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ON SEMIPRIME COMULTIPLICATION MODULES OVER PULLBACK RINGS

BY

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Abstract. Following some ideas and a technique introduced in [Algebra Discrete Math. 2009, no. 1, 1–13] and [Colloq. Math. 120 (2010), 23–42] we give a complete classification, up to isomorphism, of all indecomposable semiprime comultiplication modules with finite-dimensional top over a pullback of two valuation domains with the same residue field.

1. Introduction. One of the aims of the modern representation theory is to solve classification problems for subcategories of modules over a unitary ring R. The reader is referred to [3], [34, Chapters 1 and 14], and [38] for a detailed discussion of classification problems, representation types (finite, tame, or wild), and useful computational reduction procedures. Here we should point out that the classification of all indecomposable modules over an arbitrary unitary ring (including finite-dimensional algebras over an algebraically closed field) is an impossible task. In particular, an infinite-dimensional version of tame representation type is in fact wild representation type. For a discussion of this kind of problems the reader is referred to the papers by Ringel [32, 33] and Simson [36, 37].

Modules over pullback rings have been studied by several authors (see for example [4, 16, 23, 29, 40]). Notably, there is the important work of Levy [25], resulting in the classification of all finitely generated indecomposable modules over Dedekind-like rings. Klingler [23] extended this classification to lattices over certain non-commutative Dedekind-like rings, and Haefner and Klingler classified lattices over certain non-commutative pullback rings, which they called special quasi-triads [17, 18]. Common to all these classifications is the reduction to a "matrix problem" over a division ring: see [5], [30], [34, Section, 17.9], and [35] for background on matrix problems and their applications.

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In the present paper we introduce a new class of R-modules, called semiprime comultiplication modules (see Definition 2.1), and we study it in detail from the classification problem point of view. We are mainly interested in the cases where R is a valuation domain or a pullback of two valuation domains. First, we give a complete description of indecomposable semiprime comultiplication modules over a valuation domain, up to isomorphism. Following some ideas and a technique introduced in [6] and [14] we give a complete classification, up to isomorphism, of all indecomposable semiprime comultiplication modules with finite-dimensional top over a pullback of two valuation domains with the same residue field (for any module M we define its *top* as $M/\mathrm{rad}(R)M$). The classification is divided into two stages: the description of all indecomposable separated semiprime comultiplication R-modules and then, using their list, we show that non-separated indecomposable semiprime comultiplication R-modules with finite-dimensional top are factor modules of finite direct sums of separated indecomposable semiprime comultiplication R-modules. Then we use the classification from Section 3, together with results of Levy [25, 26] on the possibilities for amalgamating finitely generated separated modules, to classify the indecomposable non-separated semiprime comultiplication modules M with finite-dimensional top (see Theorem 4.5). We will see that non-separated modules may be represented by certain amalgamation chains of indecomposable separated semiprime comultiplication modules (where infinite length semiprime comultiplication modules can occur only at the ends) and where adjacency corresponds to amalgamation in the socles of these separated semiprime comultiplication modules.

For completeness, we state some definitions and notation used throughout. In this paper all rings are commutative with identity, and all modules are unitary. Let $v_1 : R_1 \to \bar{R}$ and $v_2 : R_2 \to \bar{R}$ be homomorphisms of two valuation domains R_i onto a common field $\bar{R}$. Denote the pullback $R = \{(r_1, r_2) \in R_1 \oplus R_2 : v_1(r_1) = v_2(r_2)\}$ by $(R_1 \xrightarrow{v_1} \bar{R} \xleftarrow{v_2} R_2)$, where $\bar{R} = R_1/J(R_1) = R_2/J(R_2)$. Then R is a ring under coordinatewise multiplication. Denote the kernel of v_i, $i = 1, 2$, by P_i. Then $\mathrm{Ker}(R \to \bar{R}) = P = P_1 \times P_2$, $R/P \cong \bar{R} \cong R_1/P_1 \cong R_2/P_2$, and $P_1P_2 = P_2P_1 = 0$ (so R is not a domain). Furthermore, for $i \neq j$, $0 \to P_i \to R \to R_j \to 0$ is an exact sequence of R-modules (see [24]).

Definition 1.1. An R-module S is defined to be *separated* if there exist R_i-modules S_i, $i = 1, 2$, such that S is a submodule of $S_1 \oplus S_2$ (the latter is made into an R-module by setting $(r_1, r_2)(s_1, s_2) = (r_1s_1, r_2s_2)$).

