. The first gives a useful sufficient condition for a function to be Denjoy integrable, and in particular provides a way to start building up the Denjoy integral step-by-step from the Lebesgue integral. Hence we dub this lemma the "Step Lemma":

LEMMA 2.1 (Step Lemma, [Gor94, Theorem 7.12, p. 111]). *Suppose that* $f \in M[a,b]$, $K \in K[a,b]$ *and* $(a,b) - K = \bigsqcup_{n=1}^{\infty}(c_n,d_n)$. *Further suppose that* $f\chi_K \in L^1[a,b]$, $f\chi_{[c_n,d_n]} \in \mathrm{Den}[a,b]$ *and* $\sum_{n=1}^{\infty}\omega(\int_{c_n}^{x} f, [c_n,d_n]) < \infty$. *Then* $f \in \mathrm{Den}[a,b]$ *and* $\int_a^b f = \int_K f + \sum_{n=1}^{\infty}\int_{c_n}^{d_n} f$.

By looking at (2.1) above, one sees immediately that the assumption that $\sum_{n=1}^{\infty}\omega(\int_{c_n}^{x} f, [c_n,d_n]) < \infty$ may be replaced by the assumption that there is $F \in AC_*(K)$ such that $F(x) = \int_{c_n}^{x} f$ on $[c_n,d_n]$. We will often apply this variant of the lemma. Finally, the Improper Integrals Lemma just says that there are no improper integrals in the context of Denjoy integration:

LEMMA 2.2 (Improper Integrals Lemma, [Gor94, Theorem 9.21, p. 150] or [Swa01, Theorem 4, pp. 25–26]). *Suppose* $f \in M[a,b]$. *If* $f\chi_{[c,b]} \in \mathrm{Den}[a,b]$ *for every* $c \in (a,b)$, *then* $f \in \mathrm{Den}[a,b]$ *with* $\int_a^b f = L$ *if and only if* $\lim_{c\searrow a^+}\int_c^b f$ *exists and is equal to* L. *Likewise, if* $f\chi_{[a,c]} \in \mathrm{Den}[a,b]$ *for every* $c \in (a,b)$, *then* $f \in \mathrm{Den}[a,b]$ *with* $\int_a^b f = L$ *if and only if* $\lim_{c\nearrow b^-}\int_a^c f$ *exists and is equal to* L.

These two lemmas can be used to define a series of subsets of $\mathrm{Den}[a,b]$ which reflect how long it takes to recover Denjoy integration from Lebesgue integration and improper integration. These subsets are closed under scalar multiplication, and they are *subinterval-closed* in the sense that if they contain f then they contain $f\chi_{(c,d)}$ for any interval (c,d). To define these subsets, let us first define two preliminary notions:

DEFINITION 2.3. Suppose that $\mathcal{X} \subseteq \mathrm{Den}[a,b]$. Then $f \in \mathrm{Den}[a,b]$ is an *improper integral* of $\mathcal{X}$ if there is a countable sequence $(a_n,b_n) \subseteq (a,b)$ such that (i) $(a,b) = \bigcup_{n=1}^{\infty}(a_n,b_n)$ and $(a_n,b_n) \subseteq (a_{n+1},b_{n+1})$, (ii) $f\chi_{(a_n,b_n)} \in \mathcal{X}$, (iii) $\lim_{c\searrow a^+}\int_c^{b_1} f$ exists, and (iv) $\lim_{c\nearrow b^-}\int_{a_1}^{c} f$ exists. Further, define $\mathrm{Lim}(\mathcal{X})$ to be the set of improper integrals of $\mathcal{X}$.

DEFINITION 2.4. Suppose that $\mathcal{X} \subseteq \mathrm{Den}[a,b]$. Then $f \in \mathrm{Den}[a,b]$ is *given by the Step Lemma from* $\mathcal{X}$ if there is a $K \in K[a,b]$ with $(a,b) - K = \bigsqcup_{n=1}^{\infty}(c_n,d_n)$ such that $f\chi_K \in L^1[a,b]$, $f\chi_{(c_n,d_n)} \in \mathcal{X}$, and there is $F \in AC_*(K)$ satisfying $F(x) = \int_a^x f$. Further, let $\mathrm{Step}(\mathcal{X})$ be the set of elements which are given by the Step Lemma 2.1 from $\mathcal{X}$.

Then we define the subsets $\mathrm{Den}_\alpha[a,b]$ of $\mathrm{Den}[a,b]$ by recursion:

DEFINITION 2.5. Define $\mathrm{Den}_0[a,b] = L^1[a,b]$, and for $\alpha > 0$ define $\mathrm{Den}_\alpha[a,b] = \mathrm{Step}(\mathrm{Lim}(\bigcup_{\beta<\alpha} \mathrm{Den}_\beta[a,b]))$.

It is routine to show that these subsets are closed under scalar multiplication, subinterval-closed, closed under a.e. differences, and non-decreasing as the ordinal α increases.

3. Three derivatives and functions of arbitrarily high rank. Now we proceed to the formal definition of our three Cantor–Bendixson-like derivatives which we mentioned in (1.5). There is a general framework for these kinds of derivatives, which is set out in [Kec95, §34.D]. Hence, let us begin by recalling the basics of this framework. Recall that $K[a,b]$ denotes the Polish space of closed subsets of $[a,b]$. Suppose that $\mathcal{B} \subseteq K[a,b]$ is closed under closed subsets, i.e., if $K \in \mathcal{B}$ and $L \in K[a,b]$ and $L \subseteq K$ then $L \in \mathcal{B}$. Then define the derivative map $D_\mathcal{B} : K[a,b] \to K[a,b]$ by

$$D_\mathcal{B}(K) = \{x \in K : \overline{U \cap K} \notin \mathcal{B} \text{ for any open } U \ni x\}. \tag{3.1}$$

Further, recursively define $D^\alpha_\mathcal{B} : K[a,b] \to K[a,b]$ by

$$\begin{aligned} D^0_\mathcal{B}(K) = K, \quad D^{\alpha+1}_\mathcal{B}(K) &= D_\mathcal{B}(D^\alpha_\mathcal{B}(K)), \\ D^\alpha_\mathcal{B}(K) &= \bigcap_{\beta<\alpha} D^\beta_\mathcal{B}(K), \quad \alpha \text{ limit}. \end{aligned} \tag{3.2}$$

Finally, for $K \in K[a,b]$, we define its *rank* $|K|_\mathcal{B}$ relative to $\mathcal{B}$ as follows:

$$|K|_\mathcal{B} = \inf\{\alpha : D^\alpha_\mathcal{B}(K) = D^{\alpha+1}_\mathcal{B}(K)\}, \tag{3.3}$$

and finally we set $D^\infty_\mathcal{B}(K) = D^{|K|_\mathcal{B}}_\mathcal{B}(K)$.

These maps have many of the same properties as the Cantor–Bendixson derivative. First, one has monotonicity: if L, K are closed sets then

$$L \subseteq K \;\Rightarrow\; D^\alpha_\mathcal{B}(L) \subseteq D^\alpha_\mathcal{B}(K). \tag{3.4}$$

Second, $|K|_\mathcal{B} < \omega_1$, so the rank is always a countable ordinal. Hence, $D^\infty_\mathcal{B}(K)$ can be written as an intersection over countable ordinals:

$$D^\infty_\mathcal{B}(K) = \bigcap_{\alpha<\omega_1} D^\alpha_\mathcal{B}(K). \tag{3.5}$$

Third and relatedly, for $K \in K[a,b]$, let us define $K \in \mathcal{B}_\sigma$ iff K is a countable union of elements from $\mathcal{B}$. Then $K \in \mathcal{B}_\sigma$ iff $D^\infty_\mathcal{B}(K) = \emptyset$. This fact is important because we will often be interested in the case where the derivative vanishes, i.e. $D^\infty_\mathcal{B}(K) = \emptyset$, and this last fact tells us that this happens exactly when K can be written as a countable union of elements from $\mathcal{B}$. For the proof of all the properties mentioned in this paragraph, see [Kec95, §34.D].

The specific derivatives we are interested in are associated to a measurable function $f \in M[a,b]$ and a continuous function $F \in C[a,b]$. Given such functions, we define

$$\begin{aligned} \mathcal{B}_f &= \{K \in K[a,b] : f\chi_K \in L^1[a,b]\}, \\ \mathcal{B}_F &= \{K \in K[a,b] : F \in AC_*(K)\}, \end{aligned} \tag{3.6}$$

and we further set $\mathcal{B}_{f,F} = \mathcal{B}_f \cap \mathcal{B}_F$. Since $\mathcal{B}_f$, $\mathcal{B}_F$, and $\mathcal{B}_{f,F}$ are closed under closed subsets, we may then use (3.1) to define

$$D_f(K) = D_{\mathcal{B}_f}(K), \quad D_F(K) = D_{\mathcal{B}_F}(K), \quad D_{f,F}(K) = D_{\mathcal{B}_{f,F}}(K). \tag{3.7}$$

These definitions are equivalent to the following, by using elementary properties of Lebesgue integrability as well as absolute continuity in the restricted sense (moreover, in these equivalent formalizations one has the freedom to restrict attention to endpoints c,d which are rational):

$$D_f(K) = \{x \in K : f\chi_{[c,d]\cap K} \notin L^1[a,b] \text{ for any } (c,d) \ni x\}, \tag{3.8}$$

$$D_F(K) = \{x \in K : F \notin AC_*([c,d] \cap K) \text{ for any } (c,d) \ni x\}, \tag{3.9}$$

$$D_{f,F}(K) = D_f(K) \cup D_F(K). \tag{3.10}$$

As one can see, $D_f(K)$ is the points of K where f is not locally Lebesgue integrable, while $D_F(K)$ is the points of K where F is not locally absolutely continuous in the restricted sense. Comparing this to the Fundamental Theorem of Calculus for Lebesgue Integrals (Theorem 1.4), one sees that these derivatives record the points at which the Fundamental Theorem locally fails for a measurable function f and a continuous function F.

