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The generalization of Jarník's identity

by

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1. Introduction. Throughout,

$$n = m + 1 > 2. \tag{1.1}$$

Given $\boldsymbol{\xi} = (\xi_1, \ldots, \xi_m) \in \mathbb{R}^m$ with

$$1, \xi_1, \ldots, \xi_m \text{ linearly independent over } \mathbb{Q}, \tag{1.2}$$

German [1] defines α, resp. α^t, as the supremum of the numbers θ such that for every $\eta < \theta$ the inequalities

$$|x| \le e^q, \quad |\xi_i x - y_i| \le e^{-\eta q} \quad (i = 1, \ldots, m), \tag{1.3}$$

resp.

$$|y_i| \le e^q \quad (i = 1, \ldots, m), \quad |\xi_1 y_1 + \cdots + \xi_m y_m - x| \le e^{-\eta q}, \tag{1.4}$$

have a solution $(x, y_1, \ldots, y_m) \in \mathbb{Z}^n \setminus \{\mathbf{0}\}$ for every large q. Often α, α^t are denoted $\hat{\omega}$, $\hat{\omega}^*$ respectively, and it is well known that

$$1/m \le \alpha \le 1, \quad \alpha^t \ge m. \tag{1.5}$$

In the context of (1.3), i.e. simultaneous approximation, German's numbers m, n, which we denote by m_G, n_G for clarity, become $1, m$ respectively, and a case of his Theorem 1 (see [1, Section 1.1]) yields $\alpha^t \ge (n_G - 1)/(1 - \alpha) = (n-2)/(1-\alpha)$, i.e.

$$n - 2 \le (1 - \alpha)\alpha^t. \tag{1.6}$$

In particular, $\alpha = 1$ precisely when $\alpha^t = \infty$.

In the context of (1.4), i.e. approximation involving a linear form, we have $m_G = m$, $n_G = 1$, and $\alpha^t \ge 1$, so that by reversing the roles of α, α^t,

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German's Theorem 1 yields $\alpha \geq (1-(\alpha^t)^{-1})/(n-2)$, i.e.

$$n-2 \geq \frac{1-(\alpha^t)^{-1}}{\alpha}. \tag{1.7}$$

Let $\Lambda = \Lambda(\xi) \subseteq \mathbb{R}^n$ be the lattice of points

$$(x, \xi_1 x - y_1, \ldots, \xi_m x - y_m),$$

where $(x, y_1, \ldots, y_m)$ runs through $\mathbb{Z}^n$. Let $\lambda_1(q), \ldots, \lambda_n(q)$ be the successive minima of Λ with respect to the box $\mathcal{K}(q)$ of points $(\zeta_0, \zeta_1, \ldots, \zeta_m)$ with

$$|\zeta_0| \leq e^{mq}, \qquad |\zeta_i| \leq e^{-q} \quad (i=1,\ldots,m),$$

and set $L_i(q) = \log \lambda_i(q)$. As pointed out in [4] and [5], the functions $L_i(q)$ are continuous and piecewise linear with slopes 1 and $-m$, and

$$-\log n! \leq L_1(q) + \cdots + L_n(q) \leq 0.$$

For $1 \leq i \leq n$, set $\varphi_i(q) = L_i(q)/q$ and

$$\underline{\varphi}_i := \liminf_{q\to\infty} \varphi_i(q), \qquad \overline{\varphi}_i := \limsup_{q\to\infty} \varphi_i(q). \tag{1.8}$$

In the present paper, we will show that (1.6), (1.7) are equivalent to the pair of inequalities

$$(n-2)\underline{\varphi}_n + \overline{\varphi}_1 \geq -\overline{\varphi}_1\underline{\varphi}_n \geq (n-2)\overline{\varphi}_1 + \underline{\varphi}_n. \tag{1.9}$$

We will give a direct proof of these inequalities, and show that they are best possible, so that (1.6), (1.7) are best possible as well. Observe that the inequalities (1.9) go into each other by interchanging $\overline{\varphi}_1, \underline{\varphi}_n$ and reversing inequalities. The case $n=3$ gives $\overline{\varphi}_1 + \underline{\varphi}_3 = -\overline{\varphi}_1\underline{\varphi}_3$, which is Jarník's identity in our present formulation.

2. The equivalence of German's (1.6), (1.7) with (1.9). We have seen in [4] that $\Lambda(\xi)$ is proper when (1.2) holds, so that there are arbitrarily large q with $\varphi_1(q) = \varphi_2(q)$, yielding

$$2\varphi_1(q) \geq -(\varphi_3(q) + \cdots + \varphi_n(q)) - \frac{\log n!}{q} \geq -(n-2) - O(1/q),$$

hence

$$\overline{\varphi}_1 \geq -\frac{n-2}{2} > -m. \tag{2.1}$$

With $\alpha = \hat{\omega}$, $\alpha^t = \hat{\omega}^*$, we obtain (see [5, equations (1.8), (1.9)])

$$(1+\alpha)(m+\overline{\varphi}_1) = (1+\alpha^t)(1-\underline{\varphi}_n) = n,$$

so that

$$\alpha = \frac{1-\overline{\varphi}_1}{m+\overline{\varphi}_1}, \qquad \alpha^t = \frac{m+\underline{\varphi}_n}{1-\underline{\varphi}_n}, \tag{2.2}$$

where the second equation means that $\underline{\varphi}_n = 1$ precisely when $\alpha^t = \infty$. In this case (1.6) is true, and so is the first relation in (1.9) by (2.1).

We will now show that (1.6) is equivalent to the first relation in (1.9) when $\underline{\varphi}_n < 1$. By (1.6) and (2.2),

$$n-2 \le \frac{n-2+2\overline{\varphi}_1}{m+\overline{\varphi}_1} \cdot \frac{m+\underline{\varphi}_n}{1-\underline{\varphi}_n}$$

or

$$(n-2)(m+\overline{\varphi}_1 - m\underline{\varphi}_n - \overline{\varphi}_1\underline{\varphi}_n) \le (n-2)m + 2m\overline{\varphi}_1 + (n-2)\underline{\varphi}_n + 2\overline{\varphi}_1\underline{\varphi}_n,$$

which by (1.1) gives

$$-n\overline{\varphi}_1\underline{\varphi}_n \le n(n-2)\underline{\varphi}_n + n\overline{\varphi}_1,$$

and hence the first inequality in (1.9).

By (2.2), $(\alpha^t)^{-1}, \alpha^{-1}$ are like α, α^t, but with the roles of $\overline{\varphi}_1$ and $\underline{\varphi}_n$ interchanged. Therefore (1.7) yields

$$n-2 \ge \frac{n-2+2\underline{\varphi}_n}{m+\underline{\varphi}_n} \cdot \frac{m+\overline{\varphi}_1}{1-\overline{\varphi}_1},$$

and eventually the second inequality in (1.9).

3. Applying Roy's fundamental work. Let $\Delta \subseteq \mathbb{R}^n$ consist of the points $(\alpha_1, \ldots, \alpha_n)$ with $\alpha_1 \le \cdots \le \alpha_n$. Observe that $\mathcal{L}_\xi = (L_1, \ldots, L_n)$ with $L_1, \ldots, L_n$ as in Section 1 is a map $(0, \infty) \to \Delta$.

