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ON BAIRE-ONE MAPPINGS WITH ZERO-DIMENSIONAL DOMAINS

BY

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Abstract. We generalize the Lebesgue–Hausdorff Theorem on Baire classification of mappings defined on strongly zero-dimensional spaces.

1. Introduction. A subset A of a topological space X is a *functionally F_σ-set* [*G_δ-set*] if A is the union [intersection] of a sequence of zero [cozero] subsets of X. If a set is functionally F_σ and functionally G_δ simultaneously, then it is called *functionally ambiguous.*

Let X and Y be topological spaces and $f : X \to Y$ be a mapping. We say that f belongs to

- *the first Baire class*, $f \in \mathrm{B}_1(X, Y)$, if f is a pointwise limit of a sequence of continuous mappings between X and Y;
- *the first* [*functional*] *Lebesgue class*, $f \in \mathrm{H}_1(X, Y)$ [$f \in \mathrm{K}_1(X, Y)$], if $f^{-1}(V)$ is [functionally] an F_σ-set in X for any open subset V of Y.

Obviously, $\mathrm{H}_1(X, Y) = \mathrm{K}_1(X, Y)$ for a perfectly normal space X and a topological space Y. It is not hard to verify that the inclusion $\mathrm{B}_1(X, Y) \subseteq \mathrm{H}_1(X, Y)$ holds for any topological space X and a perfectly normal space Y (see [16, p. 386]). But the proof of the inverse inclusion is a much more difficult problem that begins in the PhD thesis of René Baire [1].

The classical Lebesgue–Hausdorff Theorem [18, 8] says that

$$\mathrm{B}_1(X, Y) = \mathrm{H}_1(X, Y) \tag{1.1}$$

if X is a metric space and $Y = [0, 1]^\omega$, or if X is a zero-dimensional metrizable separable space and Y is a metrizable separable space (see [15, Theorem 24.10]). This result was generalized by many mathematicians in several ways. The first direction concerns the verification of (1.1) for a connected-like space Y. So, the equality (1.1) holds under the following assumptions:

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- (I) X is a metrizable space, Y is a separable convex subset of a Banach space (S. Rolewicz [21]);
- (II) X is normal, $Y = \mathbb{R}$ (M. Laczkovich [17] without proof);
- (III) X is a complete metric space, Y is a Banach space (C. Stegall [23]).

Moreover, it was proved that if

- (IV) X is a topological space and Y is a metrizable separable arcwise connected and locally arcwise connected space (O. Karlova and V. Mykhaylyuk [14]),

then

$$\mathrm{B}_1(X,Y) = \mathrm{K}_1(X,Y). \tag{1.2}$$

R. Hansell [4] (see also [5]) introduced the notion of σ-discrete mapping as a convenient tool for the investigation of Borel measurable mappings with values in non-separable metric spaces. A mapping $f : X \to Y$ is called *σ-discrete* if there exists a family $\mathcal{B} = \bigcup_{n=1}^{\infty} \mathcal{B}_n$ of subsets of X such that every family $\mathcal{B}_n$ is discrete in X and the preimage $f^{-1}(V)$ of any open set V in Y is a union of sets from $\mathcal{B}$. The class of all σ-discrete mappings between X and Y is denoted by $\Sigma(X,Y)$. It is easy to see that if Y is metrizable separable then every mapping $f : X \to Y$ is σ-discrete. The equality

$$\mathrm{B}_1(X,Y) = \mathrm{H}_1(X,Y) \cap \Sigma(X,Y) \tag{1.3}$$

holds in the following situations:

- (V) X is metrizable, Y is a convex subset of a normed space (R. Hansell [5]);
- (VI) X is collectionwise normal, Y is a closed convex subset of a Banach space (R. Hansell [7]);
- (VII) X is a metrizable space, Y is a metrizable space, every continuous function from a closed subset of X to Y can be extended continuously on X, and for each $y \in Y$ and each neighborhood V of y in Y there exists a neighborhood W of y such that each continuous function from a closed subset $F \subseteq X$ to V admits an extension $f : X \to V$ (C. A. Rogers [20]);
- (VIII) X is a perfectly normal paracompact space, Y is a Banach space (J. E. Jayne, J. Orihuela, A. J. Pallarés and G. Vera [9]);
- (IX) X is metrizable, Y is metrizable arcwise connected and locally arcwise connected (M. Fosgerau [3]).

L. Veselý [24] noticed that every Baire-one mapping f between a topological space X and a metrizable space Y is "strongly σ-discrete", i.e., there exists a family $\mathcal{B} = \bigcup_{n=1}^{\infty} \mathcal{B}_n$ of subsets of X such that for every family $\mathcal{B}_n = (B_i : i \in I_n)$ there exists a discrete family $(U_i : i \in I_n)$ of open sets in X with $\overline{B_i} \subseteq U_i$ for all $i \in I_n$, and moreover, the preimage $f^{-1}(V)$ of any open set V in Y is a union of sets from $\mathcal{B}$. Veselý denoted the collection of all such mappings by $\Sigma^*(X,Y)$, and proved the equality

$$\mathrm{B}_1(X,Y)=\mathrm{H}_1(X,Y)\cap\Sigma^*(X,Y), \tag{1.4}$$

in particular, in the case when

(X) X is a normal space, Y is a metrizable arcwise connected and locally arcwise connected space (L. Veselý [24]).