Fixing still a measurable function f and a continuous function F, one then recursively defines the iterates $D_f^\alpha(K)$ as in (3.2):

$$\begin{aligned} D_f^0(K) = K, \quad & D_f^{\alpha+1}(K) = D_f(D_f^\alpha(K)), \\ & D_f^\alpha(K) = \bigcap_{\beta<\alpha} D_f^\beta(K), \quad \alpha \text{ limit}, \end{aligned} \tag{3.11}$$

and similarly for $D_F^\alpha(K)$. To associate a rank directly to $f \in M[a,b]$ and $F \in C([a,b])$, one then employs (3.3) to define

$$|f| = |[a,b]|_{\mathcal{B}_f}, \quad |F| = |[a,b]|_{\mathcal{B}_F}, \quad |f,F| = |[a,b]|_{\mathcal{B}_{f,F}}. \tag{3.12}$$

Hence, the rank $|f|$ is the least ordinal such that $D_f^\alpha([a,b]) = D_f^{\alpha+1}([a,b])$, and similarly for the other ranks. Later (Theorem 3.2), we will show that the ranks $|f,F|$ of Denjoy integrable functions f and their indefinite integrals F may be an arbitrarily high countable ordinal.

Let us note that the derivatives eventually vanish for Denjoy integrable functions and their indefinite integrals:

PROPOSITION 3.1. *Let* $f \in \mathrm{Den}[a,b]$, $F(x) = \int_a^x f$, *and* $K \in K[a,b]$. *Then* (i) $D_f^\infty(K) = \emptyset$, (ii) $D_F^\infty(K) = \emptyset$, *and* (iii) $D_{f,F}^\infty(K) = \emptyset$.

Proof. (i) One has $D_f^\infty(K) = \emptyset$ if and only if $K \in (\mathcal{B}_f)_\sigma$, i.e. if there are $K_n \in K[a,b]$ such that $K = \bigcup_n K_n$ and $f\chi_{K_n} \in L^1[a,b]$. But this happens when $f \in \mathrm{Den}[a,b]$, as we had occasion to note immediately subsequent to (2.1).

(ii) Likewise, $D_F^\infty(K) = \emptyset$ if and only if $K \in (\mathcal{B}_F)_\sigma$, i.e. if there are $L_m \in K[a,b]$ such that $K = \bigcup_m L_m$ and $F \in AC_*(L_m)$. But this is just to say that $F \in ACG_*([a,b])$, and so this follows immediately from the Fundamental Theorem of Calculus for Denjoy Integrals (Theorem 1.5).

(iii) Now, retaining the closed sets K_n from part (i) and the closed sets L_m from part (ii), consider the sequence of closed sets $C_{n,m} = K_n \cap L_m$. Then $K = \bigcup_{n,m} C_{n,m}$. Further, since $f\chi_{K_n} \in L^1[a,b]$ and $F \in AC_*(L_m)$, we have $f\chi_{C_{n,m}} \in L^1[a,b]$ and $F \in AC_*(C_{n,m})$. This just says that $K \in (\mathcal{B}_{f,F})_\sigma$, so that $D_{f,F}^\infty(K) = \emptyset$. ∎

Finally, let us close this section by noting that there are Denjoy integrable functions whose derivatives vanish only at arbitrarily high countable ordinals. As mentioned in the introduction, this result is important for the negative parts of Theorems 1.10 and 1.6, which we prove in §5. The construction in the successor step of the following example is based on the example discussed in [Gor94, pp. 117–118], although that discussion does not treat the derivatives $D_F^\alpha([a,b])$ introduced above.

THEOREM 3.2. *For every $\alpha < \omega_1$, every $[a,b]$ and every $r > 0$, there exists an $f \in \mathrm{Den}[a,b]$ with $\int_a^b f = 0$ and $f(a) = f(b) = 0$ such that the function $F(x) = \int_a^x f$ satisfies $\omega(F,[a,b]) = r$ and $a,b \in D_F^\alpha([a,b])$. Hence, for all $\alpha < \omega_1$ there exists an $f \in \mathrm{Den}[a,b]$ such that the function $F(x) = \int_a^x f$ satisfies $\alpha < |F| \le |f,F|$.*

Proof. Suppose that $\alpha = 0$. Let $f(x) = \sin(2\pi(b-a)^{-1}(x-a))$. Since $\omega(F,[a,b]) = (b-a)/\pi > 0$ where $F(x) = \int_a^x f$, to ensure that for any $r > 0$ we can obtain $\omega(F,[a,b]) = r$, simply multiply f by an appropriate constant.

Suppose now that $\alpha = \beta + 1$. Let C be the Cantor 1/3-set on $[a,b]$, let $(a,b) - C = \bigsqcup_{n>0}(c_n,d_n)$, let C_n be the Cantor 1/3-set on $[c_n,d_n]$, and let $(c_n,d_n) - C_n = \bigsqcup_{m>0}(c_{nm},d_{nm})$. Choose $f_{nm} \in \mathrm{Den}[c_{nm},d_{nm}]$ with $F_{nm}(x) = \int_{c_{nm}}^x f_{nm}$, $\int_{c_{nm}}^{d_{nm}} f_{nm} = 0$, $f_{nm}(c_{nm}) = f_{nm}(d_{nm}) = 0$ and $c_{nm}, d_{nm} \in D_{F_{nm}}^\beta([c_{nm},d_{nm}])$, such that

$$\omega(F_{nm},[c_{nm},d_{nm}]) = \begin{cases} 2^{-n} & \text{if } m < 2^n, \\ 2^{-n}2^{-m+2^n-1} & \text{if } m \ge 2^n. \end{cases}$$

Then by fixing n we have

$$\sum_{m>0} \omega(F_{nm},[c_{nm},d_{nm}]) = (2^n-1)2^{-n} + 2^{-n}\sum_{m\ge 2^n} 2^{-m+2^n-1} = 1. \tag{3.13}$$

Still fixing n, let $f_n = f_{nm}$ on $[c_{nm}, d_{nm}]$ and $f_n = 0$ otherwise, so that $f_n \in \mathrm{Den}[c_n, d_n]$ with $\int_{c_n}^{d_n} f_n = 0$ by the Step Lemma 2.1, and set $F_n(x) = \int_{c_n}^{x} f_n$. Fixing n for the remainder of the paragraph, we claim that

$$\omega(F_n, [c_n, d_n]) \leq 2 \cdot 2^{-n}.$$

For let $\epsilon > 0$ and let $[x, y] \subseteq [c_n, d_n]$. Since F_n is continuous, choose $\delta > 0$ such that $0 < u - x < \delta$ implies $|\int_x^u f_n| < \epsilon/2$, and $0 < y - v < \delta$ implies $|\int_v^y f_n| < \epsilon/2$. Choose $u, v \notin C_n$ such that $c_n \leq x < u < v < y \leq d_n$, $0 < u - x < \delta$ and $0 < y - v < \delta$. If $[u, v] \subseteq [c_{nm}, d_{nm}]$ then $|\int_u^v f_n| \leq \omega(F_{nm}, [c_{nm}, d_{nm}]) \leq 2^{-n}$, and hence $|\int_x^y f_n| \leq \epsilon + 2^{-n}$. Otherwise, we have $c_{n\ell} \leq u \leq d_{n\ell} < c_{nm} \leq v \leq d_{nm}$, and then estimating as before we get $|\int_x^y f_n| \leq \epsilon + 2 \cdot 2^{-n} + |\int_{d_{n\ell}}^{c_{nm}} f_n|$, and so it suffices to show that $\int_{d_{n\ell}}^{c_{nm}} f_n = 0$, which follows as above from the Step Lemma 2.1. Hence the above claim is proved.