We will now recall the definition of an $(n, 0)$-*system* as introduced in [5]. For convenience we will call it an n-*system* in what follows. It is a map $\mathcal{P} : (\eta_0, \infty) \to \Delta$ for some $\eta_0 \ge 0$, where $\mathcal{P} = (P_1, \ldots, P_n)$ with each P_i continuous and piecewise linear with slopes among $1, -m$, with $P_1(q) + \cdots + P_n(q) = 0$ and with a further condition formulated below.

Numbers q where some P_i changes slopes are called *division numbers*. We will consider intervals $I = [q, q']$ or $I = [q, \infty)$ whose endpoints are division numbers, but points in their interior are not. In such an interval I, m of the functions have slope 1, and one function has slope $-m$. Put differently, if an inclining line segment is said to be *of multiplicity* l if it is part of the graph of l functions $P_j, P_{j+1}, \ldots, P_{j+l-1}$, then the combined graph of $\mathcal{P}$ in I consists of m inclining line segments of slope 1 (counted with multiplicity) and one declining line segment of slope $-m$. The inclining (resp. declining) line segments in adjacent intervals I which lie on a common line combine to form longer line segments called inclining (resp. declining) *strands*.

A number q where one declining strand ends and another one starts is called a *switch number*. In fact for such q there are two integers u, v such that at $(q, P_u(q))$ (resp. $(q, P_v(q))$) a declining strand begins (resp. a declining strand ends) and where an inclining strand ends or the multiplicity of its line segments decreases (resp. an inclining strand begins or the multiplicity of its line segments increases). We require that P_u has slope $-m$ to the right of q, and that P_v has slope $-m$ to the left of q. For an n-system there is the extra condition that always $u > v$. A switch number with given (u, v) is said to be *of type* $\binom{u}{v}$. As a consequence, if P_1 has a local maximum at q, then $(q, P_1(q))$ cannot be the endpoint of a strand, and therefore $P_1(q) = P_2(q)$.

THEOREM 3.1.

(a) *For each* $\boldsymbol{\xi} \in \mathbb{R}^m$ *there is an* n*-system* $\mathcal{P} : (\eta_0, \infty) \to \Delta$ *such that* $\mathcal{P} - \mathcal{L}_{\boldsymbol{\xi}}$ *is bounded on* (η_0, ∞).

(b) *Given an* n*-system* $\mathcal{P} : (\eta_0, \infty) \to \Delta$ *where* $q - P_n(q)$ *tends to infinity with* q, *there exixts some* $\boldsymbol{\xi} = (\xi_1, \ldots, \xi_m)$ *with* $1, \xi_1, \ldots, \xi_m$ *linearly independent over* $\mathbb{Q}$ *such that* $\mathcal{P} - \mathcal{L}_{\boldsymbol{\xi}}$ *is bounded on* (η_0, ∞).

This theorem will now be deduced from Roy's work [3]. We define a dual n-system $\mathcal{P}^* : (\eta_0, \infty) \to \Delta$ exactly like an n-system, except that its components P_i^* will have slopes -1 and m. Suppose

$$\mathcal{P} = (P_1, \ldots, P_n), \qquad \mathcal{P}^* = (-P_n, \ldots, -P_1). \tag{3.1}$$

Then it is clear that $\mathcal{P}$ is an n-system precisely when $\mathcal{P}^*$ is a dual n-system. A switch number of type $\binom{u}{v}$ for $\mathcal{P}$ will be a switch number of type $\binom{n+1-v}{n+1-u}$ for $\mathcal{P}^*$.

Let $\Lambda^*(\xi)$ be the lattice reciprocal to $\Lambda(\xi)$, consisting of the points

$$(x - \xi_1 y_1 - \cdots - \xi_m y_m, y_1, \ldots, y_m) \quad \text{with } (x, y_1, \ldots, y_m) \in \mathbb{Z}^n,$$

and let $\mathcal{K}^*(q)$ consist of the points $(\zeta_0, \zeta_1, \ldots, \zeta_m) \in \mathbb{R}^n$ with

$$|\zeta_0| \le e^{-mq}, \qquad |\zeta_i| \le e^q \quad (i = 1, \ldots, m). \tag{3.2}$$

It is well known that if $\lambda_1(q), \ldots, \lambda_n(q)$ are the successive minima of $\Lambda(\xi)$ with respect to $\mathcal{K}(q)$, and $\lambda_1^*(q), \ldots, \lambda_n^*(q)$ are the successive minima of $\Lambda^*(\xi)$ with respect to $\mathcal{K}^*(q)$, then the quotients $\lambda_i(q)/\lambda_{n+1-i}^*(q)$ are bounded from above and below by positive constants depending only on n. So if we set $L_i^*(q) = \log \lambda_i^*(q)$ $(i = 1, \ldots, n)$, the map

$$(L_1, \ldots, L_n) - (-L_n^*, \ldots, -L_1^*) \tag{3.3}$$

is bounded on $(0, \infty)$. If $q - L_n(q)$ tends to infinity with q, then so does $q + L_1^*(q)$.

We define a *Roy* n*-system* to be a map $\mathcal{P}^R : (q_0, \infty) \to \Delta$ like an n-system, but with components P_i^R having slopes 0 and 1, and with $P_1^R(q) + \cdots + P_n^R(q) = q$. If

$$P_i^R(q) = q/n + P_i^*(q/n) \qquad (i = 1, \ldots, n), \tag{3.4}$$

then $\mathcal{P}^R = (P_1^R, \ldots, P_n^R)$ is a Roy n-system on $(n\eta_0, \infty)$ precisely when $\mathcal{P}^* = (P_1^*, \ldots, P_n^*)$ is a dual n-system on (η_0, ∞). Let $\mathcal{K}^R(q)$ be the box of points $\boldsymbol{\zeta} = (\zeta_0, \zeta_1, \dots, \zeta_m)$ with

$$|\zeta_0| \le e^{-q}, \quad |\zeta_i| \le 1 \quad \text{for } i = 1, \dots, m. \tag{3.5}$$

Then $\mathcal{K}^*(q) = e^q \mathcal{K}^R(nq)$. Therefore the minima $\lambda_i^R(q)$ of $\Lambda^*(\xi)$ with respect to $\mathcal{K}^R(q)$ have $\lambda_i^*(q) = e^{-q}\lambda_i^R(nq)$, hence $\lambda_i^R(q) = e^{q/n}\lambda_i^*(q/n)$, and $L_i^R(q) = \log \lambda_i^R(q)$ is given by

$$L_i^R(q) = q/n + L_i^*(q/n), \qquad i = 1, \dots, n. \tag{3.6}$$

The proof of Theorem 3.1 can now easily be finished.