The second way of development of the Lebesgue–Hausdorff Theorem deals with the case when Y does not satisfy any properties like connectedness, but X is zero-dimensional, strongly zero-dimensional, etc. In this direction the following results were obtained: the equality (1.1) holds if

(XI) X is a normal strongly zero-dimensional space, Y is a zero-dimensional metrizable separable space (H. R. Shatery and J. Zafarani [22]);

the equality (1.2) takes place when

(XII) X is a strongly zero-dimensional space, Y is a metrizable separable space (O. Karlova [10]);

and the equality (1.3) is valid if

(XIII) X is a strongly zero-dimensional metrizable space, Y is a metrizable space (O. Karlova [11]).

Finally, the third direction is connected with the case when Y is nonmetrizable. Here we are able to prove the equality (1.1) if

(XIV) X is a hereditarily Baire separable metrizable space, Y is the strict inductive limit of a sequence of metrizable locally convex spaces (O. Karlova and V. Mykhaylyuk [13]).

However, there are many unsolved problems, in particular, the following.

Question 1.1 ([19, Question 3.3, p. 659]). *Does every* H_1*-mapping* $f:[0,1]\to C_p[0,1]$ *belong to the first Baire class?*

This question is equivalent to the following one.

Question 1.2 ([19, Question 3.4, p. 659]). *Let* $f:[0,1]\times[0,1]\to\mathbb{R}$ *be a function which is continuous with respect to the first variable and belongs to the first Baire class with respect to the second one. Is* f *the pointwise limit of a sequence of separately continuous functions?*

In this paper we develop techniques from [3], and generalize the Lebesgue–Hausdorff Theorem to σ-discrete mappings defined on strongly zero-dimensional spaces with values in metrizable spaces. In order to do this we consider the class of σ-strongly functionally discrete mappings introduced in [12]. We denote this class by $\Sigma^s(X,Y)$, and notice that $\Sigma^s(X,Y)=\Sigma^*(X,Y)$ if X

is a normal space. We prove that $\mathrm{K}_1(X,Y)\cap\Sigma^s(X,Y)=\mathrm{B}_1(X,Y)$, if X is a strongly zero-dimensional space and Y is a metrizable space. We also introduce almost strongly zero-dimensional spaces and prove that if X is a topological space and Y is a disconnected metrizable separable space, then the following conditions are equivalent: (i) X is almost strongly zero-dimensional; (ii) $\mathrm{K}_1(X,Y)=\mathrm{B}_1(X,Y)$.

2. Relations between functionally σ-discrete and B_1-mappings.

Definition 2.1. A family $\mathcal{A}=(A_i : i\in I)$ of subsets of a topological space X is said to be

1. *discrete* if every point of X has a neighborhood which intersects at most one set from $\mathcal{A}$;
2. *strongly discrete* if there exists a discrete family $(U_i : i\in I)$ of open subsets of X such that $\overline{A_i}\subseteq U_i$ for every $i\in I$;
3. *strongly functionally discrete* or, briefly, an *sfd family* if there exists a discrete family $(U_i : i\in I)$ of cozero subsets of X such that $\overline{A_i}\subseteq U_i$ for every $i\in I$.

Remark 2.2.

1. For an arbitrary space X we have (3)⇒(2)⇒(1);
2. X is collectonwise normal if and only if (1)⇔(2);
3. if X is normal, then (2)⇔(3).

Definition 2.3. Let $\mathcal{P}$ be a property of a family of sets. A family $\mathcal{A}$ is called a *σ-$\mathcal{P}$ family* if $\mathcal{A}=\bigcup_{n=1}^{\infty}\mathcal{A}_n$, where every family $\mathcal{A}_n$ has property $\mathcal{P}$.

Definition 2.4. A family $\mathcal{B}$ of subsets of a topological space X is called *a base* for a mapping $f:X\to Y$ if the preimage $f^{-1}(V)$ of an arbitrary open set V in Y is a union of sets from $\mathcal{B}$.

Clearly, we may assume that V is an element of an open base of Y in Definition 2.4.

Definition 2.5. If a mapping $f:X\to Y$ has a base which is a σ-$\mathcal{P}$ family, then we say that f is a *σ-$\mathcal{P}$ mapping.*

The collection of all σ-$\mathcal{P}$ mappings between X and Y will be denoted by

- $\Sigma(X,Y)$ if $\mathcal{P}$ is discreteness;
- $\Sigma^*(X,Y)$ if $\mathcal{P}$ is strong discreteness;
- $\Sigma^s(X,Y)$ if $\mathcal{P}$ is strong functional discreteness.