Equivalently, S is separated if it is a pullback of an R_1-module and an R_2-module, and then, using the same notation for pullbacks of modules as for rings, $S = (S/P_2S \to S/PS \leftarrow S/P_1S)$ [24, Corollary 3.3] and $S \subseteq$

$(S/P_2S) \oplus (S/P_1S)$. Also, S is separated if and only if $P_1S \cap P_2S = 0$ [24, Lemma 2.9].

If R is a pullback ring, then every R-module is an epimorphic image of a separated R-module, indeed every R-module has a "minimal" such representation: a *separated representation* of an R-module M is an epimorphism $\varphi = (S \xrightarrow{f} S' \to M)$ of R-modules where S is separated, and if φ admits a factorization $\varphi : S \xrightarrow{f} S' \to M$ with S' separated, then f is one-to-one. The module $K = \mathrm{Ker}(\varphi)$ is then an $\bar{R}$-module, since $\bar{R} = R/P$ and $PK = 0$ [24, Proposition 2.3]. An exact sequence $0 \to K \to S \to M \to 0$ of R-modules with S separated and K an $\bar{R}$-module is a separated representation of M if and only if $P_iS \cap K = 0$ for each i and $K \subseteq PS$ [24, Proposition 2.3]. Every module M has a separated representation, which is unique up to isomorphism [24, Theorem 2.8]. Moreover, R-homomorphisms lift to a separated representation, preserving epimorphisms and monomorphisms [24, Theorem 2.6].

Definition 1.2. (a) If R is a ring and N is a submodule of an R-module M, the ideal $\{r \in R : rM \subseteq N\}$ is denoted by $(N : M)$. Then $(0 : M)$ is the annihilator of M. A proper submodule N of a module M over a ring R is said to be a *primary submodule* (resp. *prime submodule*) if whenever $rm \in N$ for some $r \in R$ and $m \in M$, then $m \in N$ or $r^n \in (N : M)$ for some n (resp. $m \in N$ or $r \in (N : M)$); so $\mathrm{rad}(N : M) = P$ (resp. $(N : M) = P'$) is a prime ideal of R, and N is said to be a P-*primary* (resp. P'-*prime*) submodule. The set of all primary (resp. prime) submodules in an R-module M is denoted $\mathrm{pSpec}(M)$ (resp. $\mathrm{Spec}(M)$) [27, 28].

(b) An R-module M is a *comultiplication module* provided for each submodule N of M there exists an ideal I of R such that N is the set of elements m in M with $Im = 0$. In this case we can take $N = (0 :_M (0 :_R N))$ [1].

(c) An R-module M is defined to be a *weak comultiplication module* if either $\mathrm{Spec}(M) = \emptyset$, or for every prime submodule N of M we have $N = (0 :_M I)$ for some ideal I of R [6].

(d) An R-module M is defined to be a *primarily comultiplication module* if either $\mathrm{pSpec}(M) = \emptyset$, or for every primary submodule N of M we have $N = (0 :_M I)$ for some ideal I of R [14].

(e) A proper submodule N of an R-module M is *semiprime* if for every ideal I of R and every submodule K of M, $I^kK \subseteq N$ for some positive integer k implies that $IK \subseteq N$. The set of all semiprime submodules in an R-module M is denoted $\mathrm{seSpec}(M)$. An R-module M is defined to be a *semiprime multiplication module* if either $\mathrm{seSpec}(M) = \emptyset$, or for every semiprime submodule N of M we have $N = IM$ for some ideal I of R [9].

(f) A submodule N of an R-module M is called *pure* if any finite system of equations over N which is solvable in M is also solvable in N. A submodule

N of an R-module M is called *relatively divisible* (or an *RD-submodule*) in M if $rN = N \cap rM$ for all $r \in R$ (see [21, 39]).

(g) A module M is *pure-injective* if it has the injective property relative to all pure exact sequences (see [16, 22, 21, 31, 39]).

(h) A module M is *semiprime* if the zero submodule of M is a semiprime submodule of M.

Remark 1.3. (a) An R-module is pure-injective if and only if it is algebraically compact (see [22, 21, 19, 39]).

