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ON SPLIT REGULAR HOM-LIE COLOR ALGEBRAS

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Abstract. We introduce the class of split regular Hom-Lie color algebras as a natural generalization of split Lie color algebras. By developing techniques of connections of roots for this kind of algebras, we show that every split regular Hom-Lie color algebra L is of the form $L = U + \sum_{[j]\in\Lambda/\sim} I_{[j]}$ with U a subspace of an abelian graded subalgebra H and any $I_{[j]}$ a well-described ideal of L, satisfying $[I_{[j]}, I_{[k]}] = 0$ if $[j] \neq [k]$. Under certain conditions, in the case of L being of maximal length, the simplicity of the algebra is characterized.

1. Introduction. The notion of Hom-Lie algebra was introduced by Hartwig, Larsson and Silvestrov to describe the q-deformation of the Witt and the Virasoro algebras [9]. Since then, many authors have studied Hom-type algebras [1, 2, 10, 11, 12, 14]. The notion of Lie color algebras was introduced as generalized Lie algebras in 1960 by Ree [13]. So far, many results for this kind of algebras have been considered in the framework of enveloping algebras, representations and related problems [4, 8, 15]. In particular, Yuan [16] introduced the notion of Hom-Lie color algebra, which can be viewed as an extension of Hom-Lie (super)algebras to Γ-graded algebras, where Γ is any abelian group.

As is well-known, the class of split algebras is closely related to additive quantum numbers, graded contractions and deformations. For instance, for a physical system which displays a symmetry of L, it is interesting to know in detail the structure of the split decomposition because its roots can be seen as certain eigenvalues which are additive quantum numbers characterizing the state of the system. Determining the structure of split algebras is becoming of importance in mathematical physics. Recently, in [3, 5, 6, 7], the structures of arbitrary split Lie algebras, arbitrary split Lie color algebras, arbitrary split Leibniz triple systems and arbitrary split regular Hom-Lie algebras have been determined by the techniques of connections of roots. The purpose of this paper is to consider the structure of split regular Hom-Lie

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color algebras by the techniques of connections of roots based on some work in [3, 6]. Throughout this paper, split regular Hom-Lie color algebras L are of arbitrary dimension and over an arbitrary base field $\mathbb{K}$.

This paper is organized as follows. In Section 2, we establish the preliminaries on split regular Hom-Lie color algebras theory. In Section 3, we show that every such algebra L with a symmetric root system is of the form $L = U + \sum_{[j]\in\Lambda/\sim} I_{[j]}$ with U a subspace of an abelian subalgebra H and any $I_{[j]}$ a well-described ideal of L, satisfying $[I_{[j]}, I_{[k]}] = 0$ if $[j] \neq [k]$. In Section 4, under certain conditions, in the case of L of maximal length, the simplicity of the algebra is characterized. Moreover, Theorem 4.6 shows that under certain conditions the decomposition of L given in Section 3 is actually into simple ideals.

2. Preliminaries.

First we recall the definitions of Lie color algebras and Hom-Lie color algebras. The following definition is well-known from the theory of graded algebras.

DEFINITION 2.1 ([16]). Let Γ be an abelian group. A *bi-character* on Γ is a map $\varepsilon : \Gamma \times \Gamma \to \mathbb{K} \setminus \{0\}$ satisfying

- $\varepsilon(\alpha, \beta)\varepsilon(\beta, \alpha) = 1$,
- $\varepsilon(\alpha, \beta + \gamma) = \varepsilon(\alpha, \beta)\varepsilon(\alpha, \gamma)$,
- $\varepsilon(\alpha + \beta, \gamma) = \varepsilon(\alpha, \gamma)\varepsilon(\beta, \gamma)$,

for all $\alpha, \beta, \gamma \in \Gamma$.

It is clear that $\varepsilon(\alpha, 0) = \varepsilon(0, \alpha) = 1$ for any $\alpha \in \Gamma$, where 0 denotes the identity element of Γ.

DEFINITION 2.2 ([6]). Let $L = \bigoplus_{g\in\Gamma} L_g$ be a Γ-graded $\mathbb{K}$-vector space. For a nonzero homogeneous element $v \in L$, denote by $\bar{v}$ the unique group element in Γ such that $v \in L_{\bar{v}}$, which will be called the *homogeneous degree* of v. We shall say that L is a *Lie color algebra* if it is endowed with a $\mathbb{K}$-bilinear map $[\cdot, \cdot] : L \times L \to L$ satisfying

- $[v, w] = -\varepsilon(\bar{v}, \bar{w})[w, v]$ (skew-symmetry),
- $[v, [w, t]] = [[v, w], t] + \varepsilon(\bar{v}, \bar{w})[w, [v, t]]$ (Jacobi identity),

for all homogeneous $v, w, t \in L$.

Lie superalgebras are examples of Lie color algebras with $\Gamma = \mathbb{Z}_2$ and $\varepsilon(i, j) = (-1)^{ij}$ for any $i, j \in \mathbb{Z}_2$. We also note that L_0 is a Lie algebra.

DEFINITION 2.3 ([16]). A *Hom-Lie color algebra* is a quadruple $(L, [\cdot, \cdot], \phi, \varepsilon)$ consisting of a Γ-graded space L, an even bilinear mapping $[\cdot, \cdot] : L \times L \to L$, a linear homomorphism ϕ and a bi-character ε on Γ satisfying

- $[x, y] = -\varepsilon(\bar{x}, \bar{y})[y, x]$,

- $\varepsilon(\bar{z},\bar{x})[\phi(x),[y,z]]+\varepsilon(\bar{x},\bar{y})[\phi(y),[z,x]]+\varepsilon(\bar{y},\bar{z})[\phi(z),[x,y]]=0$,

for all homogeneous $x,y,z \in L$, where $\bar{x},\bar{y},\bar{z}$ denote the homogeneous degrees of x,y,z. When ϕ is furthermore an algebra automorphism, L is a *regular Hom-Lie color algebra.*

Clearly, Hom-Lie algebras and Lie color algebras are examples of Hom-Lie color algebras.