Moreover, $\Sigma^s_0(X,Y)$ is the collection of mappings $f:X\to Y$ that have a σ-sfd base of zero sets.

Let us observe that a continuous mapping $f:X\to Y$ is σ-discrete if either X or Y is a metrizable space, since every metrizable space has a

σ-discrete base of open sets. Moreover, it is evident that every mapping with values in a second countable space is σ-discrete. Hansell [4] proved that every Borel measurable mapping $f : X \to Y$ between a complete metric space X and a metric space Y is σ-discrete. For any metric spaces X and Y the family $\Sigma(X,Y)$ is closed under pointwise limits [6], which implies that every Baire measurable mapping between metric spaces is σ-discrete.

The following fact is a consequence of [12, Theorem 6].

Theorem 2.6. *Let X be a topological space and Y be a metrizable space. Then $\mathrm{K}_1(X,Y) \cap \Sigma^s(X,Y) = \Sigma_0^s(X,Y)$.*

The next simple lemma will be useful.

Lemma 2.7. *Let X be a topological space, $(U_i : i \in I)$ be a locally finite family of cozero subsets of X, $(F_i : i \in I)$ be a family of zero subsets of X such that $F_i \subseteq U_i$ for every $i \in I$. Then $F = \bigcup_{i\in I} F_i$ is a zero set in X.*

Proof. For every $i \in I$ we choose a continuous function $f_i : X \to [0,1]$ such that $F_i = f_i^{-1}(0)$ and $X \setminus U_i = f_i^{-1}(1)$. For every $x \in X$ let $f(x) = \min_{i\in I} f_i(x)$. Then $f : X \to [0,1]$ is continuous and $F = f^{-1}(0)$. ■

Corollary 2.8. *The union of an sfd family of zero sets in a topological space is a zero set.*

We say that a topological space X is *strongly zero-dimensional* if for any completely separated subsets A and B of X there exists a clopen set U such that $A \subseteq U \subseteq X \setminus B$.

For families $\mathcal{A}$ and $\mathcal{B}$ we write $\mathcal{A} \prec \mathcal{B}$ if for every $A \in \mathcal{A}$ there exists $B \in \mathcal{B}$ such that $A \subseteq B$.

Proposition 2.9. *Let X be a strongly zero-dimensional space, (Y,d) be a metric space, $f : X \to Y$ be a mapping, $\mathcal{F}_1, \ldots, \mathcal{F}_n$ be families of zero subsets of X such that*

1. *$\mathcal{F}_k$ is an sfd-family for every $k = 1, \ldots, n$;*
2. *$\mathcal{F}_{k+1} \prec \mathcal{F}_k$ for every $k = 1, \ldots, n-1$;*
3. *for every $k = 1, \ldots, n$, $\operatorname{diam}(f(F)) < 1/2^{k+2}$ for all $F \in \mathcal{F}_k$.*

Then there exists a continuous mapping $g : X \to Y$ such that if $x \in \bigcup \mathcal{F}_k$ for some $k = 1, \ldots, n$, then

$$d(f(x), g(x)) < 1/2^k. \tag{2.1}$$

Proof. Let $\mathcal{F}_k = (F_{i,k} : i \in I_k)$, $k = 1, \ldots, n$. We choose a discrete family $(U_{i,1} : i \in I_1)$ of cozero sets in X such that $F_{i,1} \subseteq U_{i,1}$ for every $i \in I_1$. Let $(V_{i,1} : i \in I_1)$ be a family of clopen sets such that $F_{i,1} \subseteq V_{i,1} \subseteq U_{i,1}$. Now we take a discrete family $(G_{i,2} : i \in I_2)$ of cozero sets such that $F_{i,2} \subseteq G_{i,2}$ for every $i \in I_2$. Since $\mathcal{F}_2 \prec \mathcal{F}_1$, for every $i \in I_2$ there exists a unique $j \in I_1$ such that $F_{i,2} \subseteq F_{j,1}$. We denote $U_{i,2} = G_{i,2} \cap V_{j,1}$ and choose a clopen set

$V_{i,2}$ with $F_{i,2} \subseteq V_{i,2} \subseteq U_{i,2} \subseteq V_{j,1}$. Proceeding in this way we obtain discrete families $(U_{i,k} : i \in I_k)$ and $(V_{i,k} : i \in I_k)$ of subsets of X for $k = 1, \dots, n$ such that $U_{i,k}$ is a cozero set, $V_{i,k}$ is a clopen set, for every $k = 1, \dots, n-1$ and $i \in I_{k+1}$ there exists a unique $j \in I_k$ such that

$$F_{i,k+1} \subseteq F_{j,k}, \tag{2.2}$$

$$F_{i,k+1} \subseteq V_{i,k+1} \subseteq U_{i,k+1} \subseteq V_{j,k}. \tag{2.3}$$

Observe that for every k the set $V_k = \bigcup_{i \in I_k} V_{i,k}$ is clopen by Corollary 2.8.