(a) Given $\mathbf{u} = (u_0, u_1, \dots, u_m) \neq \mathbf{0}$, we define a lattice $\Lambda^*(\mathbf{u})$ to consist of points $(u_0 x - u_1 y_1 - \dots - u_m y_m, y_1, \dots, y_m)$ with $\mathbf{x} = (x, y_1, \dots, y_m) \in \mathbb{Z}^n$. Thus $\Lambda^*(\boldsymbol{\xi}) = \Lambda^*(\mathbf{u})$ with $\mathbf{u} = (1, \xi_1, \dots, \xi_m)$. Let $\hat{\mathbf{u}} := \|\mathbf{u}\|^{-1}\mathbf{u}$ be the corresponding normalized vector, where $\|\cdot\|$ denotes the Euclidean norm. Clearly, the minima defined in terms of $\hat{\mathbf{u}}$ rather than $\mathbf{u}$ are the same except for a bounded factor, and hence their logarithms L_i^R are only changed by bounded amounts. Therefore applying Roy's Theorem 1.3 from [3] to $\hat{\mathbf{u}}$, we obtain a Roy n-system $\mathcal{P}^R$ with $\mathcal{L}^R - \mathcal{P}^R$ bounded on some range (η_0, ∞). (Note that Roy has $|\zeta_0| \le e^{-q}$, $\|(\zeta_1, \dots, \zeta_m)\| \le 1$ in place of (3.5), but this does not matter).

Given $\mathcal{P}^R$, we now define $\mathcal{P}^*$ by (3.4) and note that $\mathcal{L}^* - \mathcal{P}^*$ is bounded. Finally, given $\mathcal{P}^*$, we obtain $\mathcal{P}$ by (3.1) where $\mathcal{L} - \mathcal{P}$ is bounded since the expression in (3.3) is bounded.

(b) Let $\mathcal{P}$ be an n-system and define $\mathcal{P}^*$, $\mathcal{P}^R$ by (3.1), (3.4), so that $\mathcal{P}^R$ is a Roy n-system. By Roy's Theorem 8.1 in [3] there is some $\mathbf{u}$ of norm 1 such that $\mathcal{L}_{\mathbf{u}}^R - \mathcal{P}^R$ is bounded, where $\mathcal{L}_{\mathbf{u}}^R$ is defined in terms of the lattice $\Lambda^*(\mathbf{u})$. By (3.1) and (3.4), $\mathcal{L}_{\mathbf{u}}^* - \mathcal{P}^*$ is bounded. If the components $u_0, u_1, \dots, u_m$ were linearly dependent, say $c_0 u_0 + c_1 u_1 + \dots + c_m u_m = 0$ with $\mathbf{c} = (c_0, c_1, \dots, c_m) \in \mathbb{Z}^n \setminus \{\mathbf{0}\}$, then

$$(u_0 c_0 - u_1 c_1 - \dots - c_m u_m, c_1, \dots, c_m) = (0, c_1, \dots, c_m)$$

is in $\Lambda^*(\mathbf{u})$ and lies in $c e^{-q}\mathcal{K}^*(q)$ with $c = \max\{|c_1|, \dots, |c_m|)$, so that $\lambda_1^*(q) \ll e^{-q}$, and hence $L_1^*(q) + q$ is bounded, which was ruled out.

Therefore $u_0, u_1, \dots, u_m$ are linearly independent, and $\mathcal{L}_{\mathbf{u}}^* - \mathcal{P}^*$, hence $\mathcal{L}_{\boldsymbol{\xi}}^* - \mathcal{P}^*$, is bounded, with $\boldsymbol{\xi} = (u_1/u_0, \dots, u_m/u_0)$. In turn, by (3.1) and the boundedness of (3.3), $\mathcal{L}_{\boldsymbol{\xi}} - \mathcal{P}$ is bounded. Here $1, \xi_1, \dots, \xi_m$ are linearly independent over $\mathbb{Q}$.

4. Proof of (1.9). Given $\boldsymbol{\xi}$, let $\mathcal{P} : (\eta_0, \infty) \to \Delta$ be the n-system of Theorem 3.1 with $\mathcal{L}_{\boldsymbol{\xi}} - \mathcal{P}$ bounded. We set $\varphi_i(q) := P_i(q)/q$, $i = 1, \dots, n$, and note that the quantities $\underline{\varphi}_i, \overline{\varphi}_i$ defined by (1.8) in terms of $\mathcal{P}$ are the same as the quantities defined in terms of $\mathcal{L}_\xi$. Therefore it will be enough for us to deal with n-systems. If $\overline{\varphi}_1 = -m$ for such a system, then it is easily seen that $\underline{\varphi}_i = \overline{\varphi}_i = 1$ for $i = 2, \dots, n$. On the other hand, we have

THEOREM 4.1. *Let $\mathcal{P}$ be an $(n,0)$-system with $\overline{\varphi}_1 > -m$. Then we have*

$$\overline{\varphi}_1 \geq -\frac{n-2}{2} \tag{4.1}$$

and (1.9).

We will first establish (4.1) and the first relation in (1.9), which can be rewritten as

$$\overline{\varphi}_1 \geq F(\underline{\varphi}_n) \quad \text{and also} \quad \underline{\varphi}_n \geq G(\overline{\varphi}_1), \tag{4.2}$$

with $F : (0,1) \to \left(-\frac{n-2}{2}, 0\right)$ and its inverse $G : \left(-\frac{n-2}{2}, 0\right) \to (0,1)$ given by

$$F(x) = -\frac{(n-2)x}{1+x}, \qquad G(x) = -\frac{x}{n-2+x}. \tag{4.3}$$

The combined graph of an n-system has inclining line segments of slope 1 and declining line segments of slope $-m$ whose multiplicities sum up to m. A number f will be called a *lower critical number* if P_1 has a local maximum at f, and a number c will be called an *upper critical number* if P_n has a local minimum at c. If f is a lower critical number, then $P_1(f) = P_2(f)$ and from the point $(f, P_1(f))$ will emanate a declining line segment to the left and right, and also an inclining line segment of some multiplicity $l \geq 2$ to the left and right. If c is an upper critical number, then $P_{n-1}(c) = P_n(c)$, with the same kind of line segments emanating from $(c, P_n(c))$.

Can there be a number which is both a lower and upper critical number? If $c = f$ is such a number, then declining line segments will emanate from both sides of $(c, P_n(c))$ as well as of $(c, P_1(c))$. But there cannot be declining line segments whose projections on the q-axis contain a common interval of positive length. Therefore the line segments emanating from $(c, P_n(c))$ and $(c, P_1(c))$ are the same, so that $P_1(c) = P_n(c) = 0$. If there are arbitrarily large numbers c with that property, then $\overline{\varphi}_1 = 0 = \underline{\varphi}_n$ and (4.1), (1.9) are trivially true. We may therefore suppose

$$-(n-1) \leq \frac{P_1(q)}{q} < 0 < \frac{P_n(q)}{q} \leq 1$$

for large q, with no ambivalent critical numbers in this range.