Throughout this paper we will consider regular Hom-Lie color algebras L of arbitrary dimension and over an arbitrary base field $\mathbb{K}$. Moreover, $\mathbb{N}$ denotes the set of all nonnegative integers, and $\mathbb{Z}$ the set of all integers.

For any $x \in L$, we consider the adjoint mapping $\mathrm{ad}_x : L \to L$ defined by $\mathrm{ad}_x(z)=[x,z]$. The usual regularity concepts will be understood in the graded sense. For instance, a *subalgebra* of L is a graded subspace $A=\bigoplus_{g\in\Gamma} A_g$ of L such that $[A,A]\subset A$ and $\phi(A)=A$. A graded subspace $I=\bigoplus_{g\in\Gamma} I_g$ of L is called an *ideal* if $[I,L]\subset I$ and $\phi(I)=I$. A Hom-Lie color algebra L will be called *simple* if $[L,L]\neq 0$ and its only (graded) ideals are 0 and L.

We introduce the concept of split regular Hom-Lie color algebra in an analogous way. We begin by considering a maximal abelian graded subalgebra $H=\bigoplus_{g\in\Gamma} H_g$ of L. Observe that H is necessarily a maximal abelian subalgebra of L:

Lemma 2.4. *Let $H=\bigoplus_{g\in\Gamma} H_g$ be a maximal abelian graded subalgebra of a Hom-Lie color algebra L. Then H is a maximal abelian subalgebra of L.*

Proof. Consider an abelian subalgebra K of L such that $H\subset K$. For any $x\in K$ we have $[x,H_g]=0$ for each $g\in\Gamma$, and so by writing $x=\sum_{i=1}^{n} x_{g_i}$ with $x_{g_i}\in L_{g_i}$ for $i=1,\ldots,n$, where $g_i\in\Gamma$ and $g_i\neq g_j$ if $i\neq j$, by the grading we get $[x_{g_i},H_g]=0$. Hence, for any g_i, $i=1,\ldots,n$, we see that $(H_{g_i}+\mathbb{K}x_{g_i})\oplus\bigoplus_{g\in\Gamma\setminus\{g_i\}} H_g$ is an abelian graded subalgebra of L containing H, and so $x_{g_i}\in H_{g_i}$. Therefore $x\in H$, and so $K=H$. ∎

Let us introduce the class of split algebras in the framework of regular Hom-Lie color algebras. Denote by $H=\bigoplus_{g\in\Gamma} H_g$ a maximal abelian (graded) subalgebra (MAGSA) of a regular Hom-Lie color algebra L. For a linear functional $\alpha : H_0 \to \mathbb{K}$, we define the *root space* of L (with respect to H) associated to α as the subspace

$$L_\alpha=\{v_\alpha\in L : [h_0,v_\alpha]=\alpha(h_0)\phi(v_\alpha) \text{ for any } h_0\in H_0\}.$$

The elements $\alpha : H_0\to\mathbb{K}$ satisfying $L_\alpha\neq 0$ are called *roots* of L with respect to H. We denote $\Lambda:=\{\alpha\in H_0^*\setminus\{o\} : L_\alpha\neq 0\}$.

Definition 2.5. We say that L is a *split regular Hom-Lie color algebra*, with respect to H, if

$$L = H \oplus \bigoplus_{\alpha \in \Lambda} L_\alpha .$$

We also say that Λ is the *root system* of L.

Note that when $\phi = \mathrm{Id}$, split Lie color algebras become examples of split regular Hom-Lie color algebras. Hence, the present paper extends the results of [6]. For convenience, the mappings $\phi|_H$, $\phi|_H^{-1} : H \to H$ will be denoted by ϕ and ϕ^{-1} respectively.

It is clear that the root space L_o associated to the zero root satisfies $H \subset L_o$. Conversely, given any $v_o \in L_o$ we can write

$$v_o = h \oplus \bigoplus_{i=1}^{n} v_{\alpha_i},$$

where $h \in H$ and $v_{\alpha_i} \in L_{\alpha_i}$ for $i = 1, \ldots, n$, with $\alpha_i \neq \alpha_j$ if $i \neq j$. Hence

$$0 = \Big[h_0, h \oplus \bigoplus_{i=1}^{n} v_{\alpha_i}\Big] = \bigoplus_{i=1}^{n} \alpha_i(h_0)\phi(v_{\alpha_i})$$

for any $h_0 \in H_0$. As the sum is direct and $\alpha_i \neq 0$, we get $v_{\alpha_i} = 0$ for $i = 1, \ldots, n$. So $v_o = h \in H$. Consequently,

$$H = L_o. \tag{2.1}$$

Lemma 2.6. *Let $L = \bigoplus_{g \in \Gamma} L_g$ be a split Hom-Lie color algebra with corresponding root space decomposition $L = H \oplus \bigoplus_{\alpha \in \Lambda} L_\alpha$. Set $L_{\alpha,g} = L_\alpha \cap L_g$. Then:*

- (i) *$L_\alpha = \bigoplus_{g \in \Gamma} L_{\alpha,g}$ for any $\alpha \in \Lambda \cup \{o\}$.*
- (ii) *$H_g = L_{o,g}$. In particular $H_0 = L_{o,0}$.*
- (iii) *L_0 is a split Hom-Lie algebra, with respect to H_0, with root space decomposition $L_0 = H_0 \oplus \bigoplus_{\alpha \in \Lambda} L_{\alpha,0}$.*

Proof. (i) By the Γ-grading of L we may express any $v_\alpha \in L_\alpha$, $\alpha \in \Lambda \cup \{o\}$, in the form $v_\alpha = v_{\alpha,g_1} + \cdots + v_{\alpha,g_n}$ with $v_{\alpha,g_i} \in L_{g_i}$ for distinct $g_1, \ldots, g_n \in \Gamma$. If $h_0 \in H_0$ then $[h_0, v_{\alpha,g_i}] = \alpha(h_0)\phi(v_{\alpha,g_i})$ for $i = 1, \ldots, n$. Hence $L_\alpha = \bigoplus_{g \in \Gamma}(L_\alpha \cap L_g)$ and we can write $L_\alpha = \bigoplus_{g \in \Gamma} L_{\alpha,g}$ for any $\alpha \in \Lambda \cup \{o\}$.