Pick $y_0 \in f(X)$ and $y_{i,k} \in f(F_{i,k})$ for every k and $i \in I_k$. For all $x \in X$ let

$$g_0(x) = y_0.$$

Suppose that for some k, $1 \le k < n$, we have defined continuous mappings $g_1, \dots, g_k$ such that

$$g_k(x) = \begin{cases} g_{k-1}(x) & \text{if } x \in X \setminus V_k, \\ y_{i,k} & \text{if } x \in V_{i,k} \text{ for some } i \in I_k. \end{cases} \tag{2.4}$$

Let

$$g_{k+1}(x) = \begin{cases} g_k(x) & \text{if } x \in X \setminus V_{k+1}, \\ y_{i,k+1} & \text{if } x \in V_{i,k+1} \text{ for some } i \in I_{k+1}. \end{cases}$$

Then the mapping $g_{k+1} : X \to Y$ is continuous, since $g_{k+1}|_{V_{k+1}}$ and $g_{k+1}|_{X \setminus V_{k+1}}$ are continuous and the set V_{k+1} is clopen. Proceeding inductively we define continuous mappings $g_1, \dots, g_n$ satisfying (2.4).

We set $g = g_n$ and prove that g satisfies (2.1). We first show that

$$d(g_{k+1}(x), g_k(x)) < 1/2^{k+2} \tag{2.5}$$

for all $0 \le k < n$ and $x \in X$. Indeed, if $x \in X \setminus V_{k+1}$, then $g_{k+1}(x) = g_k(x)$ and $d(g_{k+1}(x), g_k(x)) = 0$. Assume that $x \in V_{i,k+1}$ for some $i \in I_{k+1}$. Take $j \in I_k$ such that (2.2) and (2.3) hold. Then $g_{k+1}(x) = y_{i,k+1}$ and $g_k(x) = y_{j,k}$. Since $f(F_{i,k+1}) \subseteq f(F_{j,k})$ we have $y_{i,k+1} \in f(F_{j,k})$. Hence, $d(g_{k+1}(x), g_k(x)) \le \operatorname{diam}(f(F_{j,k})) < 1/2^{k+2}$.

Let $1 \le k \le n$ and $x \in \bigcup \mathcal{F}_k$. Then $x \in F_{i,k} \subseteq V_{i,k}$ for some $i \in I_k$. It follows that $g_k(x) = y_{i,k}$ and $d(f(x), g_k(x)) \le \operatorname{diam}(f(F_{i,k})) < 1/2^{k+2}$. Taking into account (2.5), we obtain

$$d(f(x), g(x)) \le d(f(x), g_k(x)) + \sum_{i=k}^{n-1} d(g_i(x), g_{i+1}(x))$$
$$< 1/2^{k+2} + 1/2^{k+1} < 1/2^k. \ \blacksquare$$

THEOREM 2.10. *Let X be a strongly zero-dimensional space and Y be a metrizable space. Then $\Sigma_0^s(X,Y) \subseteq \mathrm{B}_1(X,Y)$.*

Proof. Fix a metric d on Y which generates the topology. For every $k \in \mathbb{N}$ we consider a covering $\mathcal{U}_k$ of Y by open sets with diameters at most $1/2^k$.

Let $f \in \Sigma_0^s(X, Y)$ and $\mathcal{B}$ be a σ-sfd base for f which consists of zero subsets of X. For every $k \in \mathbb{N}$ we set

$$\mathcal{B}_k = (B \in \mathcal{B} : \exists U \in \mathcal{U}_k,\, B \subseteq f^{-1}(U)).$$

Then $\mathcal{B}_k$ is a σ-sfd family and $X = \bigcup \mathcal{B}_k$ for every k. According to [12, Lemma 13] for every $k \in \mathbb{N}$ there exists a sequence $(\mathcal{B}_{k,n})_{n=1}^{\infty}$ of sfd families of zero subsets of X such that $\mathcal{B}_{k,n} \prec \mathcal{B}_k$, $\mathcal{B}_{k,n} \prec \mathcal{B}_{k,n+1}$ for every $n \in \mathbb{N}$ and $\bigcup \bigcup_{n=1}^{\infty} \mathcal{B}_{k,n} = X$. For all $k, n \in \mathbb{N}$ we set

$$\mathcal{F}_{k,n} = (B_1 \cap \cdots \cap B_k : B_m \in \mathcal{B}_{m,n},\, 1 \le m \le k).$$

Notice that each of the families $\mathcal{F}_{k,n}$ is strongly functionally discrete, consists of zero sets and satisfies the following conditions:

(a) $\mathcal{F}_{k+1,n} \prec \mathcal{F}_{k,n}$,
(b) $\mathcal{F}_{k,n} \prec \mathcal{F}_{k,n+1}$,
(c) $\bigcup_{n=1}^{\infty} \mathcal{F}_{k,n} = X$.