If only finitely many lower critical numbers exist, then P_1 will decrease with slope $-m$ from some point on, so that $\overline{\varphi}_1 = -m = -(n-1)$, against our hypothesis. Therefore there will be infinitely many lower critical numbers f. Such a number has $2P_1(f) + P_3(f) + \cdots + P_n(f) = 0$, hence

$$\frac{P_1(f)}{f} \geq \frac{-(n-2)}{2}, \tag{4.4}$$

and (4.1) follows. If only finitely many upper critical numbers exist, then $\underline{\varphi}_1 = 1$, and the first relation in (1.9) holds. We may therefore assume that there are infinitely many upper as well as lower critical numbers. Then for large q, say $q > q_0$, we have

$$-mq < P_1(q) < 0 < P_n(q) < q. \tag{4.5}$$

LEMMA 4.2. *Suppose $c < f$ with c an upper and f a lower critical number. Suppose there is no critical number between c and f. Then*

$$(n-2)\frac{P_n(c)}{c} + \frac{P_1(f)}{f} \geq -\frac{P_n(c)}{c}\,\frac{P_1(f)}{f}. \tag{4.6}$$

Observe that (4.6) may be written as

$$\frac{P_1(f)}{f} \geq F\left(\frac{P_n(c)}{c}\right) \quad \text{and also as} \quad \frac{P_n(c)}{c} \geq G\left(\frac{P_1(f)}{f}\right). \tag{4.7}$$

Proof of Lemma 4.2. Rising as well as declining line segments will emanate from the right of $(c, P_n(c))$. Therefore no declining line segment can emanate from the right of $(c, P_1(c))$, and a rising segment will. Also there will be a rising line segment to the left of $(f, P_1(f))$. Since there is no critical number between c and f, it easily follows that there is a rising line segment connecting $(c, P_1(c))$ and $(f, P_1(f))$. Thus P_1 has slope 1 in $[c, f]$. An analoguous argument shows that P_n has slope 1 in this interval. If some P_i rises (resp. declines) in this range, then so does φ_i, by virtue of (4.5).

Set $u_1 = P_1(c)$, $u_n = P_n(c)$ and $u = (P_2(c) + \cdots + P_{n-2}(c))/(n-3)$ when $n > 3$, but $u = 0$ when $n = 3$. Since $P_1(c) + \cdots + P_n(c) = 0$ and $P_{n-1}(c) = P_n(c)$, we have

$$u_1 + (n-3)u + 2u_n = 0. \tag{4.8}$$

Points (q, w), (q', w') with $q \leq q'$ which lie on (possibly distinct) declining line segments have $w' + mq' \geq w + mq$, and therefore $P_1(f) + mf \geq P_n(c) + mc$, which yields

$$m(f-c) \geq P_n(c) - P_1(f) = P_n(c) - P_1(c) - (f-c) = u_n - u_1 - (f-c),$$

hence

$$n(f-c) \geq u_n - u_1. \tag{4.9}$$

Let (g, v_1) be the point of intersection of the line of slope 1 through (c, u_1), and the line of slope $-m$ through (c, u_n), as depicted in Figure 1.

Setting $D = g - c$, we have

$$v_1 = u_1 + D \quad \text{and also} \quad v_1 = u_n - mD, \tag{4.10}$$

hence

$$u_n - u_1 = nD. \tag{4.11}$$

Now (4.8) yields $(n-3)u + 3u_n = nD$, so $u_n \geq D$, and $(n-2)u_n + v_1 = (n-1)u_n - mD \geq 0$ by (4.10), i.e.

$$(n-2)P_n(c) + P_1(g) \geq 0. \tag{4.12}$$

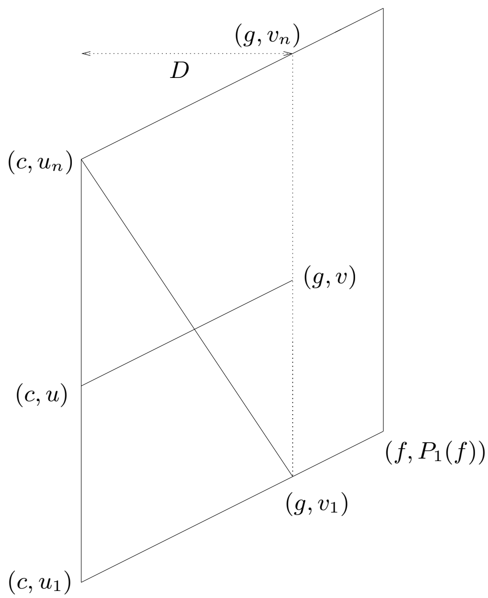

Fig. 1

As a consequence, $v_1 = -\alpha u_n$ with $0 < \alpha \le n-2$, so that

$$n - 2 \ge \frac{m\alpha}{1+\alpha} \ge \alpha. \tag{4.13}$$

Moreover,

$$mD = u_n - v_1 = -v_1(1 + 1/\alpha) = -v_1(1+\alpha)/\alpha. \tag{4.14}$$

We obtain

$$\begin{aligned}(n-2)\frac{P_n(c)}{c} + \frac{P_1(g)}{g} &= (n-2)\frac{u_n}{c} + \frac{v_1}{g} = \frac{(n-2)u_n g + v_1 c}{cg}\\ &= \frac{(n-2)u_n g - \alpha u_n c}{cg} = \frac{Au_n}{cg}\end{aligned}$$

with

$$A := (n-2)g - \alpha c \ge \frac{m\alpha}{1+\alpha}(g-c) = \frac{m\alpha}{1+\alpha}D = -v_1$$

by (4.13) and (4.14), so that

$$(n-2)\frac{P_n(c)}{c} + \frac{P_1(g)}{g} \ge -\frac{v_1 u_n}{cg} = -\frac{P_1(g)}{g}\,\frac{P_n(c)}{c}. \tag{4.15}$$

But $g \in [c, f]$, and P_1, hence $P_1(q)/q$, increases in this interval, so that

$$\frac{P_1(f)}{f} \ge \frac{P_1(g)}{g} \ge F\left(\frac{P_n(c)}{c}\right),$$

giving the first relation in (4.7), and Lemma 4.2 is proved.

In the proof of the lemma we dealt only with the part of the combined graph in the interval $[c, f]$. But to make sense of c, f being critical numbers, it is better to think of an open interval containing $[c, f]$. In what follows we will deal with a combined graph in an open interval satisfying all the usual properties including (4.5).

We will now generalize Lemma 4.2. We will no longer require that there are no upper critical numbers between c and f.

LEMMA 4.3. *Suppose $c < f$ with c an upper and f a lower critical number. Suppose there is no lower critical number between c and f. Then* (4.6) *holds.*

Proof. We will proceed by induction on the number l of upper critical numbers between c and f. The case $l = 0$ is true by Lemma 4.2. Assuming the truth for l, where $l \geq 0$, we will now establish the case of $l + 1$ upper critical numbers

$$c < c_l < c_{l-1} < \cdots < c_1$$

with $c_1 < f$. By the induction hypothesis,

$$\frac{P_n(c_l)}{c_l} \geq G\left(\frac{P_1(f)}{f}\right). \tag{4.16}$$

When $P_n(c)/c \geq P_n(c_l)/c_l$, then $P_n(c)/c \geq G(P_1(f)/f)$, hence (4.7). But sometimes $P_n(c)/c < P_n(c_l)/c_l$, so that we need a more intricate argument. By the same reasoning as for Lemma 4.2, P_1 increases with slope 1 in $[c, f]$. Set

$$R(q) = (P_2(q) + \cdots + P_{n-1}(q))/(n-2), \tag{4.17}$$

so that R will have slopes 1 and $-2/(n-2)$. Also set $t_i = P_i(c_l)$ $(1 \leq i \leq n)$, $t = R(c_l)$, so that $t_1 + (n-2)t + t_n = 0$.