(ii) A consequence of (2.1) and (i).

(iii) We have $L_g = H_g \oplus \bigoplus_{\alpha \in \Lambda} L_{\alpha,g}$ for any $g \in \Gamma$. For $g = 0$ we get $L_0 = H_0 \oplus \bigoplus_{\alpha \in \Lambda} L_{\alpha,0}$. As the sum is direct and $\alpha \neq 0$ for any $\alpha \in \Lambda$, it follows that H_0 is a MASA of the Hom-Lie algebra L_0. Hence L_0 is a split Hom-Lie algebra with respect to H_0. ■

Lemma 2.7. *For any* $\alpha, \beta \in \Lambda \cup \{o\}$*:*

(i) $\phi(L_\alpha) \subset L_{\alpha\phi^{-1}}$ *and* $\phi^{-1}(L_\alpha) \subset L_{\alpha\phi}$.
(ii) $[L_\alpha, L_\beta] \subset L_{\alpha\phi^{-1}+\beta\phi^{-1}}$.

Proof. (i) For $h_0 \in H_0$ write $h_0' = \phi(h_0)$. Then for all $h_0 \in H_0$ and $v_\alpha \in L_\alpha$, since $[h_0, v_\alpha] = \alpha(h_0)\phi(v_\alpha)$, one has

$$[h_0', \phi(v_\alpha)] = \phi([h_0, v_\alpha]) = \alpha(h_0)\phi(\phi(v_\alpha)) = \alpha\phi^{-1}(h_0')\phi(\phi(v_\alpha)).$$

Therefore $\phi(v_\alpha) \in L_{\alpha\phi^{-1}}$, and so $\phi(L_\alpha) \subset L_{\alpha\phi^{-1}}$. In a similar way, one gets $\phi^{-1}(L_\alpha) \subset L_{\alpha\phi}$.

(ii) For any $h_0 \in H_0$, $v_\alpha \in L_\alpha$ and $v_\beta \in L_\beta$, denoting $h_0' = \phi(h_0)$ and applying the Hom-Jacobi identity, we obtain

$$\begin{aligned}[h_0', [v_\alpha, v_\beta]] &= [[h_0, v_\alpha], \phi(v_\beta)] + \varepsilon(\bar{h_0}, \bar{v_\alpha})[\phi(v_\alpha), [h_0, v_\beta]] \\ &= [\alpha(h_0)\phi(v_\alpha), \phi(v_\beta)] + \beta(h_0)[\phi(v_\alpha), \phi(v_\beta)] \\ &= (\alpha+\beta)(h_0)\phi([v_\alpha, v_\beta]) = (\alpha+\beta)\phi^{-1}(h_0')\phi([v_\alpha, v_\beta]).\end{aligned}$$

Therefore $[v_\alpha, v_\beta] \in L_{\alpha\phi^{-1}+\beta\phi^{-1}}$, and so $[L_\alpha, L_\beta] \subset L_{\alpha\phi^{-1}+\beta\phi^{-1}}$. ■

Lemma 2.8. *If* $\alpha \in \Lambda$ *then* $\alpha\phi^{-z} \in \Lambda$ *for any* $z \in \mathbb{Z}$.

Proof. This is a consequence of Lemma 2.7(i). ■

Definition 2.9. A root system Λ of a split Hom-Lie color algebra is called *symmetric* if $\alpha \in \Lambda$ implies $-\alpha \in \Lambda$.

3. Decompositions. In the following, L denotes a split regular Hom-Lie color algebra with a symmetric root system Λ, and $L = H \oplus \bigoplus_{\alpha\in\Lambda} L_\alpha$ the corresponding root decomposition. Given a linear functional $\alpha : H \to \mathbb{K}$, we denote by $-\alpha : H \to \mathbb{K}$ the element in H^* defined by $(-\alpha)(h) := -\alpha(h)$ for all $h \in H$. We begin by developing the techniques of connections of roots.

Definition 3.1. Let α and β be nonzero roots. We shall say that α is *connected* to β if there exist $\alpha_1, \ldots, \alpha_k \in \Lambda$ such that:

- If $k = 1$ then $\alpha_1 \in \{\alpha\phi^{-n} : n \in \mathbb{N}\} \cap \{\pm\beta\phi^{-m} : m \in \mathbb{N}\}$. If $k \geq 2$ then $\alpha_1 \in \{\alpha\phi^{-n} : n \in \mathbb{N}\}$.
- $\alpha_1\phi^{-1} + \alpha_2\phi^{-1} \in \Lambda$,
  $\alpha_1\phi^{-2} + \alpha_2\phi^{-2} + \alpha_3\phi^{-1} \in \Lambda$,
  $\alpha_1\phi^{-3} + \alpha_2\phi^{-3} + \alpha_3\phi^{-2} + \alpha_4\phi^{-1} \in \Lambda$,
  $\ldots$
  $\alpha_1\phi^{-i} + \alpha_2\phi^{-i} + \alpha_3\phi^{-i+1} + \cdots + \alpha_{i+1}\phi^{-1} \in \Lambda$,
  $\ldots$
  $\alpha_1\phi^{-k+2} + \alpha_2\phi^{-k+2} + \alpha_3\phi^{-k+3} + \cdots + \alpha_i\phi^{-k+i} + \cdots + \alpha_{k-1}\phi^{-1} \in \Lambda$.
- $\alpha_1\phi^{-k+1} + \alpha_2\phi^{-k+1} + \alpha_3\phi^{-k+2} + \cdots + \alpha_i\phi^{-k+i-1} + \cdots + \alpha_k\phi^{-1} \in \{\pm\beta\phi^{-m} : m \in \mathbb{N}\}$.

