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Exponential sums involving the Möbius function

by

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1. Introduction and statement of results. Let $\mu(n)$ be the Möbius function, $e(x) = e^{2\pi i x}$, $k \geq 1$ an integer, and x real. The exponential sum

$$S_k(x,\alpha) = \sum_{n\leq x} \mu(n)e(n^k\alpha) \tag{1.1}$$

was first estimated by Davenport [2] in 1937 by using Vinogradov's elementary method. He proved that

$$\max_{\alpha\in[0,1]} |S_1(x,\alpha)| \ll_A x(\log x)^{-A} \quad \text{for any } A > 0. \tag{1.2}$$

Here and below, $\ll_A$ indicates that the implied constant depends at most on A. For $k \geq 2$, Hua [4] proved that

$$\max_{\alpha\in[0,1]} |S_k(x,\alpha)| \ll_A x(\log x)^{-A} \quad \text{for any } A > 0.$$

Now we consider estimating exponential sums under the following weak Generalized Riemann Hypothesis (briefly GRH): for some $0 \leq \delta < 1/2$ and every Dirichlet character χ,

$$L(s,\chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s} \text{ has no zeros in the half-plane } \sigma > 1/2 + \delta, \tag{1.3}$$

where $s = \sigma + it$. For $k = 1$, the best result in this direction is due to Baker and Harman [1], who showed in 1991 that for any $\varepsilon > 0$,

$$\max_{\alpha\in[0,1]} |S_1(x,\alpha)| \ll_\varepsilon x^{b+\varepsilon}, \tag{1.4}$$

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where

$$b = \begin{cases} \delta + 3/4 & \text{for } 0 \le \delta < 1/20, \\ 4/5 & \text{for } 1/20 \le \delta < 1/10, \\ \delta/2 + 3/4 & \text{for } 1/10 \le \delta < 1/2. \end{cases}$$

For $k \ge 2$, Liu and Zhan [7] proved that for any $k \ge 2$, $\varepsilon > 0$, under GRH,

$$\max_{\alpha \in [0,1]} |S_k(x, \alpha)| \ll_\varepsilon x^{\varphi_k + \varepsilon}, \quad \text{where} \quad \varphi_k = 1 - \frac{1}{2^{2k-1}}.$$

In this paper we combine the results of Ren [8], Kumchev [6], Wooley [11] and Zhao [12] to improve the result of Liu and Zhan [7] when $k \ge 3$. Our main result is the following.

THEOREM 1.1. *For any $k \ge 3$ and $\varepsilon > 0$, under weak GRH, we have*

$$\max_{\alpha \in [0,1]} |S_k(x, \alpha)| \ll_\varepsilon x^{b_k + \varepsilon},$$

where

$$b_k = \begin{cases} 1 - \rho_k + \varepsilon & \text{if } 0 \le \delta < 1/2 - k\rho_k, \\ 1 - \frac{1}{2k}(1 - 2\delta) + \varepsilon & \text{if } 1/2 - k\rho_k \le \delta < 1/2 - 2\rho_k, \\ 1 - \frac{1-2\delta}{2^{2k-1}} + \varepsilon & \text{if } 1/2 - 2\rho_k \le \delta < 1/2, \end{cases} \tag{1.5}$$

and

$$\rho_k = \begin{cases} \frac{1}{3 \cdot 2^{k-1}} & \text{if } 3 \le k \le 7, \\ \frac{1}{6k(k-2)} & \text{if } k \ge 8, \end{cases} \tag{1.6}$$

REMARK 1.2. When $0 \le \delta < 1/2 - k\rho_k$, the upper bound of $S_k(x, \alpha)$ is independent of δ. In particular, when $\delta = 0$, we get, under GRH,

$$\max_{\alpha \in [0,1]} |S_k(x, \alpha)| \ll_\varepsilon x^{\phi_k + \varepsilon}, \quad \text{where} \quad \phi_k = \begin{cases} 1 - \frac{1}{3 \cdot 2^{k-1}} & \text{if } 3 \le k \le 7, \\ 1 - \frac{1}{6k(k-2)} & \text{if } k \ge 8; \end{cases}$$

this improves the result of Liu and Zhan [7] when $k \ge 3$. For $k = 2$, we can get a similar result, but comparing it with Liu and Zhan's result we find that we cannot do better in this case.