The unique maximum of P_n in $[c, c_l]$ will be assumed at a number b, $c < b < c_l$. Write $s_n = P_n(b)$, $s = R(b)$. Since P_n decreases with slope $-m$ in $[b, c_l]$, and R increases with slope 1 in this interval, we have $s_n > t_n$, $s < t$.

Let (a, r) be the point of intersection of the line $\mathcal{L}$ of slope 1 through (b, s_n), and the line of slope $-2/(n-2)$ through (b, s). This point (a, r) is not necessarily on the graph $\mathcal{G}$. Figure 2 may be helpful.

The point $(c, P_n(c))$ certainly lies on $\mathcal{L}$. We claim that, as indicated in the figure, $a \leq c \leq b$. The upper bound is clear. We also know that P_1 has slope 1 on $[c, f]$, and that by construction P_n has slope 1 on $[c, b]$. So P_1 and P_n have both slope 1 on $[c, b]$, and therefore R has constant slope $-2/(n-2)$ on $[c, b]$. Since by (4.17) it is clear that $R \leq P_n$, this implies that $a \leq c$, which gives the lower bound. Moreover,

$$\begin{aligned}(n-1)r &= P_n(b) - (b-a) + (n-2)\big(R(b) + (2/(n-2))(b-a)\big)\\ &= (P_2(b) + \cdots + P_n(b)) + (b-a) > 0,\end{aligned}$$

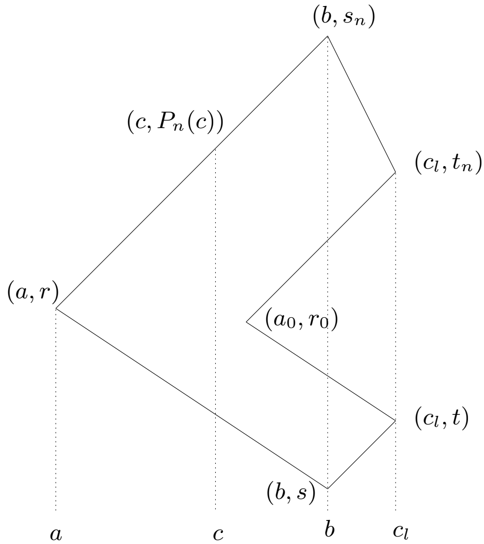

Fig. 2

hence $r > 0$, and $r \leq a$ since (a, r) lies on the line $\mathcal{L}$ through $(b, P_n(b))$. Thus $a \geq r > 0$.

Let $y = S(q)$ be the equation of $\mathcal{L}$, so that $S(q) = P_n(c) + q - c = q + P_n(c) - c$. Here $P_n(c) < c$. Therefore $S(q)/q$ will increase, yielding

$$\frac{P_n(c)}{c} = \frac{S(c)}{c} \geq \frac{S(a)}{a} = \frac{r}{a}. \tag{4.18}$$

Let (a_0, r_0) be the point of intersection of the line of slope 1 through (c_l, t_n) and the line of slope $-2/(n-2)$ through (c_l, t) (see Fig. 2 again).

If $y = S(q)$ is the equation of the combined two line segments at the top of Figure 2 connecting (a, r) and (c_l, t_n), and $y = T(q)$ the equation of the combined line segments at the bottom connecting (a, r) and (c_l, t), then $S + (n-2)T$ has slope -1 in $[a, c_l]$. Therefore

$$\begin{aligned} mr &= S(a) + (n-2)T(a) = S(c_l) + (n-2)T(c_l) + c_l - a \\ &= t_n + (n-2)t + c_l - a. \end{aligned}$$

Similarly, $mr_0 = t_n + (n-2)t + c_l - a_0$. Since $a < a_0$, we obtain $r > r_0$ and $r/a > r_0/a_0$. Therefore, in view of (4.18), it will suffice to show that

$$\frac{r_0}{a_0} \geq G\left(\frac{P_1(f)}{f}\right). \tag{4.19}$$

When $t = t_n$ (which is always true if $n = 3$), the points $(c_l, t), (c_l, t_n), (a_0, r_0)$ are the same, so that $r_0/a_0 = t_n/c_l = P_n(c_l)/c_l$, and (4.7), (4.6) is a consequence of (4.16). When $t < t_n$, we will construct a graph $\mathcal{G}'$ in an open interval containing $[a_0, f]$. This graph will coincide with $\mathcal{G}$ for $q \geq c_l$. The fun part will be for $q \leq c_l$.

When $t < t_n$, there will be some k, $2 \leq k \leq n-2$, with $t_n = t_{n-1} = \cdots = t_{k+1}$ and $t_{k+1} > t_k$. We set $(\alpha_{k+1}, s_{k+1}) = (c_l, t_n)$.

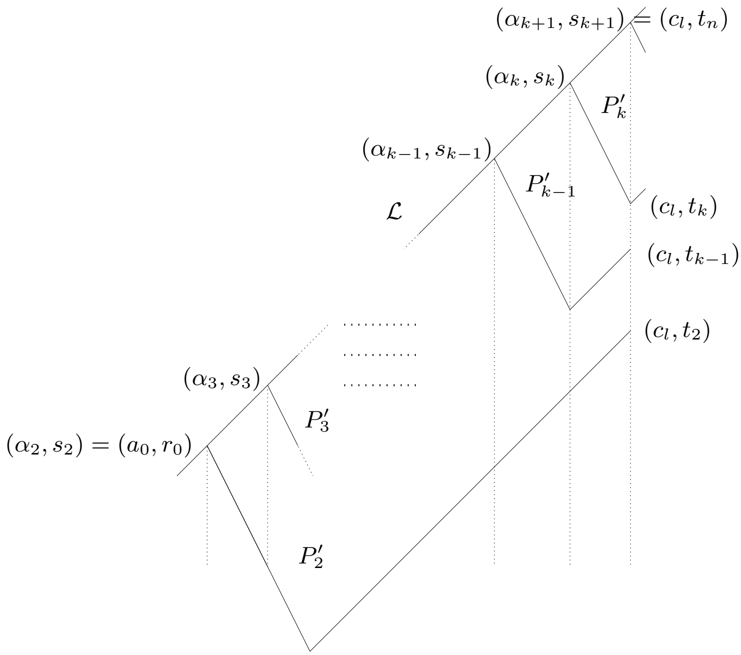

Fig. 3

As shown in Figure 3, let (α_k, s_k) be the intersection of the line $\mathcal{L}$ of slope 1 through (c_l, t_n) and the line $\mathcal{L}_k$ of slope $-m$ through (c_l, t_k). In $[\alpha_k, c_l] = [\alpha_k, \alpha_{k+1}]$ the graph of P'_k will be the line segment on $\mathcal{L}_k$ between (α_k, s_k) and (c_l, t_k), so that P'_k has slope $-m$. The P'_i with $i \neq k$ will have slope 1. Observe that $P'_{k+1}(q) = \cdots = P'_n(q)$ for $q \in [\alpha_k, c_l]$, and their graph is a rising line segment of multiplicity $n-k$. Also note that c_l is a switch number for $\mathcal{G}'$ of type $\binom{k+1}{k}$.