We shall also say that $\{\alpha_1, \ldots, \alpha_k\}$ is a *connection* from α to β.

Observe that the case $k=1$ in Definition 3.1 is equivalent to $\beta=\epsilon\alpha\phi^{z}$ for some $z\in\mathbb{Z}$ and $\epsilon\in\{\pm1\}$.

LEMMA 3.2. *For any* $\alpha\in\Lambda$, $\alpha\phi^{z_1}$ *is connected to* $\alpha\phi^{z_2}$ *for every* $z_1,z_2\in\mathbb{Z}$. *Also,* $\alpha\phi^{z_1}$ *is connected to* $-\alpha\phi^{z_2}$ *in case* $-\alpha\phi^{z_2}\in\Lambda$.

Proof. This can be proved analogously to [3, Lemma 2.2]. ■

LEMMA 3.3. *Let* $\{\alpha_1,\ldots,\alpha_k\}$ *be a connection from* α *to* β.

(i) *Suppose that* $\alpha_1=\alpha\phi^{-n}$, $n\in\mathbb{N}$. *Then for any* $r\in\mathbb{N}$ *such that* $r\geq n$, *there exists a connection* $\{\overline{\alpha}_1,\ldots,\overline{\alpha}_k\}$ *from* α *to* β *such that* $\overline{\alpha}_1=\alpha\phi^{-r}$.

(ii) *Suppose that* $\alpha_1=\epsilon\beta\phi^{-m}$ *in case* $k=1$, *while*

$$\alpha_1\phi^{-k+1}+\alpha_2\phi^{-k+1}+\alpha_3\phi^{-k+2}+\cdots+\alpha_k\phi^{-1}=\epsilon\beta\phi^{-m}$$

in case $k\geq2$, *with* $m\in\mathbb{N}$ *and* $\epsilon\in\{\pm1\}$. *Then for any* $r\in\mathbb{N}$ *such that* $r\geq m$, *there exists a connection* $\{\overline{\alpha}_1,\ldots,\overline{\alpha}_k\}$ *from* α *to* β *such that* $\overline{\alpha}_1=\epsilon\beta\phi^{-r}$ *in case* $k=1$, *while*

$$\overline{\alpha}_1\phi^{-k+1}+\overline{\alpha}_2\phi^{-k+1}+\overline{\alpha}_3\phi^{-k+2}+\cdots+\overline{\alpha}_k\phi^{-1}=\epsilon\beta\phi^{-r}$$

in case $k\geq2$.

Proof. This can be proved analogously to [3, Lemma 2.3]. ■

PROPOSITION 3.4. *The relation* $\sim$ *in* Λ, *defined by* $\alpha\sim\beta$ *if and only if* α *is connected to* β, *is an equivalence relation.*

Proof. This can be proved analogously to [3, Proposition 2.4]. ■

For any $\alpha\in\Lambda$, we denote

$$\Lambda_\alpha:=\{\beta\in\Lambda:\beta\sim\alpha\}.$$

Clearly, if $\beta\in\Lambda_\alpha$ then $-\beta\in\Lambda_\alpha$, and by Proposition 3.4, if $\gamma\notin\Lambda_\alpha$ then $\Lambda_\alpha\cap\Lambda_\gamma=\emptyset$.

Our next goal is to associate a suitable ideal L_{Λ_α} of L to any Λ_α. For $\alpha\in\Lambda$, we define

$$H_{\Lambda_\alpha}:=\operatorname{span}_{\mathbb{K}}\{[L_\beta,L_{-\beta}]:\beta\in\Lambda_\alpha\}.$$

Then H_{Λ_α} is the direct sum of

$$\sum_{\beta\in\Lambda_\alpha,\,g\in\Gamma}[L_{\beta,g},L_{-\beta,-g}]\subseteq H_0$$

and

$$\sum_{\substack{\beta\in\Lambda_\alpha\\ g,g'\in\Gamma,\,g+g'\neq0}}[L_{\beta,g},L_{-\beta,g'}]\subseteq\bigoplus_{g\in\Gamma\setminus\{0\}}H_g.$$

We also define

$$V_{\Lambda_\alpha} := \bigoplus_{\beta\in\Lambda_\alpha} L_\beta = \bigoplus_{\beta\in\Lambda_\alpha,\, g\in\Gamma} L_{\beta,g}.$$

Finally, we denote by L_{Λ_α} the following graded subspace of L,

$$L_{\Lambda_\alpha} := H_{\Lambda_\alpha} \oplus V_{\Lambda_\alpha}.$$

Proposition 3.5. *For any $\alpha \in \Lambda$, the linear subspace L_{Λ_α} is a subalgebra of L.*