This of course implies that $\sum_{n>0} \omega(F_n, [c_n, d_n]) \leq \sum_{n>0} 2 \cdot 2^{-n} \leq 2$, and so letting $f = f_n$ on $[c_n, d_n]$ and $f = 0$ otherwise, we find that $f \in \mathrm{Den}[a, b]$ with $\int_a^b f = 0$ by the Step Lemma 2.1. Now set $F(x) = \int_a^x f$. To see that $a, b \in D_F^{\alpha}([a, b])$, note that by hypothesis c_{nm}, d_{nm} are in $D_{F_{nm}}^{\beta}([c_{nm}, d_{nm}])$, and hence in $D_F^{\beta}([a, b])$ since

$$D_{F_{nm}}^{\beta}([c_{nm}, d_{nm}]) = D_{F\restriction[c_{nm}, d_{nm}]}^{\beta}([c_{nm}, d_{nm}]) \subseteq D_F^{\beta}([a, b]). \tag{3.14}$$

Since a subsequence of the c_{nm} converges to c_n, and a subsequence of the d_{nm} converges to d_n, we see that $c_n, d_n \in D_F^{\beta}([a, b])$. Then we claim that

$$c_n, d_n \in D_F(D_F^{\beta}([a, b])).$$

We give the argument for c_n since the argument for d_n is similar. Indeed, if $c_n \notin \overline{D_F(D_F^{\beta}([a, b]))}$ then there is an open $U \ni c_n$ such that $F \in AC_*(U \cap D_F^{\beta}([a, b]))$, and hence $F \in AC_*(U \cap D_F^{\beta}([a, b]))$. Then choose $\delta > 0$ corresponding to $\epsilon = 1/2$. Since U is open and intersects C, it follows that U contains infinitely many intervals (c_ℓ, d_ℓ) and so an interval (c_ℓ, d_ℓ) with length $d_\ell - c_\ell < \delta$. Applying (3.13) with $n = \ell$, choose a finite sequence $(c_{\ell 1}, d_{\ell 1}), \ldots, (c_{\ell M}, d_{\ell M})$ such that $\sum_{m=1}^{M} \omega(F_{\ell m}, [c_{\ell m}, d_{\ell m}]) > 1/2$. But this is a contradiction, since $(c_{\ell 1}, d_{\ell 1}), \ldots, (c_{\ell M}, d_{\ell M})$ is a $U \cap D_F^{\beta}([a, b])$-edged pre-partition with $\sum_{m=1}^{M} d_{\ell m} - c_{\ell m} \leq d_\ell - c_\ell < \delta$. Hence the claim is proved, which of course implies that $a, b \in D_F(D_F^{\beta}([a, b])) = D_F^{\alpha}([a, b])$, since there is a subsequence of the c_n converging to a and likewise a subsequence of the d_n converging to b.

Suppose that $\alpha < \omega_1$ is a limit ordinal. Let α_n be an enumeration of the ordinals less than α. Let w be the midpoint of $[a, b]$. Choose $u_n \searrow a^+$ with $u_0 = w$ and $v_n \nearrow b^-$ with $v_0 = w$. Choose $h : \omega \to \omega$ such that $h^{-1}(n)$ is infinite for all n. Choose $f_n \in \mathrm{Den}[u_{n+1}, u_n]$ with $F_n(x) = \int_{u_{n+1}}^{x} f_n$,

$\int_{u_{n+1}}^{u_n} f = 0$, $f(u_{n+1}) = f(u_n) = 0$, $u_{n+1}, u_n \in D_{F_n}^{\alpha_{h(n)}}([u_{n+1}, u_n])$ and $\omega(F_n, [u_{n+1}, u_n]) = 1/n$. Likewise, choose $g_n \in \mathrm{Den}[v_n, v_{n+1}]$ with $G_n(x) = \int_{v_n}^{x} g_n$, $\int_{v_n}^{v_{n+1}} g_n = 0$, $g_n(v_n) = g_n(v_{n+1}) = 0$, $v_{n+1}, v_n \in D_{G_n}^{\alpha_{h(n)}}([v_n, v_{n+1}])$ and $\omega(G_n, [v_n, v_{n+1}]) = 1/n$.

Let $f = f_n$ on $[u_{n+1}, u_n]$, $f = g_n$ on $[v_n, v_{n+1}]$ and $f(a) = f(b) = 0$. Since $\omega(F_n, [u_{n+1}, u_n]) = \omega(G_n, [v_n, v_{n+1}]) = 1/n$, we claim that

$$f \in \mathrm{Den}[a, b] \quad \text{with} \quad \int_a^b f = 0$$

by the Improper Integrals Lemma 2.2. For, to apply this lemma, it must be shown that $\lim_{c \searrow a^+} \int_c^w f$ and $\lim_{c \nearrow b^-} \int_w^c f$ exist and are zero, where recall that w is the midpoint of $[a, b]$. Without loss of generality, consider the former limit. Let $\epsilon > 0$. Choose N such that $1/N < \epsilon$ and set $\delta = u_N - a$. Suppose that $0 < c - a < \delta$, so that $a < c < u_N$. Let $n \geq N$ be such that $a < u_{n+1} \leq c < u_n \leq u_N$. Since $\omega(F_n, [u_{n+1}, u_n]) = 1/n$ and $\int_{u_{i+1}}^{u_i} f = 0$, it follows that

$$\Big|\int_c^w f\Big| \leq \Big|\int_c^{u_n} f\Big| + \sum_{i=0}^{n-1} \Big|\int_{u_{n-i}}^{u_{n-i-1}} f\Big| \leq \frac{1}{n} + 0 \leq \frac{1}{N} < \epsilon.$$

Hence the claim follows by the Improper Integrals Lemma 2.2, and so we define $F(x) = \int_a^x f$.

To show that $a \in D_F^{\alpha}([a, b])$, it suffices to show that $a \in D_F^{\alpha_n}([a, b])$ for all n. So, fixing n and recalling that $h^{-1}(n)$ is infinite, choose a sequence $u_{n_k} \searrow a^+$ such that $u_{n_k} \in D_{F_{n_k}}^{\alpha_n}([u_{n_k+1}, u_{n_k}])$. Since $u_{n_k} \in D_{F_{n_k}}^{\alpha_n}([u_{n_k+1}, u_{n_k}])$ and $D_{F_{n_k}}^{\alpha_n}([u_{n_k+1}, u_{n_k}]) = D_{F\restriction[u_{n_k+1}, u_{n_k}]}^{\alpha_n}([u_{n_k+1}, u_{n_k}]) \subseteq D_F^{\alpha_n}([a, b])$, it follows that $u_{n_k} \in D_F^{\alpha_n}([a, b])$. Since $u_{n_k} \searrow a^+$, it follows that $a \in D_F^{\alpha_n}([a, b])$. Since the α_n enumerated the ordinals below the limit ordinal α, it follows that $a \in D_F^{\alpha}([a, b])$. An analogous argument shows that $b \in D_F^{\alpha}([a, b])$. ∎

4. Calibrating rank and entry into subspaces. In the previous section, we defined three derivatives whose vanishing is related to how far one is from satisfying the Fundamental Theorem of Calculus for Lebesgue Integrals (Theorem 1.4). In §2, we defined the sequence of subsets $\mathrm{Den}_\alpha[a, b]$ which record how long it takes to define the Denjoy integral in terms of Lebesgue integration and improper integrals. The main result of this section, Theorem 4.3, calibrates entry into the subsets $\mathrm{Den}_\alpha[a, b]$ with the vanishing of the derivatives. From this theorem, we obtain Corollary 4.4 which presents an equivalent characterization of Denjoy integration in terms of the vanishing of derivatives, or equivalently entry into the subsets $\mathrm{Den}_\alpha[a, b]$. This equivalent characterization is what we use in the next subsection to obtain our main results on the descriptive set theory complexity of Denjoy integration. We begin with two preliminary propositions.

Proposition 4.1. *Suppose $f \in M[a,b]$ and $K \in K[a,b]$.*

(i) *If $D_f(K) = \emptyset$ then $f\chi_K \in L^1[a,b]$.*
(ii) *$(p,q) \cap D_f(K) = \emptyset$ iff $f\chi_{[r,s]\cap K} \in L^1[a,b]$ for all rational $[r,s] \subseteq (p,q)$.*

Proof. (i) By (3.8), if $D_f(K) = \emptyset$ then for every $x \in K$ there is an open interval $(a_x, b_x) \ni x$ such that $f\chi_{[a_x,b_x]\cap K} \in L^1[a,b]$. By the compactness of K, there is a finite subcovering of K by such intervals $(a_1, b_1), \ldots, (a_N, b_N)$. Then of course $f\chi_K \in L^1[a,b]$.

(ii) The left-to-right direction follows immediately from (i), while the converse follows directly from the equivalent characterization of $D_f(K)$ in (3.8). ∎

Proposition 4.2. *Suppose $F \in C[a,b]$ and $K \in K[a,b]$.*

(i) *If $D_F(K) = \emptyset$ then $F \in AC_*(K)$.*
(ii) *$(p,q) \cap D_F(K) = \emptyset$ iff $F \in AC_*([r,s] \cap K)$ for all rational $[r,s] \subseteq (p,q)$.*

Proof. (i) By (3.9), if $D_F(K) = \emptyset$ then for every $x \in K$ there is $(a_x, b_x) \ni x$ such that $F \in AC_*([a_x, b_x] \cap K)$. By the compactness of K, there is a finite subcovering $(a_1, b_1), \ldots, (a_N, b_N)$ of K such that $F \in AC_*([a_i, b_i] \cap K)$. Let $\eta > 0$ be strictly less than all the non-zero $|a_i - b_j|$, $|b_i - a_j|$ for $i \neq j$. Let $\epsilon > 0$ and choose $\delta_i > 0$ such that for every $[a_i, b_i] \cap K$-edged pre-partition $\mathcal{D}$ of $[a,b]$, if $\sum_{J\in\mathcal{D}} \mu(J) < \delta_i$ then $\sum_{J\in\mathcal{D}} \omega(F,J) < N^{-1}\epsilon$. Choose $\delta > 0$ such that $\delta < \delta_i$ for each i as well as $\delta < \eta$.