(b) Let R be a Dedekind domain, M an R-module, and N a submodule of M. Then N is pure in M if and only if $IN = N \cap IM$ for each ideal I of R. Moreover, N is pure in M if and only if N is an RD-submodule of M [39].

(c) Let N be an R-submodule of M. It is clear that N is an RD-submodule of M if and only if for all $m \in M$ and $r \in R$, $rm \in N$ implies that $rm = rn$ for some $n \in N$. Furthermore, if M is torsion-free, then N is an RD-submodule if and only if for all $m \in M$ and for all non-zero $r \in R$, $rm \in N$ implies that $m \in N$. In this case, N is an RD-submodule if and only if N is a prime submodule.

(d) It is easy to see that if N is a semiprime R-submodule of M, then M/N is a semiprime R-module.

2. Basic properties of semiprime comultiplication modules

Definition 2.1. Let R be a commutative ring. An R-module M is defined to be a *semiprime comultiplication module* if either $\mathrm{seSpec}(M) = \emptyset$, or for every semiprime submodule N of M we have $N = (0 :_M I)$ for some ideal I of R.

One can easily show that if M is a semiprime comultiplication module, then $N = (0 :_M \mathrm{ann}(N))$ for every semiprime submodule N of M. It is also easy to see that the class of semiprime comultiplication modules contains the class of weak comultiplication modules defined in [6]. We need the following lemma proved in [9, Lemmas 3.1 and 3.5].

Lemma 2.2.

(a) *Let $K \subseteq N$ be submodules of an R-module M. Then:*

- (i) *N is semiprime if and only if, whenever $r^k m \in N$ for some $m \in M$, $r \in R$, and a positive integer k, then $rm \in N$.*
- (ii) *N is a semiprime submodule of M if and only if N/K is a semiprime submodule of M/K.*
- (iii) *N is prime if and only if N is a primary and semiprime submodule.*

(iv) *If N is a semiprime R-submodule of M and I an ideal of R with $I \subseteq (0 : M)$, then N is a semiprime submodule of M as an R/I-module.*

(b) *Let R and R' be any commutative rings, $f : R \to R'$ a surjective homomorphism and M an R'-module. Then:*

(i) *If M is semiprime as an R-module, then M is semiprime as an R'-module.*

(ii) *If N is a semiprime R-submodule of M, then N is a semiprime R'-submodule of M.*

EXAMPLE 2.3 ([9, Example 3.3]). Let $R = M = \mathbb{Z}$ be the ring of integers. If $N = 4\mathbb{Z}$, then N is a primary submodule of M, but it is not semiprime. So a primary module need not be semiprime. If $K = 6\mathbb{Z}$, then an inspection will show that K is a semiprime submodule of M that is not primary. Hence a semiprime module need not be primary.

LEMMA 2.4. *Let M be a semiprime comultiplication module over a commutative ring R. Then:*

(i) *If N is a pure submodule of M, then M/N is a semiprime comultiplication R-module.*

(ii) *Every direct summand of M is a semiprime comultiplication submodule.*

Proof. (i) Let K/N be a semiprime submodule of M/N. Then by Lemma 2.2(a)(ii), K is a semiprime submodule of M, so $K = (0 :_M I)$ for some ideal I of R. An inspection will show that $K/N = (0 :_{M/N} I)$.

(ii) follows from (i). ■

LEMMA 2.5. *Let M be an R-module, N a proper submodule of M and $I \subseteq (0 : M)$. Then:*

(i) *N is a semiprime R-submodule M if and only if N is a semiprime submodule of M as an R/I-module.*

(ii) *M is a semiprime comultiplication R-module if and only if M is a semiprime comultiplication module as an R/I-module.*

Proof. The proof of (i) is straightforward. To see (ii), apply Lemma 2.2 and the fact that $(0 :_M J) = (0 :_M (J + I)/I)$ for every ideal J of R. ■

LEMMA 2.6. *Let R and R' be any commutative rings, $f : R \to R'$ a surjective homomorphism and M an R'-module. If M is a semiprime comultiplication R'-module, then M is a semiprime comultiplication R-module.*

Proof. Let N be a semiprime R-submodule of M. Then N is a semiprime R'-submodule of M by Lemma 2.2(b)(i), so $N = (0 :_M I')$ for some ideal

I' of R'. Set $I = f^{-1}(I')$. Then I is an ideal of R and $f(I) = f(f^{-1}(I')) = I' \cap f(R) = I'$; hence $(0 :_M I) = (0 :_M f(I)) = (0 :_M I') = N$. ■

REMARK 2.7. Let R be a valuation domain with unique maximal ideal $P = Rp$.