Proof. First we have to check that $[L_{\Lambda_\alpha}, L_{\Lambda_\alpha}] \subset L_{\Lambda_\alpha}$. Taking into account $H = L_o$, we have

$$\begin{aligned}(3.1)\qquad [L_{\Lambda_\alpha}, L_{\Lambda_\alpha}] &= [H_{\Lambda_\alpha} \oplus V_{\Lambda_\alpha}, H_{\Lambda_\alpha} \oplus V_{\Lambda_\alpha}]\\ &\subset [H_{\Lambda_\alpha}, V_{\Lambda_\alpha}] + [V_{\Lambda_\alpha}, H_{\Lambda_\alpha}] + \sum_{\beta,\delta\in\Lambda_\alpha} [L_\beta, L_\delta].\end{aligned}$$

Let us consider the first summand on the right hand side. From $H_{\Lambda_\alpha} \subset H = L_o$, given $\beta \in \Lambda_\alpha$, we have $[H_{\Lambda_\alpha}, L_\beta] \subset L_{\beta\phi^{-1}}$ with $\beta\phi^{-1} \in \Lambda_\alpha$ by Lemma 2.7(ii). Hence,

$$(3.2)\qquad [H_{\Lambda_\alpha}, V_{\Lambda_\alpha}] \subset V_{\Lambda_\alpha}.$$

Similarly, we can also get

$$(3.3)\qquad [V_{\Lambda_\alpha}, H_{\Lambda_\alpha}] \subset V_{\Lambda_\alpha}.$$

Consider now the third summand $\sum_{\beta,\delta\in\Lambda_\alpha}[L_\beta, L_\delta]$. Given $\beta, \delta \in \Lambda_\alpha$ such that $[L_\beta, L_\delta] \neq 0$, if $\delta = -\beta$ then clearly $[L_\beta, L_\delta] = [L_\beta, L_{-\beta}] \subset H_{\Lambda_\alpha}$. Suppose that $\delta \neq -\beta$. Since $[L_\beta, L_\delta] \neq 0$ together with Lemma 2.7(ii) ensures that $\beta\phi^{-1} + \delta\phi^{-1} \in \Lambda$, we see that $\{\beta, \delta\}$ is a connection from β to $\beta\phi^{-1} + \delta\phi^{-1}$. The transitivity of $\sim$ gives $\beta\phi^{-1} + \delta\phi^{-1} \in \Lambda_\alpha$, and so

$$(3.4)\qquad [L_\beta, L_\delta] \subset L_{\beta\phi^{-1}+\delta\phi^{-1}} \subset V_{\Lambda_\alpha}.$$

From (3.1)–(3.4) we conclude that $[L_{\Lambda_\alpha}, L_{\Lambda_\alpha}] \subset L_{\Lambda_\alpha}$.

Second, we have to verify that $\phi(L_{\Lambda_\alpha}) = L_{\Lambda_\alpha}$. But this is a direct consequence of Lemmas 2.7(i) and 3.2. ∎

Proposition 3.6. *If $\gamma \notin \Lambda_\alpha$ then $[L_{\Lambda_\alpha}, L_{\Lambda_\gamma}] = 0$.*

Proof. We have

$$\begin{aligned}(3.5)\qquad [L_{\Lambda_\alpha}, L_{\Lambda_\gamma}] &= [H_{\Lambda_\alpha} \oplus V_{\Lambda_\alpha}, H_{\Lambda_\gamma} \oplus V_{\Lambda_\gamma}]\\ &\subset [H_{\Lambda_\alpha}, V_{\Lambda_\gamma}] + [V_{\Lambda_\alpha}, H_{\Lambda_\gamma}] + [V_{\Lambda_\alpha}, V_{\Lambda_\gamma}].\end{aligned}$$

Consider the third summand $[V_{\Lambda_\alpha}, V_{\Lambda_\gamma}]$. Suppose that there exist $\beta \in \Lambda_\alpha$ and $\eta \in \Lambda_\gamma$ such that $[L_\beta, L_\eta] \neq 0$. As necessarily $\beta \neq -\eta$, we have $\beta\phi^{-1} + \eta\phi^{-1} \in \Lambda$. So $\{\beta, \eta, -\beta\phi^{-1}\}$ is a connection between β and η. By the transitivity of $\sim$ we have $\gamma \in \Lambda_\alpha$, a contradiction. Hence $[L_\beta, L_\eta] = 0$, and so

$$(3.6)\qquad [V_{\Lambda_\alpha}, V_{\Lambda_\gamma}] = 0.$$

Consider now the first summand $[H_{\Lambda_\alpha}, V_{\Lambda_\gamma}]$ in (3.5). Suppose there exist $\beta \in \Lambda_\alpha$ and $\eta \in \Lambda_\gamma$ such that $[[L_\beta, L_{-\beta}], \phi(L_\eta)] \neq 0$. Then

$$[[L_{\beta,g}, L_{-\beta,g'}], \phi(L_\eta)] \neq 0$$

for some $g, g' \in \Gamma$. By the Hom-Jacobi identity, either $[L_{-\beta,g'}, \phi(L_\eta)] \neq 0$ or $[L_{\beta,g}, \phi(L_\eta)] \neq 0$, and so $[V_{\Lambda_\alpha}, V_{\Lambda_\gamma}] \neq 0$ in any case, which contradicts (3.6). Hence

$$[H_{\Lambda_\alpha}, V_{\Lambda_\gamma}] = 0.$$

Finally, the same argument shows $[V_{\Lambda_\gamma}, H_{\Lambda_\alpha}] = 0$. By (3.5) we conclude that $[L_{\Lambda_\alpha}, L_{\Lambda_\gamma}] = 0$. ∎

THEOREM 3.7.