NOTATION. Throughout the paper, ε denotes a small positive real number, which may be different at each occurrence. For example, we may write

$$x^\varepsilon x^\varepsilon \ll x^\varepsilon.$$

Any statement in which ε occurs holds for each positive ε, and any implied constant in such a statement is allowed to depend on ε. The letter p, with or without subscripts, is reserved for prime numbers. We write $(a, b) = \gcd(a, b)$, and we use $m \sim M$ as an abbreviation for $M < m \le 2M$.

2. Outline of the proof. Take

$$P_1 = x^{1/2-\delta}, \quad P_2 = x^{1/2+\delta}, \quad Q = x^{k+\delta-1/2}.$$

Since $S_k(x,\alpha)$ is of period 1 with respect to α, we only need to consider $\alpha \in [1/Q, 1+1/Q]$. By Dirichlet's lemma on rational approximations, for each $\alpha \in [1/Q, 1+1/Q]$, we can write

$$\alpha = \frac{a}{q} + \lambda \text{ with } (a,q)=1,\ 1 \le a \le q,\ 1 \le q \le Q,\ |\lambda| \le \frac{1}{qQ}. \tag{2.1}$$

So $[1/Q, 1+1/Q]$ can be divided into three disjoint sets

$$E_1 = \left\{\alpha;\ \alpha = \frac{a}{q} + \lambda,\ (a,q)=1,\ 1 \le q \le P_1,\ |\lambda| \le \frac{1}{qQ}\right\},$$

$$E_2 = \left\{\alpha;\ \alpha = \frac{a}{q} + \lambda,\ (a,q)=1,\ P_1 < q \le P_2,\ |\lambda| \le \frac{1}{qQ}\right\},$$

$$E_3 = \left\{\alpha;\ \alpha = \frac{a}{q} + \lambda,\ (a,q)=1,\ P_2 < q \le Q,\ |\lambda| \le \frac{1}{qQ}\right\}.$$

We have the following three propositions, which easily imply Theorem 1.1.

PROPOSITION 2.1. *Assume weak GRH and $k \ge 3$. Then*

$$\max_{\alpha \in E_1} |S_k(x,\alpha)| \ll x^{1-\frac{1}{2k}(1-2\delta)+\varepsilon}. \tag{2.2}$$

PROPOSITION 2.2. *Assume weak GRH and $k \ge 3$. Then*

$$\max_{\alpha \in E_2} |S_k(x,\alpha)| \ll x^{c+\varepsilon}, \tag{2.3}$$

where

$$c = \begin{cases} 4/5 & \text{if } 0 \le \delta < 1/10, \\ 3/4 + \delta/2 & \text{if } 1/10 \le \delta < 1/2. \end{cases}$$

REMARK 2.3. When $\alpha \in E_2$, the upper bound of $S_k(x,\alpha)$ is independent of k.

PROPOSITION 2.4. *Assume weak GRH and $k \ge 3$. Then*

$$\max_{\alpha \in E_3} |S_k(x,\alpha)| \ll x^{d_k+\varepsilon}, \tag{2.4}$$

where

$$d_k = \begin{cases} 1 - \rho_k & \text{if } 0 \le \delta < 1/2 - 2\rho_k, \\ 1 - \frac{1-2\delta}{2^{2k-1}} & \text{if } 1/2 - 2\rho_k \le \delta < 1/2, \end{cases}$$

and ρ_k is defined in (1.6).

3. Proof of Proposition 2.1. To prove Proposition 2.1, we use analytic methods. We need the following lemmas.

Lemma 3.1 (see [7, Lemma 2]). *Let $k \ge 3$ and $\alpha = a/q+\lambda$ with $(a,q)=1$. Then for any $\varepsilon > 0$,*

$$S_k(x,\alpha) \ll q^{\eta_k+\varepsilon} \sum_{d|q} \max_{\chi_{q/d}} \Big| \sum_{\substack{m\le x/d \\ (m,q)=1}} \mu(m)\chi(m)e(m^k d^k \lambda)\Big|,$$

where $\eta_k = 1 - 1/k$. ∎

Lemma 3.2 (see [10, Theorem 14.2]). *Under weak GRH, we have*

$$L^{-1}(\sigma+it,\chi) \ll q^{\varepsilon}(|t|+1)^{\varepsilon}$$

for $\sigma \ge 1/2+\delta+\varepsilon$ and every Dirichlet character χ (mod q). ∎

Lemma 3.3. *Assume weak GRH and $k \ge 3$. Then*

$$S_k(x,\alpha) \ll q^{\eta_k} x^{1/2+\delta+\varepsilon}(1+|\lambda|^{1/2}x^{k/2}), \tag{3.1}$$

where η_k is defined in Lemma 3.1.