Suppose that $\mathcal{D}$ is a K-edged pre-partition of $[a,b]$ with $\sum_{J\in\mathcal{D}} \mu(J) < \delta$. Note that requiring $\delta < \eta$ implies that if some closed interval $J \in \mathcal{D}$ is not $[a_j, b_j] \cap K$-edged for any j, then there are non-overlapping closed intervals I_J, L_J such that $J = I_J \cup L_J$ and I_J is $[a_i, b_i] \cap K$-edged and L_J is $[a_k, b_k] \cap K$-edged for some $i \neq k$. Let $\mathcal{K}$ be a K-edged pre-partition of $[a,b]$ which contains J where $J \in \mathcal{D}$ is an $[a_j, b_j] \cap K$-edged for some j, and contains I_J, L_J where $J \in \mathcal{D}$ is not $[a_j, b_j] \cap K$-edged for any j. Then for every $J \in \mathcal{K}$ there is some j such that J is $[a_j, b_j] \cap K$-edged. Let $\mathcal{K}_j$ be an $[a_j, b_j] \cap K$-edged pre-partition of $[a,b]$ which consists of those $J \in \mathcal{K}$ such that J is $[a_j, b_j] \cap K$-edged. Then $\mathcal{K}_j$ is an $[a_j, b_j] \cap K$-edged pre-partition of $[a,b]$ such that $\sum_{J\in\mathcal{K}_j} \mu(J) < \delta < \delta_j$, so that $\sum_{J\in\mathcal{K}_j} \omega(F,J) < N^{-1}\epsilon$. Then

$$\sum_{J\in\mathcal{D}} \omega(F,J) \leq \sum_{J\in\mathcal{K}} \omega(F,J) = \sum_{j=1}^{N} \sum_{J\in\mathcal{K}_j} \omega(F,J) < \sum_{j=1}^{N} N^{-1}\epsilon = \epsilon.$$

(ii) The left-to-right direction follows immediately from (i), while the converse follows directly from the equivalent characterization of $D_F(K)$ in (3.9). ∎

With these preliminary propositions in place, let us now prove the main theorem of this section.

THEOREM 4.3. *Let* $\alpha < \omega_1$, $f \in \mathrm{Den}[a,b]$ *and* $F(x) = \int_a^x f$. *Then* $D^{\alpha+1}_{f,F}[a,b] = \emptyset$ *if and only if* $f \in \mathrm{Den}_\alpha[a,b]$.

Proof. The proof is by induction on α. Let $\alpha = 0$. First suppose that $D^{\alpha+1}_{f,F}[a,b] = \emptyset$. By Proposition 4.1(i) we have $f \in L^1[a,b] = \mathrm{Den}_0[a,b]$. Second, suppose that $f \in \mathrm{Den}_0[a,b] = L^1[a,b]$. By the Fundamental Theorem of Calculus for Lebesgue Integrals (Theorem 1.4), $F \in AC([a,b])$, and hence $F \in AC_*([a,b])$. Then $D^{\alpha+1}_{f,F}([a,b]) = D_{f,F}([a,b]) = \emptyset$.

Now let $\alpha > 0$, and suppose that the result holds for all $\beta < \alpha$. First suppose that $D^{\alpha+1}_{f,F}([a,b]) = \emptyset$. By the previous two propositions, we know that $f\chi_{D^\alpha_{f,F}([a,b])} \in L^1[a,b]$ and $F \in AC_*(D^\alpha_{f,F}([a,b]))$. Suppose that $(a,b) - D^\alpha_{f,F}([a,b]) = \bigsqcup_n (c_n,d_n)$. If $[c',d'] \subseteq (c_n,d_n)$, then

$$D^\alpha_{f,F}([c',d']) \subseteq [c',d'] \subseteq (c_n,d_n) \subseteq (a,b) - D^\alpha_{f,F}([a,b]). \tag{4.1}$$

Hence $D^\alpha_{f,F}([c',d']) = \emptyset$. Thus there is $\beta < \alpha$ such that $D^{\beta+1}_{f,F}([c',d']) = \emptyset$ and hence by induction hypothesis $f\chi_{[c',d']} \in \mathrm{Den}_\beta[a,b]$. Since we are supposing that $f \in \mathrm{Den}[a,b]$, it follows from the left-to-right direction of the Improper Integrals Lemma 2.2 that $f\chi_{[c_n,d_n]} \in \mathrm{Lim}(\bigcup_{\beta<\alpha} \mathrm{Den}_\beta[a,b])$. Since by definition $(a,b) - D^\alpha_{f,F}([a,b]) = \bigsqcup_n (c_n,d_n)$ and we have already established that $F \in AC_*(D^\alpha_{f,F}([a,b]))$, it follows from Definition 2.4 that $f \in \mathrm{Step}(\mathrm{Lim}(\bigcup_{\beta<\alpha} \mathrm{Den}_\beta[a,b])) = \mathrm{Den}_\alpha[a,b]$.

Second, suppose that $f \in \mathrm{Den}_\alpha[a,b] = \mathrm{Step}(\mathrm{Lim}(\bigcup_{\beta<\alpha} \mathrm{Den}_\beta[a,b]))$. By Definition 2.4, there is a closed set $K \in K[a,b]$ with $(a,b) - K = \bigsqcup_n (c_n,d_n)$ such that $f\chi_K \in L^1[a,b]$, $f\chi_{(c_n,d_n)} \in \mathrm{Lim}(\bigcup_{\beta<\alpha} \mathrm{Den}_\beta[a,b])$ and $F \in AC_*(K)$. We may assume without loss of generality that $a,b \in K$.

Further, since $f\chi_{(c_n,d_n)} \in \mathrm{Lim}(\bigcup_{\beta<\alpha} \mathrm{Den}_\beta[a,b])$, it follows from Definition 2.3 that $(a,b) = \bigcup_m (c_{nm},d_{nm})$ and $f\chi_{(c_n,d_n)}\chi_{(c_{nm},d_{nm})} \in \mathrm{Den}_{\beta_{nm}}[a,b]$ for some $\beta_{nm} < \alpha$. Let $[c'_{nm},d'_{nm}] = [c_n,d_n] \cap [c_{nm},d_{nm}]$, so that $f\chi_{[c'_{nm},d'_{nm}]}$ is in $\mathrm{Den}_{\beta_{nm}}[a,b]$. By induction hypothesis, one has $D^{\beta_{nm}+1}_{f,F}([c'_{nm},d'_{nm}]) = \emptyset$, and thus $D^\alpha_{f,F}([c'_{nm},d'_{nm}]) = \emptyset$. Since $a,b \in K$, it follows that

$$[a,b] \subseteq K \cup ((a,b) - K) = K \cup \bigcup_{n,m} ((c_n,d_n) \cap (c_{nm},d_{nm})). \tag{4.2}$$

By intersecting both sides with $D^\alpha_{f,F}([a,b])$, we obtain

$$D^\alpha_{f,F}([a,b]) \subseteq K \cup \bigcup_{n,m} \big(D^\alpha_{f,F}([a,b]) \cap (c_n,d_n) \cap (c_{nm},d_{nm})\big). \tag{4.3}$$

But since $D^\alpha_{f,F}(E) \cap U \subseteq D^\alpha_{f,F}(\overline{E \cap U})$ for any open U and closed E, this implies

$$D^\alpha_{f,F}([a,b]) \subseteq K \cup \bigcup_{n,m} D^\alpha_{f,F}(\overline{(c_n,d_n) \cap (c_{nm},d_{nm})}), \tag{4.4}$$

and the latter summands are all empty by the hypothesis that $D^{\alpha}_{f,F}([c'_{nm}, d'_{nm}]) = \emptyset$; thus we obtain $D^{\alpha}_{f,F}([a,b]) \subseteq K$. From this and the fact that $f\chi_K \in L^1[a,b]$ and $F \in AC_*(K)$ it follows that $D^{\alpha+1}_{f,F}([a,b]) \subseteq D_{f,F}(K) = \emptyset$, which is what we wanted to establish. ■

Here is then the equivalent characterizations of Denjoy integration. It is perhaps worth noting explicitly the *absence* from this list of a condition related purely to the derivative D_f; indeed, one can show that the vanishing of this derivative does not in general suffice for being Denjoy integrable.