(i) *For any $\alpha \in \Lambda$, the Hom-Lie color subalgebra $L_{\Lambda_\alpha} = H_{\Lambda_\alpha} \oplus V_{\Lambda_\alpha}$ of L associated to Λ_α is an ideal of L.*
(ii) *If L is simple, then there exists a connection from α to β for any $\alpha, \beta \in \Lambda$ and $H = \sum_{\alpha\in\Lambda}[L_\alpha, L_{-\alpha}]$.*

Proof. (i) Since $[L_{\Lambda_\alpha}, H] = [L_{\Lambda_\alpha}, L_o] \subset V_{[\alpha]}$, taking into account Propositions 3.5 and 3.6 we have

$$[L_{\Lambda_\alpha}, L] = \Big[L_{\Lambda_\alpha}, H \oplus \bigoplus_{\beta\in\Lambda_\alpha} L_\beta \oplus \bigoplus_{\gamma\notin\Lambda_\alpha} L_\gamma\Big] \subset L_{\Lambda_\alpha}.$$

As $\phi(L_{\Lambda_\alpha}) = L_{\Lambda_\alpha}$ by Proposition 3.5, we conclude that L_{Λ_α} is an ideal of L.

(ii) The simplicity of L implies $L_{\Lambda_\alpha} = L$. Hence clearly $\Lambda_\alpha = \Lambda$ and $H = \sum_{\alpha\in\Lambda}[L_\alpha, L_{-\alpha}]$. ∎

THEOREM 3.8. *For a vector space complement U of* $\mathrm{span}_{\mathbb{K}}\{[L_\alpha, L_{-\alpha}] : \alpha \in \Lambda\}$ *in H, we have*

$$L = U + \sum_{[\alpha]\in\Lambda/\sim} I_{[\alpha]},$$

where any $I_{[\alpha]}$ is one of the ideals of L described in Theorem 3.7(i), *satisfying $[I_{[\alpha]}, I_{[\beta]}] = 0$, whenever $[\alpha] \neq [\beta]$.*

Proof. By Proposition 3.4, we can consider the quotient set $\Lambda/\sim := \{[\alpha] : \alpha \in \Lambda\}$. Set $I_{[\alpha]} := L_{\Lambda_\alpha}$. Then $I_{[\alpha]}$ is well-defined, and by Theorem 3.7(i), it is an ideal of L. Therefore

$$L = U + \sum_{[\alpha]\in\Lambda/\sim} I_{[\alpha]}.$$

By applying Proposition 3.6 we also obtain $[I_{[\alpha]}, I_{[\beta]}] = 0$ if $[\alpha] \neq [\beta]$. ∎

DEFINITION 3.9. The *annihilator* of a Hom-Lie color algebra L is the set $\mathrm{Z}(L) = \{x \in L : [x, L] = 0\}$.

Corollary 3.10. *If* $\mathrm{Z}(L) = 0$ *and* $[L, L] = L$, *then* L *is the direct sum of the ideals given in Theorem* 3.7,

$$L = \bigoplus_{[\alpha]\in\Lambda/\sim} I_{[\alpha]}.$$

Proof. From $[L, L] = L$, it is clear that $L = \bigoplus_{[\alpha]\in\Lambda/\sim} I_{[\alpha]}$. The sum is direct because $\mathrm{Z}(L) = 0$ and $[I_{[\alpha]}, I_{[\beta]}] = 0$ if $[\alpha] \neq [\beta]$. ■

4. The simplicity of split regular Hom-Lie color algebras of maximal length. In this section we consider the simplicity of split regular Hom-Lie color algebras, centering our attention on those of maximal length. From now on, $\mathrm{char}(\mathbb{K}) = 0$.

Lemma 4.1. *Let* $L = H \oplus \bigoplus_{\alpha\in\Lambda} L_\alpha$ *be a split regular Hom-Lie color algebra. If* I *is an ideal of* L *then* $I = (I \cap H) \oplus \bigoplus_{\alpha\in\Lambda}(I \cap L_\alpha)$.

Proof. We may view $L = H \oplus \bigoplus_{\alpha\in\Lambda} L_\alpha$ as a weight module with respect to the split Hom-Lie color algebra L_0 with maximal abelian subalgebra H_0 (see Lemma 2.6(iii)), in the natural way. The characteristic property of ideals shows that I is a submodule of L. It is well-known that a submodule of a weight module is again a weight module. Hence, I is a weight module with respect to L_0 (and H_0), and so $I = (I \cap H) \oplus \bigoplus_{\alpha\in\Lambda}(I \cap L_\alpha)$. ■

Taking into account the above lemma, observe that the grading of I and Lemma 2.6(i) let us write

$$I = \bigoplus_{g\in\Gamma} I_g = \bigoplus_{g\in\Gamma}\Big((I_g \cap H_g) \oplus \bigoplus_{\alpha\in\Lambda}(I_g \cap L_{\alpha,g})\Big). \tag{4.1}$$

Lemma 4.2. *Let* L *be a split regular Hom-Lie color algebra with* $\mathrm{Z}(L) = 0$ *and* I *an ideal of* L. *If* $I \subset H$ *then* $I = \{0\}$.

Proof. Suppose there exists a nonzero ideal I of L such that $I \subset H$. We get $[I, H] \subset [H, H] = 0$. Moreover, $[I, \bigoplus_{\alpha\in\Lambda} L_\alpha] \subset I \subset H$. Then taking into account $H = L_o$, we have $[I, \bigoplus_{\alpha\in\Lambda} L_\alpha] \subset H \cap \bigoplus_{\alpha\in\Lambda} L_\alpha = 0$. Hence $I \subset \mathrm{Z}(L) = 0$, a contradiction. ■

Let us introduce the concepts of root-multiplicativity and maximal length in the framework of split Hom-Lie color algebras, in a similar way to the ones for split Hom Lie algebras (see [3]). For each $g \in \Gamma$, set $\Lambda_g := \{\alpha \in \Lambda : L_{\alpha,g} \neq 0\}$.

