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STEINHAUS' LATTICE POINT PROBLEM FOR POLYHEDRA

BY

HIROSHI MAEHARA (Nishihara)

Abstract. It is proved that for every d-dimensional polyhedron Π in $\mathbb{R}^d$, $d \geq 2$, with volume $n + \alpha$, $|\alpha| < 1$, there is a congruent copy of Π that contains exactly n lattice points.

1. Introduction. A *lattice point* in $\mathbb{R}^d$ is a point whose coordinates are all integers. The set of all lattice points in $\mathbb{R}^d$ is denoted by $\mathbb{Z}^d$. Two subsets of $\mathbb{R}^d$ are *congruent* if they differ by a *rigid motion*, i.e. one can be moved to the other by a translation followed by a rotation.

In 1957, H. Steinhaus posed an interesting problem in elementary mathematics [2, 12]: Is there a circle in the plane that contains in its interior exactly n lattice points, for any given n? W. Sierpiński [10] showed that there are circles and squares in the plane containing in their interior a given number of lattice points. J. Browkin also proved the square case (and the cube case in $\mathbb{R}^3$); see [2, p. 127]. More generally, A. Schinzel and T. Kulikowski proved (see [2, p. 125]) that, for every nonempty plane bounded convex region W and for every natural number n, there is a similarity f of $\mathbb{R}^2$ such that $f(W)$ contains exactly n lattice points. The same result is given in [3] and [4].

It is also proved in [4] that if a polygon (not necessarily convex) in the plane has area n, then there is a polygon congruent to it that covers exactly n lattice points. In this paper, we generalize this result to higher dimensions.

By a *closed polyhedral surface* in $\mathbb{R}^d$ we mean a $(d-1)$-dimensional closed manifold that is contained in the union of a finite number of hyperplanes in $\mathbb{R}^d$. The Jordan–Brouwer separation theorem (see [8], for example) shows that a closed polyhedral surface divides $\mathbb{R}^d$ into two connected components, one bounded and the other unbounded. The closure of the bounded one is called a d-dimensional *polyhedron* in $\mathbb{R}^d$.

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THEOREM 1.1. *If a d-dimensional polyhedron in $\mathbb{R}^d$ has volume $n+\alpha$, $|\alpha|<1$, then it is possible to move it, by rotation and translation, to a position where it covers exactly n lattice points.*

REMARK 1.2. We cannot drop "rotation" in this theorem. For example consider the rectangle $[0,1/2]\times[0,2n]$ of area n in $\mathbb{R}^2$. No matter how we translate this rectangle, it either contains no lattice points, or contains $2n$ or $2n+1$ lattice points.

REMARK 1.3. It is possible to replace a d-dimensional polyhedron in Theorem 1.1 by a union of finitely many d-dimensional polyhedra.

PROBLEM 1.4. Is there a compact set in $\mathbb{R}^d$ with volume n no congruent copy of which covers exactly n lattice points?

Let us state some other works related to Steinhaus' problem. A. Schinzel [9] proved that there is a circle in the plane that *passes through* exactly n lattice points (see also [7]). Kuwata–Maehara [3] considered a similar problem for quadratic curves. They showed, for example, that for any $0\le n<5$, there is a parabola in $\mathbb{R}^2$ passing through exactly n lattice points, but if a parabola passes trough five lattice points, then it passes through infinitely many lattice points. Bannai–Miezaki [1] proved that for any 2-dimensional integral lattice $\Lambda\subset\mathbb{R}^2$, and for any $n>0$, there is a circle in $\mathbb{R}^2$ that passes through exactly n points of Λ. (An *integral lattice* is a discrete additive subgroup of $\mathbb{R}^d$ in which the squared Euclidean distance between any two elements is an integer.) Maehara [5] proved that there is a sphere in $\mathbb{R}^d$ that passes through exactly n lattice points and whose convex hull is a d-dimensional polytope.

P. Zwoleński [13] considered a Steinhaus type problem in a more general setting. In a metric space M, a countable subset $X\subset M$ is called a *quasi-finite* set if every ball in M contains only finitely many points of X. Zwoleński proved that if X is a quasi-finite subset of a Hilbert space, then there is a dense subset Y of the Hilbert space such that for every $y\in Y$ and for every positive integer n, there is a ball with center y that contains exactly n points of X. In [6], our Theorem 1.1 is generalized to the case of a quasi-finite subset in $\mathbb{R}^d$ instead of $\mathbb{Z}^d$.