(α_{k-1}, s_{k-1}) will be the point of intersection of $\mathcal{L}$ and the line $\mathcal{L}_{k-1}$ of slope $-m$ through $(\alpha_k, P'_{k-1}(\alpha_k))$. On $[\alpha_{k-1}, \alpha_k]$, P'_{k-1} will have slope $-m$, and the P'_i with $i \neq k-1$ will have slope 1. Continuing in this way we construct $\alpha_{k-2}, s_{k-2}, P'_{k-2}, \ldots, \alpha_2, s_2, P'_2$. We will have

$$P'_2(\alpha_2) = \cdots = P'_n(\alpha_2) = s_2.$$

Now P'_1, P'_n have slope 1 in $[\alpha_2, c_l]$, and $R' = (P'_2 + \cdots + P'_{n-1})/(n-2)$ has slope $-2/(n-2)$. Therefore $(\alpha_2, s_2) = (a_0, r_0)$.

We have $t_n = P_n(c_l) < c_l$, and since $\mathcal{L}$ has slope 1, also $r_0 < a_0$. We now extend $\mathcal{G}'$ a little to the left of a_0, with P'_n of slope $-m$, each P'_i with $i \neq n$ of slope 1, but little enough to guarantee that still $P_n(q) < q$ throughout $\mathcal{G}'$.

The graph $\mathcal{G}'$ satisfies all the usual conditions, including (4.5). The number $a_0 = s_2$ is an upper extreme number for $\mathcal{G}'$, but c_l is not. Therefore we may apply our induction hypothesis for $l-1$ to $P'_n(a_0)/a_0$ and obtain

$$\frac{r_0}{a_0} = \frac{P'_n(a_0)}{a_0} \geq G\left(\frac{P_1(f)}{f}\right),$$

i.e. (4.19), hence (4.7), (4.6). Lemma 4.3 is proved.

The proof of the first inequality in (1.9) will now be easily finished. There are infinitely many upper as well as lower critical numbers, hence infinitely many pairs $c < f$ as in Lemma 4.2. Let $c_0 < f_0$ be such a pair with $c_0 > q_0$, and $c_0 < f_0$, $c_1 < f_1, \ldots$ with $c_0 < f_0 < c_1 < f_1 < \cdots$ the sequence of such pairs with $c > q_0$. There may be critical numbers between f_{i-1} and c_i, but there will be a number h_i, $f_{i-1} < h_i < c_i$, with no upper critical numbers in $[f_{i-1}, h_i]$, and no lower critical numbers in $[h_i, c_i]$.

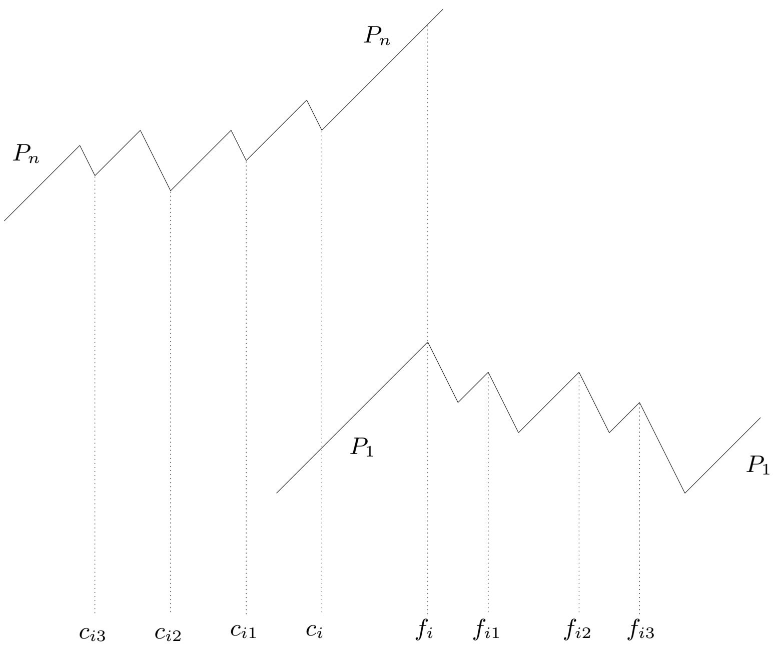

Fig. 4

Let $\varphi_{n,i}$ be the minimum of $\varphi_n(q)$ over the numbers in $[h_i, h_{i+1}]$, i.e. the minimum of $\varphi_n(q)$ over the critical numbers in $[h_i, c_i]$:

$$\min\{\varphi_n(c_{il}), \ldots, \varphi_n(c_{i1}), \varphi_n(c_i)\},$$

where $c_{il}, \ldots, c_{i1}$ are the numbers formerly denoted by $c_l, \ldots, c_1$. Let $\varphi_{1,i}$ be the maximum of $\varphi_1(q)$ over $q \in [h_i, h_{i+1}]$, so that

$$\varphi_{1,i} = \max\{\varphi_1(f_i), \varphi_1(f_{i1}), \ldots, \varphi_1(cf_{il'})\}$$

in obvious notation. Then

$$\underline{\varphi}_n = \liminf_{i\to\infty} \varphi_{n,i} \le \liminf_{i\to\infty} \varphi_n(c_i), \tag{4.20}$$

$$\overline{\varphi}_1 = \limsup_{i\to\infty} \varphi_{1,i} \ge \limsup_{i\to\infty} \varphi_1(f_i). \tag{4.21}$$

Since $\varphi_{n,i} \ge G(\varphi_1(f_i))$ by Lemma 4.3, and G is decreasing, we have

$$\underline{\varphi}_n \ge \liminf G(\varphi_1(f_i)) = G(\limsup \varphi_1(f_i)) \ge G(\overline{\varphi}_1),$$

hence the first inequality in (1.9).

It remains for us to prove the dual inequality, i.e. the second inequality in (1.9), which may also be written as

$$\overline{\varphi}_1 \le G(\underline{\varphi}_n).$$

It will follow from (4.20), (4.21) once we establish

$$\varphi_{1,i} \le G(\varphi_n(c_i)) \quad (i = 1, 2, \ldots). \tag{4.22}$$

Let $c < f$ be as in Lemma 4.2, and set $w_1 = P_1(f)$, $w_n = P_n(f)$ and

$$w = (P_3(f) + \cdots + P_{n-1}(f))/(n-3) \quad \text{when } n > 3,$$

but $w = 0$ when $n = 3$. Reflection on a point $(a, 0)$ with $c < a < f$ will reverse the roles of $(c, P_1(c)), \ldots, (c, P_n(c))$ and $(f, P_n(f)), \ldots, (f, P_1(f))$, and we obtain $2w_1 + (n-3)w + w_n = 0$ in place of (4.8). We define (g, v_n) to be the point of intersection of the line of slope 1 through (f, w_n) and the line of slope $-m$ through (f, w_1). To stress the duality we set $D := g - f$, and note that $D < 0$. Proceeding in a dual way to the one in the proof of Lemma 4.2, we obtain $c \le g < f$, $v_n = P_n(g)$, as well as $w_1 \le D$ and $(n-2)w_1 + v_n \le 0$, i.e.

$$(n-2)P_1(f) + P_n(g) \le 0,$$

in analogy to (4.12).