Proof. By Lemma 3.1 we know that the conclusion will follow if we can prove that for any $\varepsilon > 0$ and $d \mid q$, $1 < H \le x/d$,

$$\sum_{\substack{m\sim H \\ (m,q)=1}} \mu(m)\chi(m)e(m^k d^k\lambda) \ll d^{-1/2-\delta}x^{1/2+\delta+\varepsilon}(1+|\lambda|^{1/2}x^{k/2}), \tag{3.2}$$

uniformly for all $\chi = \chi_{q/d}$.

Let I_1 denote the left-hand side of (3.2), and

$$F(s,\chi) = F_q(s,\chi) = \sum_{\substack{m=1 \\ (m,q)=1}}^{\infty} \mu(m)\chi(m)m^{-s}, \qquad \sigma > 1,$$

$$H(s,\chi) = H_q(s,\chi) = \prod_{p|q}\Big(1 - \frac{\chi(p)}{p^s}\Big)^{-1}.$$

Then

$$F(s,\chi) = L^{-1}(s,\chi)H(s,\chi). \tag{3.3}$$

By (3.3) we know that under weak GRH the function $F(s,\chi)$ is analytic in the region $\mathrm{Re}(s) \ge 1/2+\delta+\varepsilon$ for any $\varepsilon > 0$. Furthermore,

$$H(s,\chi) \ll \prod_{p|q}\Big(1 - \frac{1}{\sqrt{p}}\Big)^{-1} \ll q^{\varepsilon}, \qquad \mathrm{Re}(s) \ge 1/2+\delta+\varepsilon. \tag{3.4}$$

By Perron's summation formula, for $u \le x$ we have

$$\sum_{\substack{m\le u \\ (m,q)=1}} \mu(m)\chi(m) = \frac{1}{2\pi i}\int_{1+\varepsilon-iT}^{1+\varepsilon+iT} F(s,\chi)\frac{u^s}{s}\,ds + O\Big(\frac{x^{1+\varepsilon}}{T} + \log x\Big).$$

Take $T = x^k$ and shift the path of integration above to $\mathrm{Re}(s) = 1/2 + \delta + \varepsilon$:

$$\sum_{\substack{m\le u\\(m,q)=1}} \mu(m)\chi(m) = \frac{1}{2\pi}\int_{-x^k}^{x^k} F(1/2+\delta+\varepsilon+it,\chi)\frac{u^{1/2+\delta+\varepsilon+it}}{1/2+\delta+\varepsilon+it}\,dt + O(x^\varepsilon).$$

Then

$$\begin{aligned}
I_1 &= \int_{H/2}^{H} e(d^k u^k\lambda)\,d\Big(\sum_{\substack{m\le u\\(m,q)=1}} \mu(m)\chi(m)\Big)\\
&= \frac{1}{2\pi}\int_{-(dH)^k}^{(dH)^k} F(1/2+\delta+\varepsilon+it,\chi)\\
&\quad\times\int_{H/2}^{H} u^{-1/2+\delta+\varepsilon/2} e\Big(d^k u^k\lambda + \frac{t}{2\pi}\log u\Big)\,du\,dt + O(|\lambda|(dH)^{k+\varepsilon}+x^\varepsilon)\\
&\ll d^{-1/2-\delta}\int_{-(dH)^k}^{(dH)^k} |F(1/2+\delta+\varepsilon+it,\chi)|\\
&\quad\times\Big|\int_{(dH)^k/2^k}^{(dH)^k} v^{-1+1/(2k)+\delta/k+\varepsilon/(2k)} e\Big(v\lambda + \frac{t}{2k\pi}\log v\Big)\,dv\Big|\,dt\\
&\quad+ O(|\lambda|(dH)^{k+\varepsilon}+x^\varepsilon).
\end{aligned}$$