Definition 4.3. We say that a split regular Hom-Lie color algebra L is *root-multiplicative* if given $\alpha \in \Lambda_{g_i}$ and $\beta \in \Lambda_{g_j}$ with $g_i, g_j \in \Gamma$ such that $\alpha + \beta \in \Lambda$, we have $[L_{\alpha,g_i}, L_{\beta,g_j}] \neq 0$.

Definition 4.4. We say that a split regular Hom-Lie color algebra L is of *maximal length* if for any $\alpha \in \Lambda_g$, $g \in \Gamma$, we have $\dim L_{\kappa\alpha,\kappa g} = 1$ for $\kappa \in \{\pm 1\}$.

Observe that if L is of maximal lenth, then (4.1) entails that for any nonzero ideal I of L,

$$I = \bigoplus_{g\in\Gamma}\Big((I_g \cap H_g) \oplus \bigoplus_{\alpha\in\Lambda_g^I} L_{\alpha,g}\Big) \tag{4.2}$$

where $\Lambda_g^I := \{\alpha \in \Lambda : I_g \cap L_{\alpha,g} \neq 0\}$ for each $g \in \Gamma$.

Theorem 4.5. *Let L be a split regular Hom-Lie color algebra of maximal length, root multiplicative and with* $\mathrm{Z}(L) = 0$. *Then L is simple if and only if it has all its nonzero roots connected and* $H = \sum_{\alpha\in\Lambda}[L_\alpha, L_{-\alpha}]$.

Proof. The direct implication is Theorem 3.7(ii). To prove the converse, let I be a nonzero ideal of L. By Lemma 4.2 and (4.2) we can write $I = \bigoplus_{g\in\Gamma}\big((I_g \cap H_g) \oplus \bigoplus_{\alpha\in\Lambda_g^I} L_{\alpha,g}\big)$ with $\Lambda_g^I \subset \Lambda_g$ for any $g \in \Gamma$ and some $\Lambda_g^I \neq \emptyset$. Hence, we may choose $\alpha_0 \in \Lambda_g^I$ with

$$0 \neq L_{\alpha_0,g} \subset I. \tag{4.3}$$

The fact that $\phi(I) = I$ together with Lemma 2.7(i) implies that

$$\text{if} \quad \alpha \in \Lambda_I \quad \text{then} \quad \{\alpha\phi^z : z \in \mathbb{Z}\} \subset \Lambda_I, \tag{4.4}$$

that is,

$$\{L_{\alpha_0\phi^z,g} : z \in \mathbb{Z}\} \subset I. \tag{4.5}$$

Now, take any $\beta \in \Lambda \setminus \{\pm\alpha_0\phi^z : z \in \mathbb{Z}\}$. Since α_0 and β are connected, we have a connection $\{\gamma_1, \dots, \gamma_k\}$, $k \geq 2$, from α_0 to β satisfying:

$$\begin{aligned}
&\gamma_1 = a_0\phi^{-n} \text{ for some } n \in \mathbb{N},\\
&\gamma_1\phi^{-1} + \gamma_2\phi^{-1} \in \Lambda,\\
&\gamma_1\phi^{-2} + \gamma_2\phi^{-2} + \gamma_3\phi^{-1} \in \Lambda,\\
&\cdots\\
&\gamma_1\phi^{-i} + \gamma_2\phi^{-i} + \gamma_3\phi^{-i+1} + \cdots + \gamma_{i+1}\phi^{-1} \in \Lambda,\\
&\cdots\\
&\gamma_1\phi^{-k+2} + \gamma_2\phi^{-k+2} + \gamma_3\phi^{-k+3} + \cdots + \gamma_i\phi^{-k+i} + \cdots + \gamma_{k-1}\phi^{-1} \in \Lambda,\\
&\gamma_1\phi^{-k+1} + \gamma_2\phi^{-k+1} + \gamma_3\phi^{-k+2} + \cdots + \gamma_i\phi^{-k+i-1} + \cdots + \gamma_k\phi^{-1} = \epsilon\beta\phi^{-m}
\end{aligned}$$

for some $m \in \mathbb{N}$ and $\epsilon \in \{\pm 1\}$.

Consider γ_1, γ_2 and $\gamma_1 + \gamma_2$. Since $\gamma_2 \in \Lambda$, there exists $g_1 \in \Gamma$ such that $L_{\gamma_2,g_1} \neq 0$. The root-multiplicativity and maximal length of L show

$0 \neq [L_{\gamma_1,g}, L_{\gamma_2,g_1}] = L_{(\gamma_1+\gamma_2)\phi^{-1},g+g_1}$, and by (4.5),

$$0 \neq L_{(\gamma_1+\gamma_2)\phi^{-1},g+g_1} \subset I.$$

We can argue in a similar way for $\gamma_1\phi^{-1} + \gamma_2\phi^{-1}$, γ_3 and $\gamma_1\phi^{-2} + \gamma_2\phi^{-2} + \gamma_3\phi^{-1}$ to get

$$0 \neq L_{\gamma_1\phi^{-2}+\gamma_2\phi^{-2}+\gamma_3\phi^{-1},g_2} \subset I$$

for some $g_2 \in \mathit{\Gamma}$. Following this process with the connection $\{\gamma_1, \dots, \gamma_k\}$ we obtain

$$0 \neq L_{\gamma_1\phi^{-k+1}+\gamma_2\phi^{-k+1}+\gamma_3\phi^{-k+2}+\dots+\gamma_k\phi^{-1},g_3} \subset I,$$

and so either $0 \neq L_{\beta\phi^{-m},g_3} \subset I$ or $0 \neq L_{-\beta\phi^{-m},g_3} \subset I$ for some $g_3 \in \mathit{\Gamma}$. That is,

$$0 \neq L_{\epsilon\beta\phi^{-m},g_3} \subset I \quad \text{for some } \epsilon \in \{\pm 1\},\ g_3 \in \mathit{\Gamma}, \tag{4.6}$$

for any fixed $\beta \in \mathit{\Lambda}$. By Lemma 2.7(i), we can get

$$0 \neq L_{\epsilon\beta,g_3} \subset I \quad \text{for some } \epsilon \in \{\pm 1\},\ g_3 \in \mathit{\Gamma}, \tag{4.7}$$

for any fixed $\beta \in \mathit{\Lambda}$.