For every $n \in \mathbb{N}$ we apply Proposition 2.9 to f and to $\mathcal{F}_{1,n}, \ldots, \mathcal{F}_{n,n}$. We get a sequence of continuous mappings $g_n : X \to Y$ such that if $x \in \bigcup \mathcal{F}_{k,n}$ for some $k \le n$ then $d(f(x), g_n(x)) < 1/2^k$. It is easy to see that properties (b) and (c) imply that $g_n \to f$ pointwise on X. Hence, $f \in \mathrm{B}_1(X, Y)$. ■

Proposition 2.11. *Let X be a topological space and Y be a metrizable space. Then* $\mathrm{B}_1(X, Y) \subseteq \Sigma_0^s(X, Y)$.

Proof. Let $f \in \mathrm{B}_1(X, Y)$ and $(f_n)_{n=1}^{\infty}$ be a sequence of continuous mappings $f_n : X \to Y$ such that $f(x) = \lim_{n\to\infty} f_n(x)$ for all $x \in X$. Let $\mathcal{V} = \bigcup_{m=1}^{\infty} \mathcal{V}_m$ be a σ-discrete open base for Y. For every $V \in \mathcal{V}$ we choose a sequence $(G_{k,V})_{k=1}^{\infty}$ of open sets such that $\overline{G_{k,V}} \subseteq G_{k+1,V}$ for every $k \in \mathbb{N}$ and $V = \bigcup_{k=1}^{\infty} \overline{G_{k,V}}$. It is not hard to verify that

$$f^{-1}(V) = \bigcup_{k=1}^{\infty} \bigcap_{n=k}^{\infty} f_n^{-1}(\overline{G_{k,V}}). \tag{2.6}$$

Denote $F_{k,V} = \bigcap_{n=k}^{\infty} f_n^{-1}(\overline{G_{k,V}})$ and notice that every $F_{k,V}$ is a zero set in X. For all $k, m \in \mathbb{N}$ we set $\mathcal{B}_{k,m} = (F_{k,V} : V \in \mathcal{V}_m)$ and $\mathcal{B} = \bigcup_{k,m=1}^{\infty} \mathcal{B}_{k,m}$. Then $\mathcal{B}$ is a base for f. Moreover, every family $\mathcal{B}_{k,m}$ is strongly functionally discrete, since $F_{k,V} \subseteq f_k^{-1}(V)$ and the family $(f_k^{-1}(V) : V \in \mathcal{V}_m)$ is discrete and consists of cozero sets. ■

Combining Theorems 2.6, 2.10 and Proposition 2.11, we get

Theorem 2.12. *Let X be a strongly zero-dimensional space and Y be a metrizable space. Then* $\mathrm{K}_1(X, Y) \cap \Sigma^s(X, Y) = \mathrm{B}_1(X, Y)$.

According to [4, Theorem 3] we have $\mathrm{K}_1(X, Y) \subseteq \Sigma^s(X, Y)$ for any completely metrizable X and metrizable Y. This fact and Theorem 2.12 immediately imply the following result.

Theorem 2.13. *Let X be a completely metrizable strongly zero-dimensional space and Y be a metrizable space. Then* $\mathrm{K}_1(X,Y)=\mathrm{B}_1(X,Y)$.

We show that the metrizability of Y in Theorem 2.13 is essential.

Example 2.14. There exists a completely metrizable strongly zero-dimensional space X and a Lindelöf strongly zero-dimensional space Y such that $\mathrm{K}_1(X,Y)\setminus \mathrm{B}_1(X,Y)\neq\emptyset$.

Proof. Let X be the set of all irrational numbers with the euclidian topology, and Y be the same set with the topology induced from the Sorgenfrey line (recall that the *Sorgenfrey line* is the real line $\mathbb{R}$ endowed with the topology generated by the base consisting of all half-closed intervals $[a,b)$, where $a<b$).

Take a countable dense in X set $Q=\{q_n : n\in\mathbb{N}\}$, and for all $x\in X$ let

$$f(x)=\begin{cases} x-1/n & \text{if } x=q_n,\\ x & \text{if } x\in X\setminus Q.\end{cases}$$

To show that $f\in \mathrm{K}_1(X,Y)$ it is enough to verify that $f^{-1}([a,b)\cap Y)$ is an F_σ-set in X for every $a<b$, since Y is Lindelöf. Denote $E=f^{-1}([a,b)\cap Y)$. Notice that the sets $A=(X\cap[a,b))\setminus E$ and $B=E\setminus(a,b)$ are at most countable. Therefore, A and B are both F_σ in X. Since $A\subseteq Q$, we can choose an increasing sequence $(n_k)_{k=1}^\infty$ of numbers such that $A=\{q_{n_k} : k\in\mathbb{N}\}$. Then $a\le q_{n_k}\le a+1/n_k$ for every k. Therefore, $\lim_{k\to\infty} q_{n_k}=a$ in X. Hence, the set A is G_δ in X. Then the equality $E=(X\cap[a,b)\setminus A)\cup B$ implies that E is an F_σ-set in X.