COROLLARY 4.4. *Let $f \in M[a,b]$ and let $F \in C[a,b]$ with $F(a) = 0$. Then the following are equivalent:*

(i) *$f \in \mathrm{Den}[a,b]$ and $F(x) = \int_a^x f$.*
(ii) *There is $\alpha < \omega_1$ such that $f \in \mathrm{Den}_\alpha[a,b]$ and $F(x) = \int_a^x f$.*
(iii) *There is $\alpha < \omega_1$ such that $D^{\alpha+1}_{f,F}([a,b]) = \emptyset$ and $F' = f$ a.e.*
(iv) *There is $\alpha < \omega_1$ such that $D^{\alpha+1}_{F}([a,b]) = \emptyset$ and $F' = f$ a.e.*

Proof. (i)⇒(ii). Suppose that $f \in \mathrm{Den}[a,b]$ with $F(x) = \int_a^x f$. Then by Proposition 3.1 there is $\alpha < \omega_1$ such that $D^{\alpha+1}_{f,F}([a,b]) = \emptyset$, and so by the left-to-right direction of the previous theorem, $f \in \mathrm{Den}_\alpha[a,b]$.

(ii)⇒(iii). Since f is in $\mathrm{Den}_\alpha[a,b]$, it is an element of $\mathrm{Den}[a,b]$, and hence $D^{\alpha+1}_{f,F}([a,b]) = \emptyset$ by the right-to-left direction of Theorem 4.3.

(iii)⇒(iv). Simply note that it follows from (3.10) that $D^{\beta}_{F}([a,b]) \subseteq D^{\beta}_{f,F}([a,b])$ for all ordinals β.

(iv)⇒(i). By the remark at the beginning of §3 (pertaining to the notation $\mathcal{B}_\sigma$), from $D^{\alpha+1}_{F}([a,b]) = \emptyset$ we can infer that there is a sequence $E_n \in K[a,b]$ such that $[a,b] = \bigcup_n E_n$ and $F \in AC_*(E_n)$, which is just the definition of $ACG_*([a,b])$. ■

5. The three derivatives are Borel. In this section, we undertake the analysis of the complexity of the notions related to the Denjoy integral which we have defined in the previous sections. So we build towards showing that the derivatives D_f, D_F, and $D_{f,F}$ from (3.8)–(3.10) are Borel maps in Propositions 5.2 and 5.4. Then, at the close of this section, we derive the main Theorems 1.6, 1.7, and 1.10.

Let us begin by taking a brief survey of the Polish spaces with which we shall be working. Recall that $K[a,b]$, the space of compact subsets of $[a,b]$, is a Polish space, where the topology is generated by the "miss" sets $\{K \in K[a,b] : K \cap U^c = \emptyset\}$ and the "hit" sets $\{K \in K[a,b] : K \cap U \neq \emptyset\}$, with $U \subseteq [a,b]$ open [Kec95, §4.F, pp. 24 ff]. Likewise, as mentioned in §1, the space $C[a,b]$ of continuous real-valued functions on $[a,b]$ is a Polish space, where the topology is given by the sup-metric $\|F - G\|_u = \sup\{x \in [a,b] : |F(x) - G(x)|\}$ [Kec95, §4.E, p. 24].

The Polish space structure on $M[a,b]$, the space of real-valued measurable functions on $[a,b]$ (where functions which are equal a.e. are identified), is less widely used. It is given by the metric $d(f,g) = \int_a^b \min(1, |f-g|)$, which has the effect that $f_n \to f$ in $M[a,b]$ if and only if $f_n \to f$ in measure, that is, $\lim_n \mu(\{x \in [a,b] : |f_n(x) - f(x)| > \epsilon\}) = 0$ for all $\epsilon > 0$. For the proof that it is a Polish space, see [Doo94, §§11–12, pp. 65–68] or [Ban87, p. 6]. Since we are dealing with measurable functions on the interval $[a,b]$ as opposed to the entire real line, it follows that addition and multiplication are continuous functions on $M[a,b]$ [Fol99, p. 63]. Further, absolute value is continuous on $M[a,b]$ since if $f_n \to f$ in measure, then $|f_n| \to |f|$ in measure because we have $\big\{x \in [a,b] : \big||f_n(x)| - |f(x)|\big| > \epsilon\big\} \subseteq \big\{x \in [a,b] : |f_n(x) - f(x)| > \epsilon\big\}$. Finally, recall that if $f_n \to f$ in measure, then $f_{n_k} \to f$ a.e. for some subsequence f_{n_k} of f_n [Fol99, Theorem 2.30, p. 61].

In what follows we will show that various maps are Borel, and it is helpful in this connection to recall that if Y is second-countable metrizable then a necessary and sufficient condition for $f : X \to Y$ to be Borel is for $f^{-1}(B(y,\epsilon))$ to be Borel for all $y \in \mathrm{Im}(f)$ and all $\epsilon > 0$.

We begin with arguments pertaining to the derivative map $K \mapsto D_f(K)$.

PROPOSITION 5.1.

(i) *The map* $E \mapsto \chi_E$ *from* $K[a,b]$ *into* $M[a,b]$ *is Borel.*
(ii) $L^1[a,b]$ *is Borel in* $M[a,b]$.

Proof. (i) It suffices to show that $\{D \in K[a,b] : d(\chi_D, \chi_E) < \epsilon\}$ is Borel. But

$$d(\chi_D, \chi_E) = \int_a^b |\chi_D - \chi_E| = \mu(D \triangle E) = \mu(D \cup E) - \mu(D \cap E),$$

and the maps $(D,E) \mapsto D \cup E$, $(D,E) \mapsto D \cap E$ and $E \mapsto \mu(E)$ are Borel [Kec95, pp. 27, 71 and 114].

(ii) Let $C_N = \{f \in L^1[a,b] : \int_a^b |f| \le N\}$ for each $N \ge 1$. Since $L^1[a,b]$ is the union of the C_N, it suffices to show that each C_N is closed in $M[a,b]$. So suppose that f_n is a sequence in C_N with $f_n \to f$ in measure. Since absolute value is continuous on $M[a,b]$, one then has $|f_n| \to |f|$ in measure. Hence some subsequence $|f_{n_k}|$ converges to $|f|$ a.e. Then by Fatou's Lemma, $\int_a^b |f| \le \liminf_k \int_a^b |f_{n_k}| \le N$, and so f is in C_N as well. ∎

PROPOSITION 5.2. *The map* $(f,K) \mapsto D_f(K)$ *is Borel from* $M[a,b] \times K[a,b]$ *to* $K[a,b]$.

Proof. It suffices to show that the graph $G = \{(f, K, E) \in M[a,b] \times (K[a,b])^2 : E = D_f(K)\}$ of the map is Borel. Since K, E are closed sets, $(f, K, E) \in G$ precisely when for all rationals $p < q$ one has

$$(p,q) \cap E = \emptyset \quad \text{iff} \quad (p,q) \cap D_f(K) = \emptyset.$$

But the left-hand side of this biconditional is Borel in $K[a,b]$ by definition of the topology on $K[a,b]$, while the right-hand side is Borel in $M[a,b] \times K[a,b]$ by Propositions 4.1(ii) and 5.1, and the fact that the map $(D, L) \mapsto D \cap L$ from $K[a,b] \times K[a,b]$ to $K[a,b]$ is Borel [Kec95, p. 71]. ∎

Let us turn now to the derivative $K \mapsto D_F(K)$ and show that it too is a Borel map. First let us note the following:

PROPOSITION 5.3. *The relation $F \in AC_*(E)$ is Borel on $C[a,b] \times K[a,b]$.*

Proof. Since F is continuous, we may replace E by a countable dense subset. But maps $d_n : K[a,b] \to [a,b]$ with $\{d_n(E) : n \geq 0\}$ dense in E for all $E \in K[a,b]$ may be chosen to be Borel [Kec95, Theorem 12.13, p. 76]. Moreover, consider the closed subset $\triangle = \{(c,d) \in \mathbb{R} \times \mathbb{R} : c \leq d\}$ which is thus a Polish space, and note that the map $(F, c, d) \mapsto \omega(F, [c,d])$ from $C([a,b]) \times \triangle$ to $\mathbb{R}$ is continuous.

Let $\epsilon, \delta > 0$ and let $\sigma = (n_1, m_1, \ldots, n_\ell, m_\ell)$ be a finite string of natural numbers of even length, and define $X_{\epsilon,\delta,\sigma}$ to be the set of pairs (F, E) such that if

$$d_{n_1}(E) < d_{m_1}(E) \leq d_{n_2}(E) < d_{m_2}(E) \leq \cdots \leq d_{n_\ell}(E) < d_{m_\ell}(E) \tag{5.1}$$

then

$$\sum_{i=1}^{\ell} (d_{m_i}(E) - d_{n_i}(E)) < \delta \;\Rightarrow\; \sum_{i=1}^{\ell} \omega(F, [d_{n_i}(E), d_{m_i}(E)]) < \epsilon. \tag{5.2}$$

Since the maps $E \mapsto d_n(E)$ and $(F, c, d) \mapsto \omega(F, [c,d])$ are Borel, the set $X_{\epsilon,\delta,\sigma}$ is Borel.