Taking into account $H = \sum_{\gamma\in\mathit{\Lambda}}[L_\gamma, L_{-\gamma}]$, the grading of L gives

$$H_0 = \sum_{\gamma\in\mathit{\Lambda},\, g\in\mathit{\Gamma}} [L_{\gamma,g}, L_{-\gamma,-g}].$$

Hence, there exist $\gamma \in \mathit{\Lambda}$ and $g_4 \in \mathit{\Gamma}$ such that

$$\big[[L_{\gamma,g_4}, L_{-\gamma,-g_4}], \phi(L_{\epsilon\beta,g_3})\big] \neq 0. \tag{4.8}$$

By the Hom-Jacobi identity, either $[L_{\gamma,g_4}, \phi(L_{\epsilon\beta,g_3})] \neq 0$ or $[L_{-\gamma,-g_4}, \phi(L_{\epsilon\beta,g_3})] \neq 0$, and so $L_{\gamma\phi^{-1}+\epsilon\beta\phi^{-2},g_4+g_3} \neq 0$ or $L_{-\gamma\phi^{-1}+\epsilon\beta\phi^{-2},-g_4+g_3} \neq 0$. That is,

$$0 \neq L_{\kappa\gamma\phi^{-1}+\epsilon\beta\phi^{-2},\kappa g_4+g_3} \subset I \tag{4.9}$$

for some $\kappa \in \{\pm 1\}$. Since $\epsilon\beta \in \mathit{\Lambda}_{g_3}$, from the maximal length of L we deduce that $-\epsilon\beta \in \mathit{\Lambda}_{-g_3}$. By (4.9) and the root-multiplicativity and maximal length of L, we obtain

$$0 \neq [L_{\kappa\gamma\phi^{-1}+\epsilon\beta\phi^{-2},\kappa g_4+g_3}, L_{-\epsilon\beta\phi^{-2},-g_3}] = L_{\kappa\gamma\phi^{-2},\kappa g_4} \subset I. \tag{4.10}$$

By Lemma 2.7(i), we get

$$L_{\kappa\gamma,\kappa g_4} \subset I. \tag{4.11}$$

Taking into account (4.10) and the fact that (4.8) gives

$$\beta\phi^{-1}([L_{\gamma,g_4}, L_{-\gamma,-g_4}]) \neq 0,$$

we find that for any $g_5 \in \mathit{\Gamma}$ such that $L_{\epsilon\beta,g_5} \neq 0$ necessarily

$$0 \neq \big[[L_{\gamma,g_4}, L_{-\gamma,-g_4}], \phi(L_{\epsilon\beta,g_5})\big] = L_{\epsilon\beta\phi^{-1},g_5} \subset I,$$

and so $L_{\epsilon\beta\phi^{-1}} \subset I$. That is,

$$L_{\epsilon\beta} \subset I \tag{4.12}$$

for any $\beta \in \Lambda$ with some $\epsilon \in \{\pm 1\}$. Since $H = \sum_{\beta \in \Lambda}[L_\beta, L_{-\beta}]$, we get

$$H \subset I. \tag{4.13}$$

Now, given any $-\epsilon\beta \in \Lambda$, as $-\epsilon\beta \neq 0$, $H \subset I$ and L has maximal length, we have

$$[H, L_{-\epsilon\beta\phi}] = L_{-\epsilon\beta} \subset I. \tag{4.14}$$

From (4.12)–(4.14) we conclude that $I = L$, so L is simple. ∎

Theorem 4.6. *Let L be a split regular Hom-Lie color algebra of maximal length, root-multiplicative and satisfying* $\mathrm{Z}(L) = 0$, $[L, L] = L$. *Then L is the direct sum of the family of its minimal ideals, each one being a simple split regular Hom-Lie color algebra having all its nonzero roots connected.*

Proof. By Corollary 3.10, $L = \bigoplus_{[\alpha] \in \Lambda/\sim} I_{[\alpha]}$ is the direct sum of the ideals $I_{[\alpha]} = H_{\Lambda_\alpha} \oplus V_{\Lambda_\alpha} = (\sum_{\beta \in [\alpha]}[L_\beta, L_{-\beta}]) \oplus \bigoplus_{\beta \in [\alpha]} L_\beta$ such that any $I_{[\alpha]}$ has root system Λ_α with all of its roots connected. It is easy to check that Λ_α has all of its roots Λ_α-connected (i.e. connected through roots in Λ_α). Moreover, each $I_{[\alpha]}$ is root-multiplicative since L is. Clearly, $I_{[\alpha]}$ is of maximal length, and finally $\mathrm{Z}_{I_{[\alpha]}}(I_{[\alpha]}) = 0$ (where $\mathrm{Z}_{I_{[\alpha]}}(I_{[\alpha]})$) denotes the center of $I_{[\alpha]}$ in $I_{[\alpha]}$, as a consequence of $[I_{[\alpha]}, I_{[\beta]}] = 0$ if $[\alpha] \neq [\beta]$ (Theorem 3.8) and $\mathrm{Z}(L) = 0$. We can apply Theorem 4.5 to any $I_{[\alpha]}$ to conclude that $I_{[\alpha]}$ is simple. It is clear that the decomposition $L = \bigoplus_{[\alpha] \in \Lambda/\sim} I_{[\alpha]}$ satisfies the assertions of the theorem. ∎

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