Now we prove that $f\notin \mathrm{B}_1(X,Y)$. Assume that there exists a sequence of continuous mappings $f_n: X\to Y$ such that $f_n(x)\to f(x)$ for every $x\in X$. Let

$$A_n=\{x\in X : \forall k\ge n,\ f_k(x)\ge f(x)\}$$

for $n\in\mathbb{N}$. It is easy to see that $\bigcup_{n=1}^\infty A_n=X$. Since $X\setminus Q$ is of the second category in X, there exist n and an interval $[a,b]$ such that $[a,b]\cap X\subseteq \overline{A_n\setminus Q}$. Notice that $A_n\setminus Q\subseteq F$, where $F=\bigcap_{k=n}^\infty\{x\in X : f_k(x)\ge x\}$. Since F is closed in X, we have $[a,b]\cap X\subseteq F$. Then $f(x)=\lim_{k\to\infty} f_k(x)\ge x$ for all $x\in[a,b]\cap X$, a contradiction. ∎

3. Almost strongly zero-dimensional spaces and characterization theorems. In this section we find necessary conditions on a space X under which the equality $\mathrm{K}_1(X,Y)\cap\Sigma^s(X,Y)=\mathrm{B}_1(X,Y)$ holds for any disconnected metrizable space Y.

Definition 3.1. A subset F of a topological space X is called a *C-set* if there exists a sequence $(U_n)_{n=1}^\infty$ of clopen sets in X such that $F=\bigcap_{n=1}^\infty U_n$. A set is called a *$C_\sigma$-set* if it is the union of a sequence of C-sets.

Definition 3.2. We say that a topological space X is *almost strongly zero-dimensional* if every zero subset of X is a C_σ-set.

Notice that every strongly zero-dimensional space is almost strongly zero-dimensional.

Lemma 3.3. *Let X be a topological space, C_1 and C_2 be disjoint C-subsets of X. Then there exists a clopen set G in X such that $C_1 \subseteq G \subseteq X \setminus C_2$.*

Proof. Let $(U_n)_{n=1}^\infty$ and $(V_n)_{n=1}^\infty$ be sequences of clopen subsets of X such that $X \setminus C_1 = \bigcup_{n=1}^\infty U_n$ and $X \setminus C_2 = \bigcup_{n=1}^\infty V_n$. For every $n \in \mathbb{N}$ set $G_n = V_n \setminus \bigcup_{k=1}^n U_k$ and $G = \bigcup_{n=1}^\infty G_n$. Clearly, $C_1 \subseteq G \subseteq X \setminus C_2$ and G is open in X. It remains to show that G is closed. Let $x \in \overline{G}$. If $x \in C_1$, then $x \in G$. If $x \notin C_1$, then there is $N \in \mathbb{N}$ such that $x \in U_N$. Notice that $U_N \cap G_n = \emptyset$ for all $n \geq N$. Then $x \in \overline{\bigcup_{n=1}^{N-1} G_n} = \bigcup_{n=1}^{N-1} \overline{G_n} = \bigcup_{n=1}^{N-1} G_n \subseteq G$. ■

Corollary 3.4. *Let X be a topological space and $C_1, \dots, C_n$ be disjoint C-subsets of X, $n \in \mathbb{N}$. Then there exist disjoint clopen sets $G_1, \dots, G_n$ in X such that $X = G_1 \cup \dots \cup G_n$ and $C_i \subseteq G_i$ for every $i = 1, \dots, n$.*

Proposition 3.5. *Every almost strongly zero-dimensional completely regular space X is totally separated.*

Proof. Let $x, y \in X$ be distinct points, and U and V be disjoint neighborhoods of x and y, respectively. Since X is almost strongly zero-dimensional, there exist C-sets C_x and C_y such that $x \in C_x \subseteq U$ and $y \in C_y \subseteq V$. By Lemma 3.3 there exists a clopen set G such that $C_x \subseteq G$ and $G \cap C_y = \emptyset$. Hence, x and y can be separated by a clopen set, which implies that X is totally separated. ■

Lemma 3.6. *Let X be a topological space, $F \subseteq X$ be a countably compact C_σ-set, $C \subseteq X$ be a C-set and $F \cap C = \emptyset$. Then there exists a clopen set G in X such that $F \subseteq G \subseteq X \setminus C$.*

Proof. Let $(C_n)_{n=1}^\infty$ be an increasing sequence of C-sets such that $F = \bigcup_{n=1}^\infty C_n$. Lemma 3.3 implies that for every n there exists a clopen set G_n in X such that $C_n \subseteq G_n \subseteq X \setminus C$. Since F is countably compact, we choose a finite subcovering $\mathcal{G}$ of the covering $(G_n : n \in \mathbb{N})$ of F. It remains to set $G = \bigcup \mathcal{G}$. ■

In the same manner we can prove the following result.

Lemma 3.7. *Let X be a topological space and $F, E \subseteq X$ be disjoint countably compact C_σ-sets. Then there exists a clopen set G in X such that $F \subseteq G \subseteq X \setminus E$.*

Taking into account that every closed subset of a countably compact space is countably compact, we obtain the following corollary from Lemma 3.7.