Further, $F \in AC_*(E)$ iff for all positive rational $\epsilon > 0$ there is a positive rational $\delta > 0$ such that $(F, E) \in X_{\epsilon,\delta,\sigma}$ for all finite strings σ of natural numbers of even length. Hence the relation $F \in AC_*(E)$ is Borel. ∎

PROPOSITION 5.4. *The map $(F, K) \mapsto D_F(K)$ is Borel from $C[a,b] \times K[a,b]$ to $K[a,b]$.*

Proof. It suffices to show that the graph $G = \{(F, K, E) \in C[a,b] \times (K[a,b])^2 : E = D_F(K)\}$ is Borel. But since K, E are closed sets, (F, K, E) is in G if and only if for all rationals $p < q$ we have

$$(p,q) \cap E = \emptyset \quad \text{iff} \quad (p,q) \cap D_F(K) = \emptyset.$$

But the left-hand side of this biconditional is Borel in $K[a,b]$ by definition of the topology on $K[a,b]$, while the right-hand side is Borel in $C[a,b] \times K[a,b]$

by Propositions 4.2(ii) and 5.3, and the fact that the map $(D, L) \mapsto D \cap L$ from $K[a, b] \times K[a, b]$ to $K[a, b]$ is Borel [Kec95, p. 71]. ■

Now, from Propositions 5.2 and 5.4, and the fact that $(D, E) \mapsto D \cup E$ is continuous [Kec95, p. 27], we conclude that the third derivative $D_{f,F}$ from (3.10) is also Borel.

Since the derivatives are Borel, we can deduce the following from [Kec95, Theorem 34.10 & p. 275]:

PROPOSITION 5.5. *The following sets are coanalytic, and* $|K|_f$, $|K|_F$, *and* $|K|_{f,F}$ *are coanalytic ranks on these sets:*

(i) $\{(f, K) \in M[a, b] \times K[a, b] : \exists \alpha < \omega_1\ D^\alpha_f(K) = \emptyset\}$,
(ii) $\{(F, K) \in C[a, b] \times K[a, b] : \exists \alpha < \omega_1\ D^\alpha_F(K) = \emptyset\}$,
(iii) $\{(f, F, K) \in M[a, b] \times C[a, b] \times K[a, b] : \exists \alpha < \omega_1\ D^\alpha_{f,F}(K) = \emptyset\}$.

Finally, before turning to the proofs of the main theorems, we need only verify that the partial operation of a.e. differentiation is Borel too:

PROPOSITION 5.6. *The class of* (F, f) *in* $C[a, b] \times M[a, b]$ *such that* F *is differentiable a.e. and* $F' = f$ *is Borel.*

Proof. Consider the function $\gamma : B \to M[a, b]$ given by $\gamma(F) = F'$ where $B = \{F \in C[a, b] : F' \text{ exists a.e.}\}$. Let us show that B and the graph of γ are Borel. To this end, let us define

$$E^r = \{(F, x) \in C[a, b] \times [a, b] : F'(x) \text{ exists } \& \ |F'(x)| > r\},$$
$$E = \bigcup_{r \in \mathbb{Q}} E^r = \{(F, x) \in C[a, b] \times [a, b] : F'(x) \text{ exists}\}.$$

Further, for $F \in C[a, b]$, $x \in [a, b]$ and $|h| > 0$, define

$$\triangle_{(F,x)}(h) = \frac{F(x + h) - F(x)}{h}.$$

Then E^r is analytic, since for $F \in C([a, b])$ we have $(F, x) \in E^r$ iff

$$\exists |L| > r\ \forall \epsilon \in \mathbb{Q}^+\ \exists \delta \in \mathbb{Q}^+\ \forall |h| \in \mathbb{Q} \cap (0, \delta) \quad |\triangle_{(F,x)}(h) - L| < \epsilon.$$

Likewise, E^r is coanalytic, since $(F, x) \in E^r$ iff

$$\begin{aligned} \forall h_n, h'_n \to 0\ [&\triangle_{(F,x)}(h_n), \triangle_{(F,x)}(h'_n) \text{ Cauchy} \\ &\& \lim_n |\triangle_{(F,x)}(h_n) - \triangle_{(F,x)}(h'_n)| = 0 \\ &\& (\exists q \in \mathbb{Q}^+\ \exists N\ \forall n > N\ |\triangle_{(F,x)}(h_n)| > r + q)]. \end{aligned}$$

So E^r is Borel and hence E too is Borel. Since E is Borel, the set $\{F \in C([a, b]) : \mu(E_F) = b - a\}$ is Borel by [Kec95, Theorem 17.25], where E_F denotes the projection $E_F = \{x \in [a, b] : (F, x) \in E\}$. But this set is precisely B, so B too is Borel.

Now let us show that the function $\gamma : B \to M[a,b]$ is Borel, where again $\gamma(F) = F'$. It suffices to show that for F in B, the following set is Borel:

$$\{G \in B : \mu(\{x \in [a,b] : F'(x), G'(x) \text{ exists } \ \& \ |F'(x) - G'(x)| > r\}) < r\}.$$

But this set is equal to $\{G \in B : \mu((E^r)_{F-G}) < r\}$, which is Borel since E^r is Borel. ∎

Now we turn to the proof of our main theorems:

THEOREM 1.10. *The graph of the indefinite Denjoy integral $f \mapsto \int_a^x f$, viewed as a subset of the product space $M[a,b] \times C[a,b]$, is coanalytic but not Borel.*

Proof. Equivalently, the claim is that the set of (f, F) in $M[a,b] \times C[a,b]$ such that $f \in \mathrm{Den}[a,b]$ and $F(x) = \int_a^x f$ is coanalytic but not Borel. The coanalyticity follows immediately from Propositions 5.5 and 5.6 and Corollary 4.4. That the set is not Borel follows from the fact that if the set is Borel then there is $\alpha < \omega_1$ such that $|f, F| \leq \alpha$ for all f, F in the set [Kec95, Theorem 35.23]. But this contradicts Theorem 3.2. ∎

THEOREM 1.6. *The set $ACG_*([a,b])$ is coanalytic but not Borel in $C[a,b]$.*

Proof. The coanalyticity follows from Proposition 5.5 and the observation that a function F is in $ACG_*([a,b])$ iff $[a,b] \in (\mathcal{B}_F)_\sigma$ iff there is $\alpha < \omega_1$ such that $D_F^{\alpha+1}([a,b]) = \emptyset$. (For the notation $(\mathcal{B}_F)_\sigma$, see the outset of §3). That the set is not Borel follows, as in the previous proof, from the fact that if the set is Borel then there is $\alpha < \omega_1$ such that $|F| \leq \alpha$ for all F in the set. But this again contradicts Theorem 3.2. ∎

THEOREM 1.7. *The set* $\mathrm{Den}[a,b]$ *of Denjoy integrable functions is a $\mathbf{\Sigma}^1_2$-subset of the Polish space $M[a,b]$ and is not analytic.*

Proof. The set $\mathrm{Den}[a,b]$ is $\mathbf{\Sigma}^1_2$ since it is the image of the coanalytic set $ACG_*([a,b])$ under the Borel function $F \mapsto \gamma(F)$, where $\gamma(F) = f$ if F' is differentiable a.e. and $F' = f$ a.e., and $\gamma(F) = 0$ otherwise (Proposition 5.6). Suppose now that $\mathrm{Den}[a,b]$ is analytic. Theorem 1.5 shows that $F \in ACG_*([a,b])$ iff there is $f \in \mathrm{Den}[a,b]$ such that $F' = f$ a.e. Since this last condition is Borel (Proposition 5.6), it follows that $ACG_*([a,b])$ would be analytic, which contradicts Theorem 1.6. ∎

Again, for the obvious questions about how to sharpen these results, see the discussion in §8.

6. Indices of subgroups and stability. In this section, we begin our study of $\mathrm{Den}[a,b]$ from the perspective of model theory, where we view $\mathrm{Den}[a,b]$ as a $\mathbb{Q}[X]$-module (resp. $\mathbb{R}[X]$-module) and where we interpret the map $f \mapsto Xf$ as the indefinite integral, so that $Xf = \int_a^x f$. It is also natural to consider various submodules like $C[a,b]$ and $L^1[a,b]$, where the integrals are respectively the Riemann and Lebesgue integrals. Further, our results also hold for a broad class of submodules of $\mathrm{Den}[a,b]$. If $\mathcal{X}$ is a subset of an R-module M, then let $\langle \mathcal{X} \rangle$ be the R-submodule of M generated by $\mathcal{X}$. Our results hold in particular for the submodules $\langle \mathrm{Den}_\alpha[a,b] \rangle$ of $\mathrm{Den}[a,b]$ (Definition 2.5).

Recall that the signature of R-modules is simply the signature of abelian groups equipped with linear maps r for each element r of R. Hence, e.g., the signature of $\mathbb{R}[X]$-modules is uncountable, whereas the signature of $\mathbb{Q}[X]$-modules is countable. Likewise, since elements r of R correspond to linear maps in an R-module M, subsets of M such as $rM = \{ra : a \in M\}$ and $\ker(r) = \{a \in M : ra = 0\}$ are definable without parameters in M.

We begin with a theorem on indices of subgroups which is important for the derivation of Theorem 1.11 given in the next section. Recall from the end of §2 that M is subinterval-closed if whenever $f \in M$ then $f\chi_{(c,d)} \in M$ for any interval (c,d).