(a) Let $M = R$ (as R-modules). For a semiprime submodule PM of M, we have $(0 :_M (0 :_R PM)) = R$. So $M = R$ is not a semiprime comultiplication R-module.

(b) By [9, Proposition 3.6], $\text{seSpec}(E(R/P)) = \emptyset$ and $\text{seSpec}(Q(R)) = \{0\}$, where $E = E(R/P)$ is the injective hull of R/P and $Q(R)$ is the field of fractions of R. Thus they are semiprime comultiplication R-modules.

THEOREM 2.8. *Let R be a valuation domain with a unique maximal ideal $P = Rp$. Then, up to isomorphism, the indecomposable semiprime comultiplication modules over R are the following:*

(i) *R/P^n, $n \geq 1$, the indecomposable torsion modules;*
(ii) *$E(R/P)$, the injective hull of R/P;*
(iii) *$Q(R)$, the field of fractions of R.*

Proof. First we note that each of the preceding modules is indecomposable by [10, Proposition 1.3] and a semiprime comultiplication module. In the case of R/P^n this follows because R/P^n is a comultiplication module (see [7]). Moreover, $Q(R)$ and $E(R/P)$ are semiprime comultiplication by Remark 2.7.

Now let M be an indecomposable semiprime comultiplication and choose any non-zero $a \in M$. Let $h(a) = \sup\{n : a \in P^n M\}$ (so $h(a)$ is a non-negative integer or ∞). Also let $(0 : a) = \{r \in R : ra = 0\}$; thus $(0 : a)$ is an ideal of the form P^m or 0. Because $(0 : a) = P^{m+1}$ implies that $p^m a \neq 0$ and $p.p^m a = 0$, we can choose a so that $(0 : a) = P$ or 0. Now we consider the various possibilities for $h(a)$ and $(0 : a)$.

CASE 1: $\text{seSpec}(M) = \emptyset$. Since $\text{Spec}(M) \subseteq \text{seSpec}(M)$, it follows from [27, Lemma 1.3 and Proposition 1.4] that M is a torsion divisible R-module with $PM = M$ and M is not finitely generated. We may assume that $(0 : a) = P$. By an argument like that in [11, Proposition 2.7], $M \cong E(R/P)$. So we may assume that $\text{seSpec}(M) \neq \emptyset$.

CASE 2. If $h(a) = n$, then $(0 : a) = P$. Suppose not. Then $(0 : a) = 0$. Say $a = p^n b$. Then $rb = 0$ implies $ra = 0$, and so $r = 0$. Thus $Rb \cong R$. We also see that Rb is pure in M (see [13, Theorem 2.12, Case 1]). As M is a torsion-free R-module by [20, Theorem 10], necessarily Rb is a prime submodule of M (see Remark 1.3(c)) (so a semiprime submodule); hence $R \cong Rb = (0 :_M 0) = M$, which is a contradiction by Remark 2.7(a). So we may assume that $h(a) = n$ and $(0 : a) = P$. Say $a = p^n b$. Then

$Rb \cong R/P^{n+1}$. Furthermore, Rb is pure in M. Hence, since Rb is a pure submodule of bounded order of M, we deduce that Rb is a direct summand of M by [20, Theorem 5]; hence $M = Rb \cong R/P^{n+1}$.

CASE 3: $h(a) = \infty$, $(0 : a) = P$. By an argument like that in [13, Theorem 2.5, Case 2], we get $M \cong E(R/P)$; hence $\mathrm{seSpec}(M) = \emptyset$ by Remark 2.7, a contradiction.