Proposition 3.8. *Let X be a countably compact space. Then X is almost strongly zero-dimensional if and only if it is strongly zero-dimensional.*

The following question is open.

Question 3.9. *Does there exist a completely regular almost strongly zero-dimensional space which is not strongly zero-dimensional?*

Proposition 3.10. *Let X be a topological space and Y be a disconnected space such that $\mathrm{K}_1(X,Y)\cap\Sigma^s(X,Y)\subseteq \mathrm{B}_1(X,Y)$. Then X is almost strongly zero-dimensional.*

Proof. Let U and V be clopen disjoint non-empty subsets of Y such that $Y=U\cup V$, $F\subseteq X$ be a zero set, $y_1\in U$, $y_2\in V$ and let $f:X\to Y$ be a mapping such that $f(x)=y_1$ for all $x\in F$, and $f(x)=y_2$ for all $x\in X\setminus F$. It is easy to see that $f\in \mathrm{K}_1(X,Y)\cap\Sigma^s(X,Y)$. Then there exists a sequence $(f_n)_{n=1}^\infty$ of continuous mappings $f_n:X\to Y$ such that $\lim_{n\to\infty} f_n(x)=f(x)$ for every $x\in X$. So $F=f^{-1}(U)=\bigcup_{n=1}^\infty\bigcap_{m=n}^\infty f_m^{-1}(U)$. Hence, F is a C_σ-set. ■

Theorem 3.11. *Let Y be a disconnected metrizable space. If*

(a) *X is locally compact paracompact Hausdorff space, or*
(b) *X is a countably compact space,*

then the following conditions are equivalent:

(1) *$\mathrm{K}_1(X,Y)\cap\Sigma^s(X,Y)=\mathrm{B}_1(X,Y)$;*
(2) *X is a strongly zero-dimensional space.*

Proof. (1)⇒(2). According to Proposition 3.10, X is almost strongly zero-dimensional. It follows that X is strongly zero-dimensional in case (b) by Proposition 3.8. In case (a), X is completely regular, and hence totally separated by Proposition 3.5. It remains to apply [2, Theorem 6.2.10].

The implication (2)⇒(1) follows from Theorem 2.12. ■

A sequence $(f_n)_{n=1}^\infty$ of mappings $f_n:X\to Y$ is called *stably convergent* to a mapping $f:X\to Y$ if for every $x\in X$ there exists a number n_0 such that $f_n(x)=f(x)$ for all $n\ge n_0$. We denote this fact by $f_n\xrightarrow{\mathrm{st}} f$.

Lemma 3.12. *Let X be an almost zero-dimensional space, Y be a T_1-space and $f\in \mathrm{K}_1(X,Y)$ be a finite-valued mapping. Then there exists a sequence of continuous finite-valued mappings $f_n:X\to Y$ which is stably convergent to f on X.*

Proof. Denote $f(X)=\{y_1,\dots,y_m\}$. Since for every $1\le i\le m$ the set $A_i=f^{-1}(y_i)$ is functionally F_σ in X, there exists an increasing sequence $(C_{i,n})_{n=1}^\infty$ of C-subsets of X such that $A_i=\bigcup_{n=1}^\infty C_{i,n}$. According to Corollary 3.4, for every $n\in\mathbb{N}$ there are disjoint clopen sets $G_{1,n},\dots,G_{m,n}$ such that $C_{i,n}\subseteq G_{i,n}$ for every $i=1,\dots,m$ and $X=G_{1,n}\cup\dots\cup G_{m,n}$. Now for every $n \geq 1$ we set $f_n(x) = y_i$ if $x \in G_{i,n}$ for some $i = 1, \ldots, m$. It is easy to see that $f_n \xrightarrow{\text{st}} f$ on X. ■

LEMMA 3.13. *Let X be an almost zero-dimensional space, (Y, d) be a metric space and $(f_n)_{n=1}^{\infty}$ be a sequence of finite-valued mappings $f_n \in \mathrm{K}_1(X, Y)$ which is uniformly convergent to $f : X \to Y$. Then $f \in \mathrm{B}_1(X, Y)$.*

Proof. Without loss of generality we may assume that

$$d(f_{n+1}(x), f_n(x)) \leq 1/2^{n+1} \tag{3.1}$$

for all $x \in X$ and $n \in \mathbb{N}$. By Lemma 3.12 for every $n \in \mathbb{N}$ there exists a sequence $(f_{n,m})_{m=1}^{\infty}$ of continuous finite-valued mappings $f_{n,m} : X \to Y$ such that

$$f_{n,m} \xrightarrow{\text{st}} f_n. \tag{3.2}$$

For all $x \in X$ and $m \in \mathbb{N}$ we set

$$h_{0,m}(x) = h_{1,m}(x) = f_{1,m}(x).$$

Now assume that for some $k \in \mathbb{N}$ we have already defined sequences $(h_{1,m})_{m=1}^{\infty}, \ldots, (h_{k,m})_{m=1}^{\infty}$ of continuous finite-valued mappings such that

$$h_{n,m} \xrightarrow{\text{st}} f_n \quad \text{for all } n = 1, \ldots, k \tag{3.3}$$

and

$$d(h_{n+1,m}(x), h_{n,m}(x)) \leq 1/2^{n+1} \text{ for all } x \in X,\ m \in \mathbb{N},\ n = 0, \ldots, k-1. \tag{3.4}$$