2. Blichfeldt's lemma. For a subset $X\subset\mathbb{R}^d$ and a point $v\in\mathbb{R}^d$, denote by $v+X$ the translate of X along $\vec{v}:=\overrightarrow{Ov}$, and denote by X^* the set that is symmetric to X with respect to the origin O. Thus

$$v+X=\{v+x:x\in X\},\qquad X^*=\{-x:x\in X\}.$$

Note that $(X^*)^*=X$ and

$$(2.1)\qquad p\in v+X \;\Leftrightarrow\; p-v\in X \;\Leftrightarrow\; v-p\in X^* \;\Leftrightarrow\; v\in p+X^*.$$

The cardinality of X is denoted by $|X|$. Thus, the number of lattice points in X is $|\mathbb{Z}^d \cap X|$.

The following is the d-dimensional version of Blichfeldt's lemma. Intuitive proofs of the 2-dimensional case are given in [2] and [11].

LEMMA 2.1. *If W is a bounded set with d-dimensional volume $n+\alpha$ in $\mathbb{R}^d$, where $-1<\alpha<1$, then*

(i) *there is a point $u \in [0,1]^d$ such that $|\mathbb{Z}^d \cap (u+W)| \geq n$, and*
(ii) *there is a point $v \in [0,1]^d$ such that $|\mathbb{Z}^d \cap (v+W)| \leq n$.*

Proof. We show only (i) in the case $d=3$. We may suppose that W is contained in the cube $Q=[0,N]^3$, where N is a large integer. Let $\chi(x,y,z)$ be the characteristic function of W. That is, $\chi(x,y,z)=1$ if $(x,y,z)\in W$, and $\chi(x,y,z)=0$ if $(x,y,z)\notin W$. The volume $\mathrm{vol}(W)$ is then equal to $\iiint_Q \chi(x,y,z)\,dx\,dy\,dz$. Hence

$$\begin{aligned}\mathrm{vol}(W) &= \sum_{i=0}^{N-1}\sum_{j=0}^{N-1}\sum_{k=0}^{N-1}\int_0^1\!\!\int_0^1\!\!\int_0^1 \chi(x+i,y+j,z+k)\,dx\,dy\,dz\\ &= \int_0^1\!\!\int_0^1\!\!\int_0^1 \sum_{i=0}^{N-1}\sum_{j=0}^{N-1}\sum_{k=0}^{N-1} \chi(x+i,y+j,z+k)\,dx\,dy\,dz.\end{aligned}$$

Since $\mathrm{vol}(W)>n-1$, there exists $(x_0,y_0,z_0)\in[0,1]^3$ such that

$$\sum_i\sum_j\sum_k \chi(x_0+i,y_0+j,z_0+k)\geq n,$$

for otherwise, $\mathrm{vol}(W)$ would be less than or equal to $n-1$. This means that there are at least n triples $(i,j,k)\in[0,N-1]^3$ such that $\chi(x_0+i,y_0+j,z_0+k)=1$. Now, under the translation that maps (x_0,y_0,z_0) to $(0,0,0)$, each $(x_0+i,y_0+j,z_0+k)\in W$ is mapped to the lattice point (i,j,k). Hence under this translation, W goes to a position where it covers at least n lattice points. ∎

As already seen in Remark 1.2, it is not always possible to translate a bounded set with d-dimensional volume n in $\mathbb{R}^d$ to a position where it covers exactly n lattice points.

3. A few more lemmas. If we remove a finite number of points from a planar disk, the remaining set (a disk with finitely many pinholes) is clearly path-connected. Similarly, the following holds. For $m<d$, an m-dimensional *flat* in $\mathbb{R}^d$ is a translate of an m-dimensional linear subspace of $\mathbb{R}^d$. Thus a hyperplane is a $(d-1)$-dimensional flat.

LEMMA 3.1. *Let $F_1,\ldots,F_k$ be $(d-2)$-dimensional flats in $\mathbb{R}^d$, $d\geq 3$, and B be an open ball in $\mathbb{R}^d$. Then, for any $\mathcal{S}\subset\bigcup_{j=1}^k F_j$, $B-\mathcal{S}$ is path-connected.*