THEOREM 6.1. *Suppose that M is a submodule of* $\mathrm{Den}[a,b]$ *which contains $C[a,b]$. Suppose further that* (i) $M = C[a,b]$ *or* (ii) *M is subinterval-closed. Then $[X^k M : X^{k+1} M]$ is infinite.*

Proof. First we show this for M satisfying hypothesis (i). For each $f \in M$ we may choose $g \in C[a,b]$ such that $f = g$ a.e., and so M may be identified with $C[a,b]$. Then for $k \geq 0$ we have

$$X^k M = \{f \in C^k[a,b] : \forall i < k \ f^{(i)}(a) = 0\} \tag{6.1}$$

where we stipulate $X^0 M = M$ and $C^0[a,b] = C[a,b]$. Indeed, for $k = 0$, this follows by the stipulation. Suppose that (6.1) holds for k. To see it holds for $k+1$, consider first the left-to-right containment. Suppose that $f \in X^{k+1}M$. Then $f = \int_a^x g$ where $g \in X^k M \subseteq M = C[a,b]$. Since this is the Riemann integral applied to a continuous function, it follows that f is differentiable everywhere and $f' = g$. Then for $i = 0$, one has $f^{(i)}(a) = f(a) = \int_a^a g = 0$, while for $0 < i < k+1$, one has $i - 1 < k$ and $f^{(i)}(a) = g^{(i-1)}(a) = 0$ by induction hypothesis. For the right-to-left containment of (6.1), suppose that $f \in C^{k+1}[a,b]$ and $f^{(i)}(a) = 0$ for all $i < k+1$. Let $g = f'$, which by hypothesis is in $C[a,b] = M$. Then by induction hypothesis, $g \in X^k M \subseteq M = C[a,b]$, so that from $\int_a^x g = \int_a^x f' = f(x) - f(a) = f(x)$ we may infer $f \in X^{k+1}M$. Hence, in fact (6.1) holds for all $k \geq 0$.

Now $C^k[a,b]$ is a Banach space with norm $\|f\|_{u,k} = \sum_{0 \leq i \leq k} \|f^{(i)}\|_u$ where $\|\cdot\|_u$ is the sup-norm on $C[a,b]$ [Fol99, p. 155]. From this and (6.1) it follows that $X^k M$ is a closed subgroup of $C^k[a,b]$, and hence a Polish group. Now, for all $k \geq 0$, $X^k M$ and $X^{k+1} M$ are homeomorphic via the map $f \mapsto Xf$. By induction on $k \geq 0$, we prove that $X^{k+1}M$ is meager in $X^k M$. For $k = 0$, $XM = XC[a,b]$ is meager in $M = C[a,b]$ since the nowhere differentiable functions are comeager in M and contained in $M \setminus XM$. Suppose now that $X^{k+1}M$ is meager in X^kM. Since meagerness is preserved under homeomorphisms, it follows that $X^{k+2}M$ is meager in $X^{k+1}M$, as desired.

Hence $[X^kM : X^{k+1}M]$ is infinite, and indeed uncountable. For suppose that $[X^kM : X^{k+1}M]$ were countable. Then $X^kM = \bigsqcup_n g_n + X^{k+1}M$, where $g_n \in X^kM$. Since X^kM is a Polish group and each $X^{k+1}M$ is meager in X^kM, we infer that each $g_n + X^{k+1}M$ is meager in X^kM. Hence, the Polish space X^kM would be a countable union of meager subsets, contradicting the Baire Category Theorem.

Now we show the result for M satisfying hypothesis (ii). Suppose that $[X^kM : X^{k+1}M]$ is finite. Then $X^kM = \bigsqcup_{i=1}^n X^k f_i + X^{k+1}M$, where $f_i \in M$. Choose a continuous nowhere differentiable function $g \in C[a,b] \subseteq M$ and a partition $[a,b] = [a_1,b_1] \sqcup \cdots \sqcup [a_n,b_n]$, and let $h = X^k[g + \sum_{i=1}^n f_i \chi_{[a_i,b_i]}]$, which is in X^kM since M is subinterval-closed. So, by hypothesis, there is $j \in [1,n]$ such that $h \in X^k f_j + X^{k+1}M$. Then

$$X^k\Big[g + \Big(\sum_{i=1}^n f_i \chi_{[a_i,b_i]}\Big) - f_j\Big] = h - X^k f_j \in X^{k+1}M.$$

Hence $g + (\sum_{i=1}^n f_i \chi_{[a_i,b_i]}) - f_j \in XM$. But this function is differentiable a.e., and so differentiable a.e. on each $[a_i,b_i]$. But on $[a_j,b_j]$, this function is equal to g, which contradicts the choice of g. ∎

Let us note an immediate consequence of this theorem for the model-theoretic complexity of Denjoy integration. The underlying vector space of $\mathrm{Den}[a,b]$ is model-theoretically a very well understood object and is stable and indeed superstable. By contrast, the next corollary tells us that the addition of the integral adds to the complexity of $\mathrm{Den}[a,b]$:

COROLLARY 6.2. *Suppose that M is a submodule of* $\mathrm{Den}[a,b]$ *which contains $C[a,b]$. Suppose further that* (i) $M = C[a,b]$ *or* (ii) *M is subinterval-closed. Then M is stable but not superstable.*

Proof. It is a classical result that all modules are stable [Pre88, Theorem 3.1(a), p. 55]. Further, a module M is superstable if and only if there is no infinite descending sequence of definable subgroups, each of infinite index in its predecessor ([Pre88, Theorem 3.1(b), p. 55] or [Zie84, Theorem 2.1, p. 156]). But the previous theorem tells us that there are such sequences of subgroups in this case. ∎

7. Elementary equivalence and decidability. The aim of this section is to establish our Theorem 1.11. To do this, we must first recall some basic facts from the model theory of modules.

If M is a module over a ring R, then a *pp-formula* $\varphi(x_1, \dots, x_j)$ is a formula of the form $\exists y_1, \dots, y_k \ \bigwedge_{i=1}^{n} \varphi_i(x_1, \dots, x_j, y_1, \dots, y_k)$ where φ_i is an atomic formula without parameters. Any subset $G \subseteq M^j$ defined by a pp-formula is a subgroup of M^j, and the *invariant sentences* of $\mathrm{Th}(M)$ are of the form $[G : G \cap H] = k$ or $[G : G \cap H] > k$, where $k \geq 0$ and $G, H \subseteq M$ are pp-definable subgroups of M. The following theorem then tells us that the invariant sentences determine the complete first-order theory of the module:

THEOREM 7.1 (pp-Elimination of Quantifiers, [Pre88, Corollaries 2.16 & 2.19, p. 37] or [Hod93, p. 655]).

(i) *Every set definable without parameters in an R-module M is a Boolean combination of pp-definable sets.*
(ii) *For an R-module M, the theory* $\mathrm{Th}(M)$ *is axiomatized by the R-module axioms and the invariant sentences of M.*

The main idea of the proof of Theorem 1.11 is to isolate the invariant sentences in the modules related to the Denjoy integral, which we do in Corollary 7.6.

To this end, it will be helpful to briefly take note of some special cases of pp-formulas. A pp-formula $\varphi(x_1, \dots, x_j)$ is said to be *basic* if it can be written as $\sum_{\ell=1}^{j} r_\ell x_\ell = 0$ or $\exists y \ (\sum_{\ell=1}^{j} r_\ell x_\ell) + sy = 0$. That is, over an R-module M, the basic pp-formula definable sets are $\ker(\overline{r})$ or $\overline{r}^{-1} sM$. In this section, if $\overline{r} = (r_1, \dots, r_j)$ is a tuple of ring elements, we will write $\overline{r}^{-1} Y$ to denote the inverse image of Y under the map $\overline{r} : M^j \to M$ given by $\overline{x} \mapsto \sum_{\ell=1}^{j} r_\ell x_\ell$. This notation ought not be confused with multiplicative inverses in the ring. Further, if R is a PID, then every pp-formula formula is equivalent to a finite conjunction of basic pp-formulas, and if R is countable, then given a pp-formula one can compute from R the finite conjunction of basic pp-formulas [Pre88, Theorem 2.$\mathbb{Z}$1, pp. 46–47]. Finally, if M is a module over a commutative ring R, then multiplication by an element of R is a homomorphism of M, and when the map is bijective, it is an automorphism of M.

To calculate the pp-definable subgroups of modules related to the Denjoy integral, we briefly recall some elements of the Riesz theory of integral equations.

Suppose that M is a normed space. Then a *compact linear operator* $q : M \to M$ is a linear operator which maps bounded sets to sets with compact closure. The Riesz Theorem says that if M is a normed space and $q : M \to M$ is a compact linear operator, then the map $1 + q$ is surjective if and only if $1 + q$ is injective [Kre99, pp. 29–30]. Using this theorem we can show:

Proposition 7.2. *Suppose that* $p \in \mathbb{R}[X]$ *such that* $X \nmid p$. *Then* $p : C[a,b] \to C[a,b]$ *is an automorphism of the* $\mathbb{R}[X]$*-module* $C[a,b]$.