Next, $v_n = -\alpha w_1$ with $0 < \alpha \le n - 2$, hence again (4.13). On the other hand, $mD = -v_n(1+\alpha)/\alpha$ in place of (4.14). We obtain

$$(n-2)\frac{w_1}{f} + \frac{v_n}{g} = \frac{(n-2)w_1 g + v_n f}{fg} = \frac{(n-2)w_1 g - \alpha f w_1}{fg} = \frac{A w_1}{fg},$$

with

$$A := (n-2)g - \alpha f > \frac{m\alpha}{1+\alpha}(g - f) = \frac{m\alpha}{1+\alpha} D = -v_n.$$

Therefore

$$(n-2)\frac{P_1(f)}{f} + \frac{P_n(g)}{g} \le -\frac{P_1(f)}{f}\,\frac{P_n(g)}{g}$$

in analogy to (4.15). As a consequence,

$$\frac{P_1(f)}{f} \le G\Big(\frac{P_n(g)}{g}\Big) \le G\Big(\frac{P_n(c)}{c}\Big)$$

since $P_n(c)/c \le P_n(g)/g$ and G is decreasing. By the method of proof of Lemma 4.3, but reversing left and right, we also obtain $P_1(f_{ij})/f_{ij} \le G(P_n(c_i)/c_i)$ for the numbers f_{ij} occurring in Figure 4, and (4.22) follows.

5. The inequalities in (1.9) are best possible. An n-system $\mathcal{P}$ will be said to be *invariant* if it is invariant under dilation by some factor $\rho > 1$.

We now begin with the first inequality in (1.9). By Theorem 3.1 it will suffice to construct for every X with $0 < X < 1$ an invariant n-system $\mathcal{P}$ such that

$$\underline{\varphi}_n = X, \quad \overline{\varphi}_1 = F(X) = -\frac{(n-2)X}{1+X}, \tag{5.1}$$

for then $\overline{\varphi}_1 = F(\underline{\varphi}_n)$ so that the first part of (4.2) holds with equality. The graph $\mathcal{G}$ will be as follows. The number $q = 1$ will be an upper critical number with $P_2(1) = \dots = P_n(1) = X$, so that a declining as well as an inclining line segment of multiplicity $n-2$ pass through $(1, X)$. Moreover, $P_1(1) = -mX = -(n-1)X$. In the interval $[1, 1+X]$, P_2 declines with slope $-m$, the P_i with $i \ne 2$ incline.

We now pick a number $\delta > 0$ to be specified later. In $[1+X, 1+X+\delta]$, P_1 will decline, each P_i with $i \ne 1$ will incline. Setting $s_2 = 1 + X + \delta$, we have

$$P_2(s_2) = -mX + X + \delta = -(n-2)X + \delta,$$
$$P_3(s_2) = \dots = P_n(s_2) = P_2(s_2) + nX = 2X + \delta.$$

For $2 < j \le n$ we set $s_j = s_2 + (j-2)X$. In the interval $[s_j, s_{j+1}]$, where $2 \le j < n$, P_{j+1} will decline, but P_i with $i \ne j+1$ will incline. For $2 < j < n$,

$$P_2(s_j) = P_3(s_j) = \dots = P_j(s_j) = P_2(s_2) + (j-2)X,$$
$$P_{j+1}(s_j) = P_{j+2}(s_j) = \dots = P_n(s_j) = P_2(s_j) + nX,$$

and moreover,

$$P_2(s_n) = \dots = P_n(s_n) = P_2(s_2) + (n-2)X = \delta.$$

Set $\rho = s_n = s_2 + (n-2)X = 1 + (n-1)X + \delta$, and let $\mathcal{G}_1$ be the graph in $[1, \rho]$ we just constructed. Figure 5 shows this construction in the case $n = 5$.

We want to set

$$\mathcal{G} = \bigcup_{t \in \mathbb{Z}} \rho^t \mathcal{G}_1,$$

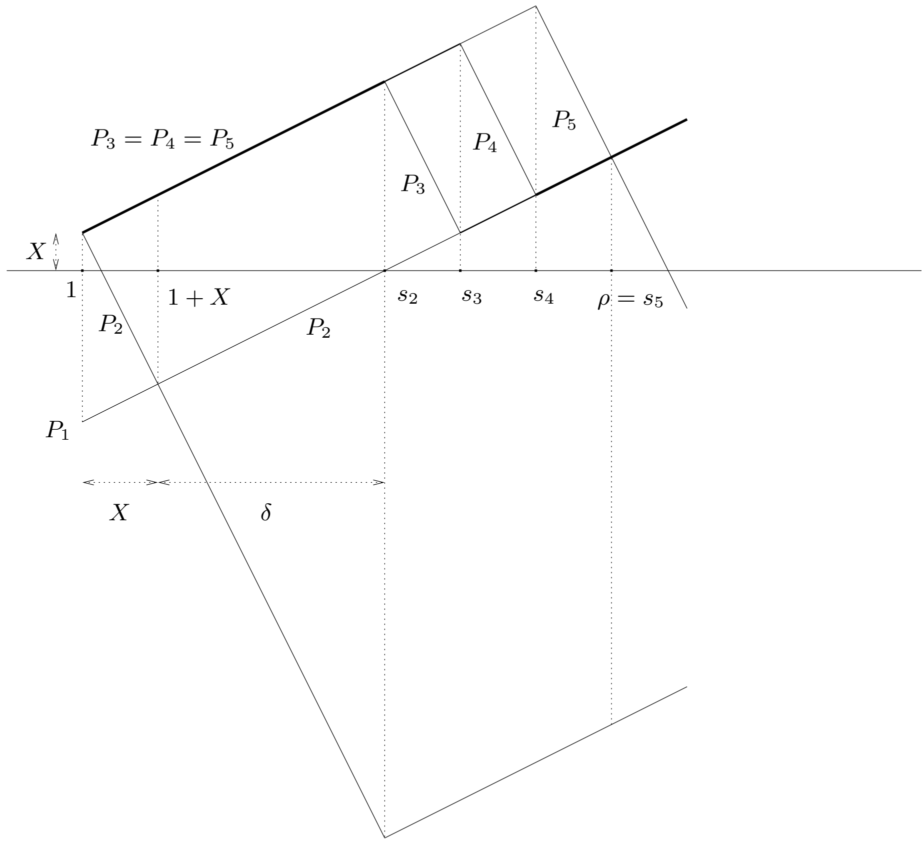

Fig. 5

so that $\mathcal{G}$ is invariant with factor ρ. For this it is necessary that $\mathcal{G}$ at $q = \rho$ is, up to the factor ρ, as at $q = 1$. So we need

$$\frac{\delta}{\rho} = \frac{P_n(\rho)}{\rho} = \frac{P_n(1)}{1} = X,$$

i.e. $\delta = X\rho = X(1 + mX + \delta)$, hence

$$\delta = \frac{X + mX^2}{1 - X}.$$

We indeed have $\underline{\varphi}_n = X$ and

$$\overline{\varphi}_1 = \frac{P_1(1 + X)}{1 + X} = \frac{-mX + X}{1 + X} = -\frac{(n - 2)X}{1 + X},$$

giving (5.1).