Since

$$\Big(v\lambda + \frac{t}{2k\pi}\log v\Big)' = \frac{t+2k\pi\lambda v}{2k\pi v} \gg \frac{\min_{(dH)^k/2^k\le v\le (dH)^k}|t+2k\pi\lambda v|}{(dH)^k},$$

$$-\Big(v\lambda + \frac{t}{2k\pi}\log v\Big)'' = \frac{t}{2k\pi v^2} \gg \frac{|t|}{(dH)^{2k}},$$

by Lemma 3.2 and (3.4), we get

$$\begin{aligned}
I_1 &\ll H^{1/2+\delta+\varepsilon}\int_{-(dH)^k}^{(dH)^k} |F(1/2+\delta+\varepsilon+it,\chi)|\\
&\quad\times\min\Big(\frac{1}{\sqrt{|t|+1}}, \frac{1}{\min_{(dH)^k/2^k\le v\le (dH)^k}|t+2k\pi\lambda v|}\Big)\,dt + O(|\lambda|(dH)^{k+\varepsilon}+x^\varepsilon)\\
&\ll H^{1/2+\delta+\varepsilon}\int_{-(dH)^k}^{(dH)^k} \min\Big(\frac{1}{\sqrt{|t|+1}}, \frac{1}{\min_{(dH)^k/2^k\le v\le (dH)^k}|t+2k\pi\lambda v|}\Big)\,dt\\
&\quad+ O(|\lambda|(dH)^{k+\varepsilon}+x^\varepsilon).
\end{aligned}$$

On noting that

$$|\lambda|(dH)^k \le d^{-1/2}|\lambda|^{1/2}(dH)^{(k+1)/2},$$

it suffices now to show that

$$(3.5)\qquad \int_{-(dH)^k}^{(dH)^k} \min\biggl(\frac{1}{\sqrt{|t|+1}}, \frac{1}{\min_{(dH)^k/2^k\le v\le (dH)^k}|t+2k\pi\lambda v|}\biggr)\,dt \ll (1+|\lambda|^{1/2}(dH)^{k/2})\log x.$$

Denote by I_2 the left-hand side of (3.5). If $|\lambda| > (dH)^{-k}$, then

$$\begin{aligned} I_2 &\ll \int_{|t|\le 2^{-k}\pi|\lambda|(dH)^k} \frac{dt}{|\lambda|(dH)^k} + \int_{4k\pi|\lambda|(dH)^k<|t|\le (dH)^k} \frac{dt}{|t|} \\ &\quad + \int_{2^{-k}\pi|\lambda|(dH)^k<|t|\le 4k\pi|\lambda|(dH)^k} \frac{dt}{\sqrt{|t|+1}} \\ &\ll \log x + |\lambda|^{1/2}(dH)^{k/2}. \end{aligned}$$

If $|\lambda| \le (dH)^{-k}$, we have

$$I_2 \ll \int_{|t|\le 4k\pi} 1dt + \int_{4k\pi<|t|\le (dH)^k} \frac{dt}{|t|} \ll \log x.$$

This proves (3.5), and the result follows. ∎

Proof of Proposition 2.1. Apply Lemma 3.3 on E_1. ∎

4. Proof of Proposition 2.2. Using an analytic method, Ren [8] obtained a new type upper bound for exponential sums over primes which is also true for exponential sums involving the Möbius function.

LEMMA 4.1 (Ren (see [8, Theorem 1.1])). *Fix $k \ge 1$, and let $\beta_k = 1/2 + \log k/\log 2$. Then*

$$S_k(x,\alpha) \ll (d(q))^{\beta_k}(\log x)^c\biggl(x^{1/2}\sqrt{q(1+|\lambda|x^k)} + x^{4/5} + \frac{x}{\sqrt{q(1+|\lambda|x^k)}}\biggr). \ ∎$$

REMARK 4.2. As pointed out in [9], one can replace the middle term $x^{4/5}$ by $x^{3/4+\varepsilon}$ under GRH.

Proof of Proposition 2.2. Apply Lemma 4.1 on E_2. ∎

5. Proof of Proposition 2.4. Combining [6, Theorem 3] and [11, Theorem 11.1] we get the following result.

LEMMA 5.1. *Let $k \geq 4$, let $\rho = \rho_k$ be defined in* (1.6) *and suppose that α satisfies* (2.1) *with $Q = x^{\frac{k^2-2k\rho}{2k-1}}$. Then*

$$S_k(x,\alpha) \ll x^{1-\rho+\varepsilon} + \frac{q^{\varepsilon} x L^c}{\sqrt{q(1+|\lambda|x^k)}}, \tag{5.1}$$

where the implied constant depends at most on k and ε. ∎

The next result is due to Zhao [12]. When $k = 3$, he gives a better upper bound for a larger range of Q. We can prove a similar result when $k \geq 4$.