Proof. We may suppose that the center of B is the origin O. Let $X = B - \bigcup_{j=1}^{k} F_j$ and let $u_1, u_2 \in X$, $u_1 \neq u_2$. First, we show that X contains a path that connects u_1 and u_2. Since X is open in B, there are open balls $U(u_i) \subset X$ with center u_i for $i = 1, 2$. For each $1 \leq j \leq k$, let E_j be the $(d-2)$-dimensional linear subspace of $\mathbb{R}^d$ obtained as a translate of the flat F_j. Take a point $v_1 \in U(u_1) - \bigcup_{j=1}^{k} E_j$. The subspace H_j spanned by $\vec{v}_1$ and E_j is a $(d-1)$-dimensional subspace of $\mathbb{R}^d$. Since a finite number of hyperplanes cannot cover $U(u_2)$, there is a point $v_2 \in U(u_2) - \bigcup_{j=1}^{k} H_j$. Then $\vec{v}_1, \vec{v}_2$ and E_j span $\mathbb{R}^d$. Let V be the 2-dimensional subspace V spanned by $\vec{v}_1, \vec{v}_2$. Then, for all j, we have $V \cap E_j = \{O\}$. Therefore V intersects each flat F_j in a single point. Thus $V \cap X$ is a disk with finitely many pinholes, and it is path-connected. Let J be a path in $V \cap X$ that connects v_1 and v_2. Joining to J the line segments $u_i v_i$ in $U(u_i)$, $i = 1, 2$, we get a path in X that connects u_1 and u_2.

Now, suppose $\mathcal{S} \subsetneq \bigcup_{j=1}^{k} F_j$, and let $u_1, u_2 \in B - \mathcal{S}$. For each $i = 1, 2$, there is a line L_i containing u_i that is not covered by $\bigcup_{j=1}^{k} F_j$. Such a line intersects $\bigcup_{j=1}^{k} F_j$ in at most k points. Hence there is a point $w_i \in B \cap L_i$ such that the intersection of the line segment $u_i w_i$ and $\bigcup_{j=1}^{k} F_j$ is either $\{u_i\}$ or $\emptyset$. Then the line segment $u_i w_i$ is contained in $B - \mathcal{S}$. Since X is path-connected, there is a path $J \subset X$ that connects w_1 and w_2. By joining J and the line segments $u_i w_i$, $i = 1, 2$, we get a path in $B - \mathcal{S}$ that connects u_1 and u_2. ■

For a 2-dimensional subspace V of $\mathbb{R}^d$, let G_V denote the subgroup of the rotation group $\mathrm{SO}(d)$ of $\mathbb{R}^d$ consisting of those elements $\sigma \in \mathrm{SO}(d)$ that pointwise fix the orthogonal complement $V^\perp$ of V, and map V onto itself. This subgroup G_V is clearly isomorphic to $\mathrm{SO}(2)$.

Lemma 3.2. *Let H_1, H_2 be hyperplanes in $\mathbb{R}^d$ and p, q be two distinct points. If V is a 2-dimensional subspace of $\mathbb{R}^d$ that intersects H_2 in a line, then the number of those $\sigma \in G_V$ that satisfy*

$$p + \sigma(H_1) = q + \sigma(H_2) \tag{3.1}$$

is at most two.

Proof. Unless H_1 is parallel to H_2 (or $H_1 = H_2$), no $\sigma \in G_V$ can satisfy (3.1). So, we suppose H_1 is parallel to H_2. Set $w = p - q$, and let v be a point in H_1. Note that

$$\begin{aligned} (3.1) \;&\Leftrightarrow\; w + \sigma(H_1) = \sigma(H_2) \;\Leftrightarrow\; \sigma^{-1}(w) + H_1 = H_2 \\ &\Leftrightarrow\; \sigma^{-1}(w) \in (-v) + H_2. \end{aligned}$$

Since $\{\sigma^{-1}(w) : \sigma \in G_V\}$ is a circle lying on a 2-dimensional flat parallel to V, and since V intersects H_2 in a line, this circle intersects the hyperplane

$(-v) + H_2$ in at most two points. Therefore, the number of $\sigma \in G_V$ that satisfy (3.1) is at most two. ∎

In the following, Π denotes a d-dimensional polyhedron in $\mathbb{R}^d$, and Σ denotes the closed polyhedral surface that bounds Π. Let B be an open ball with center O such that $\Pi \subset B$, $[0,1]^d \subset B$, and let

$$K = \{p \in \mathbb{Z}^d : B \cap (p + B) \neq \emptyset\}.$$

It is obvious that K is a finite set. For a $\sigma \in \mathrm{SO}(d)$, let

$$\mathcal{S}(\sigma) = \bigcup\{(p + \sigma(\Sigma^*)) \cap (q + \sigma(\Sigma^*)) : p, q \in K,\ p \neq q\}.$$

Since $p \in v + \sigma(\Sigma) \;\Leftrightarrow\; v \in p + \sigma(\Sigma^*)$ by (2.1), it follows that

$$p, q \in v + \sigma(\Sigma) \;\Leftrightarrow\; v \in (p + \sigma(\Sigma^*)) \cap (q + \sigma(\Sigma^*)).$$