It remains to show that the second inequality in (1.9) is best possible as well. It will suffice to see that for every Y with $0 < Y < 1$, there exists an invariant n-system $\mathcal{P}$ with

$$\underline{\varphi}_n = Y, \quad \overline{\varphi}_1 = G(Y) = -\frac{Y}{n-2+Y}. \tag{5.2}$$

It will be convenient to set $X := Y/(n-2)$, so that (5.2) becomes

$$\underline{\varphi}_n = (n-2)X, \quad \overline{\varphi}_1 = -\frac{X}{1+X}. \tag{5.3}$$

As before, we construct an appropriate graph $\mathcal{G}$. The number $q = 1$ will be an upper critical number with $P_{n-1}(1) = P_n(1) = (n-2)X$, so that a declining as well as an inclining line segment of multiplicity 1 pass through $(1,(n-2)X)$. Moreover, $P_1(1) = \cdots = P_{n-2}(1) = -2X$. In the interval $[1, 1+X]$, P_{n-1} declines with slope $-m$, the P_i with $i \neq n-1$ incline.

We again pick a number δ to be specified later. In $[1+X, 1+X+\delta]$, P_1 will decline, each P_i with $i \neq 1$ will incline. Setting $s_2 = 1+X+\delta$, we get

$$P_1(s_2) = -X - (n-1)\delta.$$

We set $s_j = s_2 + (j-2)\delta$ for $2 < j < n$, but $s_n = s_{n-1} + X$. In the interval $[s_j, s_{j+1}]$, where $2 \leq j < n-1$, P_j will decline, each P_i with $i \neq j$ will incline. But in $[s_{n-1}, s_n]$, P_n will decline and P_i with $i < n$ will incline. Observe that

$$\begin{aligned} P_{n-1}(s_n) &= P_1(1+X) + (n-2)\delta + X = (n-2)\delta, \\ P_n(s_n) &= P_n(s_{n-1}) - mX = (n-2)X + X + (n-2)\delta - mX = (n-2)\delta, \end{aligned}$$

so that $P_{n-1}(s_n) = P_n(s_n)$ as in Figure 6, which deals with the case $n = 5$.

We have

$$P_1(s_n) = \cdots = P_{n-2}(s_n) = P_1(s_2) + (n-3)\delta + X = -2\delta.$$

Set $\rho = s_n = s_2 + (n-3)\delta + X = 1 + 2X + (n-2)\delta$, and let $\mathcal{G}_2$ be the graph in $[1,\rho]$ we just constructed.

In order to obtain an invariant graph with factor ρ by setting

$$\mathcal{G} = \bigcup_{t\in\mathbb{Z}} \rho^t \mathcal{G}_2,$$

it is necessary that $\mathcal{G}$ at $q = \rho$ is, up to the factor ρ, like at $q = 1$. So we need

$$\frac{-2\delta}{\rho} = \frac{P_1(\rho)}{\rho} = \frac{P_1(1)}{1} = -2X,$$

i.e. $(1 + 2X + (n-2)\delta)X = \delta$, hence

$$\delta = \frac{X + 2X^2}{1-(n-2)X}.$$

Then we have $\underline{\varphi}_n = (n-2)X$ and

$$\overline{\varphi}_1 = \frac{P_1(1+X)}{1+X} = \frac{-2X+X}{1+X} = -\frac{X}{1+X},$$

and hence (5.3).

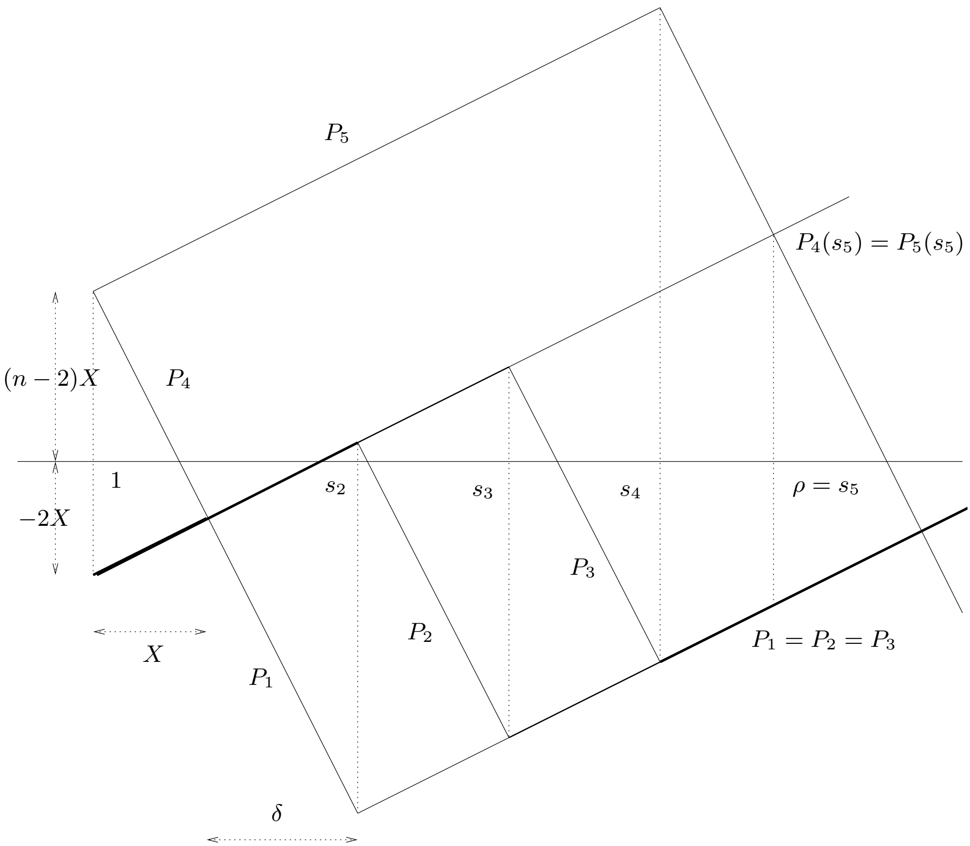

Fig. 6

Acknowledgments. An analogous result has independently been obtained by A. Marnat [2] who applies more directly the concepts introduced by Roy [3]. The authors also wish to thank the referee for the careful reading of the manuscript.

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