LEMMA 5.2 (see [12, Lemma 8.5]). *Suppose that α satisfies* (2.1) *and $x^{1/2} \leq Q \leq x^{5/2}$. Then*

$$S_3(x,\alpha) \ll x^{1-1/12+\varepsilon} + \frac{q^{-1/6} x^{1+\varepsilon}}{\sqrt{(1+x^3|\lambda|)}}. \ ∎$$

REMARK 5.3. Following the proof of Lemma 5.2, we can show that when $x^{1/2} \leq Q \leq x^{17/6-\varepsilon}$, Lemma 5.2 is also true. This will be used in our result.

LEMMA 5.4. *Let $k \geq 4$, let $\rho = \rho_k$ be as defined in* (1.6) *and suppose that α satisfies* (2.1) *with $x^{2\rho+\varepsilon} \leq Q \leq x^{k-2\rho-\varepsilon}$. Then*

$$S_k(x,\alpha) \ll x^{1-\rho+\varepsilon} + \frac{q^{\varepsilon} x L^c}{\sqrt{q(1+|\lambda|x^k)}}, \tag{5.2}$$

where the implied constant depends at most on k and ε.

Proof. For any $\alpha \in \mathbb{R}$, there exist $b \in \mathbb{Z}$ and $r \in \mathbb{N}$ with

$$(b,r) = 1, \quad 1 \leq r \leq x^{\frac{k^2-2k\rho}{2k-1}} \quad \text{and} \quad |r\alpha - b| \leq x^{-\frac{k^2-2k\rho}{2k-1}}.$$

Hence

$$S_k(x,\alpha) \ll x^{1-\rho+\varepsilon} + \frac{r^{\varepsilon} x L^c}{\sqrt{r(1+|\alpha - b/r|x^k)}}. \tag{5.3}$$

We assume that

$$r \leq x^{2\rho-\varepsilon} \quad \text{and} \quad |\alpha - b/r| \leq r^{-1} x^{2\rho-k-\varepsilon}; \tag{5.4}$$

otherwise $S_k(x,\alpha) \ll x^{1-\rho+\varepsilon}$ by (5.3). Combining (2.1) and (5.4), we have

$$\begin{aligned} |bq - ar| = |q(b - r\alpha) + r(q\alpha - a)| &\leq qr\left|\frac{b}{r} - \alpha\right| + qr\left|\frac{a}{q} - \alpha\right| \\ &\leq Qx^{2\rho-k-\varepsilon} + \frac{x^{2\rho-\varepsilon}}{Q} < 1, \end{aligned}$$

provided that $x^{2\rho+\varepsilon} \leq Q \leq x^{k-2\rho-\varepsilon}$, hence $a = b,\ q = r$. ∎

When δ is large, we cannot use Lemmas 5.2 and 5.4 for $\alpha \in E_3$, but we can use the next lemma unconditionally.

LEMMA 5.5 (see [3, Theorem 1]). *For $k \geq 3$ and α satisfy* (2.1) *we have unconditionally*

$$|S_k(x,\alpha)| \ll x^{1+\varepsilon}\left(\frac{1}{q}+\frac{1}{x^{1/2}}+\frac{q}{x^k}\right)^{2^{2-2k}}. \ \blacksquare$$

REMARK 5.6. Lemmas 4.1, 5.1, 5.2, 5.4 and 5.5 in the relevant references are about sums over primes. However, we can get the same bounds for our $S_k(x,\alpha)$ by a similar argument with Heath-Brown's identity for $\mu(n)$ instead of one for $\Lambda(n)$. In [5, Section 6.3] there is a similar argument, but it just uses Vaughan's identity and it is for short intervals. We therefore omit the details.

Proof of Proposition 2.4. When $0 \leq \delta < 1/2 - 2\rho_k$, for $k = 3$, applying Lemma 5.2 on E_3 yields Proposition 2.4; for $k \geq 4$, applying Lemma 5.4 on E_3, we get Proposition 2.4.

When $1/2 - 2\rho_k \leq \delta < 1/2$, applying Lemma 5.5 on E_3, we obtain Proposition 2.4. ∎

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