For every $m \in \mathbb{N}$ let

$$U_m = \{x \in X : d(f_{k+1,m}(x), h_{k,m}(x)) \leq 1/2^{k+1}\}.$$

Then U_m is clopen in X. Moreover, conditions (3.1)–(3.3) imply that

$$X = \bigcup_{n=1}^{\infty} \bigcap_{m=n}^{\infty} U_m. \tag{3.5}$$

Define a sequence of finite-valued continuous mappings $(h_{k+1,m})_{m=1}^{\infty}$ by

$$h_{k+1,m}(x) = \begin{cases} f_{k+1,m}(x) & \text{if } x \in U_m, \\ h_{k,m}(x) & \text{if } x \notin U_m. \end{cases}$$

Notice that (3.2) and (3.5) imply $h_{k+1,m} \xrightarrow{\text{st}} f_{k+1}$ on X.

We prove (3.4). Fix $m \in \mathbb{N}$ and $x \in X$. If $x \in U_m$, then $h_{k+1,m}(x) = f_{k+1,m}(x)$ and $d(h_{k+1,m}(x), h_{k,m}(x)) = d(f_{k+1,m}(x), h_{k,m}(x)) \leq 1/2^{k+1}$. If $x \notin U_m$, then $h_{k+1,m}(x) = h_{k,m}(x)$ and $d(h_{k+1,m}(x), h_{k,m}(x)) = 0$.

Finally, we show that $\lim_{m\to\infty} h_{m,m}(x) = f(x)$. Let $x \in X$, $\varepsilon > 0$ and n_0 be a number such that

$$1/2^{n_0} < \varepsilon/2 \quad \text{and} \quad d(f_{n_0}(x), f(x)) < \varepsilon/2.$$

Take $m_0 > n_0$ with $h_{n_0,m}(x) = f_{n_0}(x)$ for all $m \geq m_0$. Then

$$\begin{aligned} &d(h_{m,m}(x), f(x)) \\ &\quad \leq \sum_{i=n_0+1}^{m} d(h_{i-1,m}(x), h_{i,m}(x)) + d(h_{n_0,m}(x), f_{n_0}(x)) + d(f_{n_0}(x), f(x)) \\ &\quad < \sum_{i=n_0+1}^{m} \frac{1}{2^i} + \frac{\varepsilon}{2} < \frac{1}{2^{n_0}} + \frac{\varepsilon}{2} < \varepsilon \end{aligned}$$

for all $m \geq m_0$. Hence, $f \in \mathrm{B}_1(X, Y)$. ∎

THEOREM 3.14. *Let X be an almost zero-dimensional space and Y be a metrizable separable space. Then $\mathrm{K}_1(X, Y) = \mathrm{B}_1(X, Y)$.*

Proof. The inclusion $\mathrm{B}_1(X, Y) \subseteq \mathrm{K}_1(X, Y)$ follows from (2.6).

Let $f \in \mathrm{K}_1(X, Y)$ and d be a metric on Y such that (Y, d) is completely bounded. For every $n \in \mathbb{N}$ we take a finite $\frac{1}{n}$-network $Y_n = \{y_{i,n} : i \in I_n\}$ in Y and set $A_{i,n} = \{x \in X : d(f(x), y_{i,n}) < 1/n\}$ for $n \in \mathbb{N}$, $i \in I_n$. Notice that for every n the family $(A_{i,n} : i \in I_n)$ is a covering of X by functionally F_σ-sets. Similarly to the proof of the Reduction Theorem [16, p. 350] we take a sequence of disjoint functionally ambiguous sets $F_{i,n}$ in X such that $F_{i,n} \subseteq A_{i,n}$ and $\bigcup_{i \in I_n} F_{i,n} = X$. For every $n \in \mathbb{N}$ we set $f_n(x) = y_{i,n}$ if $x \in F_{i,n}$ for some $i \in I_n$. Then $(f_n)_{n=1}^\infty$ is a sequence of finite-valued mappings $f_n \in \mathrm{K}_1(X, Y)$ which is uniformly convergent to f on X. It remains to apply Lemma 3.12. ∎

Combining Proposition 3.10 and Theorem 3.14 we obtain the following result.

THEOREM 3.15. *If X is a topological space and Y is a disconnected metrizable separable space, then the following conditions are equivalent:*

1. *X is almost strongly zero-dimensional;*
2. *$\mathrm{K}_1(X, Y) = \mathrm{B}_1(X, Y)$.*

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