Proof. It suffices to show that it is a bijection. Since $X \nmid p$, we may without loss of generality write $p = 1 + a_1 X + \cdots + a_k X^k$. Note that by the Arzelà–Ascoli Theorem, $p - 1$ is a compact linear operator. Then by the Riesz Theorem, it suffices to show that p is injective, or what is the same, that the only solution to $p \cdot f = 0$ is $f = 0$. For this, it in turn suffices to show that any solution to $p \cdot f = 0$ would be a solution to a certain higher-order differential equation which has only one solution, namely $f = 0$.

Indeed, suppose $p \cdot f = 0$, or explicitly $f + a_1 Xf + \cdots + a_k X^k f = 0$. First, since f is in $C[a,b]$ and $Xf = \int_a^x f$, it follows that $f(a) = 0$. Second, if f satisfies this equation then it is differentiable, and by differentiating we obtain $f' + a_1 f + \cdots + a_k X^{k-1} f = 0$. Iterating this an additional $k - 1$ more times shows that f satisfies the initial value problem $f(a) = f'(a) = \cdots = f^{(k-1)}(a) = 0$ and $f^{(k)} + a_1 f^{(k-1)} + \cdots + a_k f = 0$. Then by the uniqueness theorems for this higher-order differential equation, any solution f to this equation is equal to zero, which is what we wanted to establish. ∎

The following trick of lifting the Riesz theory to $\mathrm{Den}[a,b]$ is from [FB02, proof of Theorem 3.10, pp. 103 ff], although Federson and Bianconi restrict themselves to the case of $\mathrm{Den}[a,b]$ and do not frame this in the language of modules.

Proposition 7.3. *Suppose that* M *is a submodule of* $\mathrm{Den}[a,b]$ *which contains* $C[a,b]$. *Suppose that* $p \in \mathbb{R}[X]$ *is such that* $X \nmid p$. *Then* $p : M \to M$ *is an automorphism of the* $\mathbb{R}[X]$*-module* M.

Proof. Again, it suffices to show that p is bijective. And again, we may assume that $p = 1 + a_1 X + \cdots + a_k X^k$. To see that p is injective, note that if $pf = 0$ then $f = -a_1 Xf - \cdots - a_k X^k f$. Since $XM \subseteq C[a,b]$, we have $f \in C[a,b]$ and $pf = 0$ in $C[a,b]$. But by the previous proposition, $p : C[a,b] \to C[a,b]$ is an injection, and hence $f = 0$.

To see that $p : M \to M$ is a surjection, pick $g \in M$. Since $XM \subseteq C[a,b]$, we have $(p-1)g \in C[a,b]$, and hence $-(p-1)g \in C[a,b]$. By the previous proposition, $p : C[a,b] \to C[a,b]$ is a surjection, and hence there is $f \in C[a,b]$ such that $pf = -(p-1)g$. Then $p(f+g) = g$. ∎

In the statement below, recall the convention that p^{-1} denotes inverse image and not multiplicative inverse.

Proposition 7.4. *Suppose that* M *is a submodule of* $\mathrm{Den}[a,b]$ *which contains* $C[a,b]$. *Suppose further that* $p, q \in \mathbb{R}[X]$. *Then* $p^{-1}qM$ *is either* M *or* $X^\ell M$ *for some* $\ell > 0$. *Further, there is a computable procedure which*

(i) *given* $p, q \in \mathbb{Q}[X]$ *determines which of these occurs, and* (ii) *returns* $\ell > 0$ *if the latter occurs.*

Proof. Compute the largest k such that X^k divides both p and q. Let $p = X^k p_0$ and $q = X^k q_0$. Then $p^{-1}qM = p_0^{-1}q_0M$ since $pf + qg = 0$ if and only if $X^k(p_0f + q_0g) = 0$ if and only if $p_0f + q_0g = 0$. Now either $X \mid q_0$ or $X \nmid q_0$, and we can compute which of these occurs.

If $X \mid q_0$ then by definition of k we have $X \nmid p_0$, and so p_0 is an automorphism of M as a $\mathbb{R}[X]$-module. Further, if $X \mid q_0$ then compute the largest $\ell > 0$ such that $X^\ell \mid q_0$. Let $q_0 = X^\ell q_1$, where $X \nmid q_1$. Then q_1 is an automorphism of M as an $\mathbb{R}[X]$-module. Consequently,

$$p^{-1}qM = p_0^{-1}q_0M = p_0^{-1}X^\ell q_1M = p_0^{-1}X^\ell M = X^\ell M, \tag{7.1}$$

where the last equality is due to the fact that automorphisms fix definable sets. On the other hand, suppose that $X \nmid q_0$. Then q_0 is an automorphism of M as an $\mathbb{R}[X]$-module. Then $p^{-1}qM = p_0^{-1}q_0M = p_0^{-1}M = M$. ∎

Proposition 7.5. *Suppose that* M *is a submodule of* $\mathrm{Den}[a,b]$ *which contains* $C[a,b]$. *Suppose further that* $p \in \mathbb{R}[X]$. *Then* $\ker(p)$ *is either* 0 *or* M. *Further, there is a computable procedure which given* $p \in \mathbb{Q}[X]$ *determines which of these occurs.*

Proof. If p is zero then $\ker(p) = M$, and we can compute whether this occurs. If p is non-zero, then compute the largest k such that X^k divides p. Let $p = X^k p_0$. Then $\ker(p) = \ker(p_0)$ since $X^k p_0 f = 0$ if and only if $p_0 f = 0$. Thus $X \nmid p_0$, hence p_0 is an automorphism of M as an $\mathbb{R}[X]$-module, and so $\ker(p_0) = 0$. ∎

Corollary 7.6. *Suppose that* M *is a submodule of* $\mathrm{Den}[a,b]$ *which contains* $C[a,b]$. *Suppose further that* (i) $M = C[a,b]$ *or* (ii) M *is subinterval-closed. Suppose finally that* G, H *are pp-definable subgroups of* M. *Then* $[G : G \cap H] = 1$ *or* $[G : G \cap H]$ *infinite, and from formulas defining* G *and* H *we can compute which of these occurs. Further, this procedure is uniform in such* M, *in that formulas for* G *and* H *will return the same values for* $[G : G \cap H]$ *for all such* M.

Proof. By the previous two propositions, G and H are finite conjunctions of the subgroups 0, $X^\ell M$, and M, and hence themselves are among the subgroups 0, $X^\ell M$, and M. Further by those propositions, given formulas defining G we can computably determine whether G (resp. H) is 0, $X^\ell M$, or M. So there are nine possible cases to consider. The cases in which 0 occurs are trivial, and so there are really only four interesting cases. Case one: $G = M$ and $H = M$. Then $[G : G \cap H] = 1$. Case two: $G = M$ and $H = X^k M$. Then $[G : G \cap H]$ is infinite by Theorem 6.1. Case three: $G = X^\ell M$ and $H = M$. Then $[G : G \cap H] = 1$. Case four: $G = X^\ell M$ and

$H = X^k M$. Then $[G : G \cap H] = 1$ if $\ell \geq k$ and $[G : G \cap H]$ is infinite if $\ell < k$ by Proposition 6.1. ■

From this corollary and the fact mentioned at the outset of this section (Theorem 7.1) that the invariant sentences determine the complete theory of a module, we can immediately deduce Theorem 1.11.

8. Further questions. In addition to Questions 1.8–1.9 mentioned in §1, a couple of other questions are left open by our study:

QUESTION 8.1. In Theorem 1.7, it was shown that $\mathrm{Den}[a,b]$ is a $\mathbf{\Sigma}^1_2$-definable non-analytic subset of $M[a,b]$. Can it be shown that $\mathrm{Den}[a,b]$ is not coanalytic? If it is not coanalytic, can it be shown that it is not Δ^1_2?

QUESTION 8.2. In Theorems 1.6 and 1.10, certain sets are shown to be coanalytic but not Borel. Can one show that these sets are coanalytic complete?

QUESTION 8.3. We showed that $C[a,b]$, $L^1[a,b]$, $\langle \mathrm{Den}_\alpha[a,b]\rangle$, and $\mathrm{Den}[a,b]$ are elementarily equivalent as $\mathbb{R}[X]$-modules (or $\mathbb{Q}[X]$-modules). Are they non-isomorphic in this signature? Obviously, the elementary equivalence result all by itself—in abstraction from the non-superstability and decidability results—would be less interesting if it turned out that they were all isomorphic.

QUESTION 8.4. Do the non-superstability, elementary equivalence, and decidability results from the last sections still hold if one views $C[a,b]$, $L^1[a,b]$, $\langle \mathrm{Den}_\alpha[a,b]\rangle$, and $\mathrm{Den}[a,b]$ as $\mathbb{R}[X]$- or $\mathbb{Q}[X]$-modules, where alternatively $Xf \mapsto \int_a^b K(x,y)f(y)\,dy$ for some appropriate real-valued continuous function $K(x,y)$? Note that some care has to be exercised with respect to the choice of K, since $\mathrm{Den}[a,b]$ is not closed under multiplication [Swa01, Example 14, p. 43].

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Sean Walsh
Department of Logic and Philosophy of Science
University of California, Irvine
5100 Social Science Plaza
Irvine, CA 92697-5100, U.S.A.
E-mail: swalsh108@gmail.com, walsh108@uci.edu