Hence, if $v \in B$, then

$$|\mathbb{Z}^d \cap (v + \sigma(\Sigma))| \geq 2 \;\Rightarrow\; v \in \mathcal{S}(\sigma). \tag{3.2}$$

LEMMA 3.3. *There is a $\sigma \in \mathrm{SO}(d)$ such that $B - \mathcal{S}(\sigma)$ is path-connected.*

Proof. Since Σ^* is a closed polyhedral surface in $\mathbb{R}^d$, there are hyperplanes $H_1, \ldots, H_k$ in $\mathbb{R}^d$ such that $\Sigma^* \subset \bigcup_{j=1}^{k} H_j$. Let

$$\mathcal{T}(\sigma) = \bigcup\{(p + \sigma(H_i)) \cap (q + \sigma(H_j)) : p, q \in K,\ p \neq q,\ 0 \leq i \leq j \leq k\}.$$

Then $\mathcal{S}(\sigma) \subset \mathcal{T}(\sigma)$. Let L be a line through O that pierces all H_j, and let V be any 2-dimensional subspace of $\mathbb{R}^d$ that contains L. Since $V \not\subset H_j$ and $\dim H_j + \dim V = n - 1 + 2$, it follows that V intersects every H_j in a line. By Lemma 3.2, the set of those $\sigma \in G_V$ that satisfy $p + \sigma(H_i) = q + \sigma(H_j)$ for some 4-tuple (p, q, i, j) (where $p, q \in K$, $p \neq q$, and $0 \leq i \leq j \leq k$) is also a finite set. Hence there is a $\sigma_0 \in G_V$ that is not contained in this set. Then, since $p + \sigma_0(H_i) \neq q + \sigma_0(H_j)$ for all (p, q, i, j), the intersection $(p + \sigma_0(H_i)) \cap (q + \sigma_0(H_j))$ is either $\emptyset$ or a $(d-2)$-dimensional flat. Therefore, $\mathcal{T}(\sigma_0)$ is a union of finitely many $(d-2)$-dimensional flats, and hence $B - \mathcal{S}(\sigma_0)$ is path-connected by Lemma 3.1. ∎

4. Proof of the theorem. Let Π be a d-dimensional polyhedron of volume $n + \alpha$, $|\alpha| < 1$, in $\mathbb{R}^d$. We use the same notation Σ, Σ^*, B, K, $\mathcal{S}(\sigma)$ as in the previous sections. By Lemma 3.3, we can choose $\sigma \in \mathrm{SO}(d)$ so that $B - \mathcal{S}(\sigma)$ is path-connected. Let Π° denote the interior of Π. Since the volumes of $\sigma(\Pi^\circ)$ and $\sigma(\Pi)$ are both equal to $n + \alpha$, it follows from Lemma 2.1 that there exist $v_1, v_2 \in B$ such that

$$|\mathbb{Z}^d \cap (v_1 + \sigma(\Pi))| \leq n, \qquad |\mathbb{Z}^d \cap (v_2 + \sigma(\Pi^\circ))| \geq n.$$

Let δ be the minimum distance from a lattice point lying in the exterior of $\sigma(\Pi)$ to $\sigma(\Sigma)$. Since $\sigma(\Sigma)$ is compact, δ is positive, and for any point v_1' within distance $\delta/2$ from v_1, we have $|\mathbb{Z}^d \cap (v_1' + \sigma(\Pi))| \leq n$. Hence, by replacing v_1 with some other point if necessary, we may assume that $v_1 \in B - \mathcal{S}(\sigma)$. Similarly, we may assume that $v_2 \in B - \mathcal{S}(\sigma)$.

To prove the theorem, we consider the case

$$|\mathbb{Z}^d \cap (v_1 + \sigma(\Pi))| < n \quad \text{and} \quad |\mathbb{Z}^d \cap (v_2 + \sigma(\Pi))| > n. \tag{4.1}$$

Since $B - \mathcal{S}(\sigma)$ is path-connected, there is a path J in $B - \mathcal{S}(\sigma)$ that connects v_1 to v_2. If $w \in J$, then $|\mathbb{Z}^d \cap (w + \sigma(\Sigma))| \leq 1$ by (3.2). This implies that when w moves from v_1 to v_2 along the path J, $|\mathbb{Z}^d \cap (w + \sigma(\Pi))|$ changes one by one. Hence, by (4.1), there exists a $w \in J$ such that $|\mathbb{Z}^d \cap (w + \sigma(\Pi))| = n$.

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Hiroshi Maehara
Ryukyu University
Nishihara, Okinawa 903-0213, Japan
E-mail: hmaehara@edu.u-ryukyu.ac.jp