CASE 4: $h(a) = \infty$, $(0 : a) = 0$. By an argument like that in [13, Theorem 2.12, Case 3], we obtain $M \cong Q(R)$. ∎

THEOREM 2.9. *Let M be a semiprime comultiplication module over a valuation domain with maximal ideal $P = Rp$. Then M is of the form $M = N \oplus K$, where N is a direct sum of copies of R/P^n $(n \geq 1)$, and K is a direct sum of copies of $E(R/P)$ and $Q(R)$. In particular, every semiprime comultiplication R-module is pure-injective.*

Proof. Let M_i denote an indecomposable summand of M. Then by Lemma 2.4(ii), M_i is an indecomposable semiprime comultiplication module. Now the assertion follows from Theorem 2.8 and [10, Proposition 1.3]. ∎

3. The separated case. Throughout this section we shall assume, unless otherwise stated, that

$$R = (R_1 \xrightarrow{v_1} \bar{R} \xleftarrow{v_2} R_2) \tag{3.1}$$

is the pullback of two valuation domains R_1, R_2 with maximal ideals P_1, P_2 generated respectively by p_1, p_2, that P denotes $P_1 \oplus P_2$, and that $R_1/P_1 \cong R_2/P_2 \cong R/P \cong \bar{R}$ is a field. In particular, R is a commutative Noetherian local ring with unique maximal ideal P. The other prime ideals of R are easily seen to be P_1 (that is, $P_1 \oplus 0$) and P_2 (that is, $0 \oplus P_2$).

REMARK 3.1 ([14, Remark 3.1]). Let R be a pullback ring as in (3.1), and let T be an R-submodule of a separated module $S = (S_1 \xrightarrow{f_1} \bar{S} \xleftarrow{f_2} S_2)$, with projection maps $\pi_i : S \twoheadrightarrow S_i$. Set

$$T_1 = \{t_1 \in S_1 : (t_1, t_2) \in T \text{ for some } t_2 \in S_2\},$$
$$T_2 = \{t_2 \in S_2 : (t_1, t_2) \in T \text{ for some } t_1 \in S_1\}.$$

Then for $i = 1, 2$, T_i is an R_i-submodule of S_i, and $T \leq T_1 \oplus T_2$. Moreover, we can define a mapping $\pi'_1 = \pi_1|T : T \twoheadrightarrow T_1$ by sending (t_1, t_2) to t_1; hence $T_1 \cong T/(0 \oplus \mathrm{Ker}(f_2) \cap T) \cong T/(T \cap P_2S) \cong (T + P_2S)/P_2S \subseteq S/P_2S$. So we may assume that T_1 is a submodule of S_1. Similarly, we may assume that T_2 is a submodule of S_2 (note that $\mathrm{Ker}(f_1) = P_1S_1$ and $\mathrm{Ker}(f_2) = P_2S_2$).

PROPOSITION 3.2. *Let R be a pullback ring as in* (3.1), *and let $S = (S/P_2S = S_1 \xrightarrow{f_1} \bar{S} = S/PS \xleftarrow{f_2} S_2 = S/P_1S)$ be any separated R-module.*

(i) *If T is a non-zero semiprime submodule of S, then T_i is a semiprime submodule of S_i for $i = 1, 2$.*

(ii) *If L is a non-zero semiprime submodule of S_1, then there exists a separated submodule T of S such that $T + (0 \oplus P_2)S$ is a semiprime submodule of S.*

(iii) *If L' is a non-zero semiprime submodule of S_2, then there exists a separated submodule T' of S such that $T' + (P_1 \oplus 0)S$ is a semiprime submodule of S.*

Proof. (i) Let $r_1^n s_1 \in T_1$ for some $r_1 \in R$, $s_1 \in S_1$ and a positive integer n. If $r_1 \notin P_1$, then $s_1 \in T_1$ since r_1 is invertible; hence $r_1 s_1 \in T_1$. So we may assume that $r_1 \in P_1$. Then $v_1(r_1) = v_2(0) = 0$; so $(r_1, 0) \in R$. By assumption, there is an element $s_2 \in S_2$ such that $(s_1, s_2 \in S$. As $f_1(r_1^n s_1) = f_2(0) = 0$, we get $(r_1, 0)^n (s_1, s_2) \in T$. Then T semiprime gives $r_1 s_1 \in T_1$. Similarly, T_2 is a semiprime submodule of S_2.

(ii) If L is a non-zero semiprime submodule of S_1, then there exists a separated submodule $T = (T_1 \to \bar{T} \leftarrow T_2)$ of S, where $T_1 = L$. By Remark 3.1,

$$T_1 \cong (T + (0 \oplus P_2)S)/(0 \oplus P_2)S \subseteq S/(0 \oplus P_2)S.$$

Thus $(T + (0 \oplus P_2)S)/(0 \oplus P_2)S$ is a semiprime R-submodule of $S/(0 \oplus P_2)S$; hence $T + (0 \oplus P_2)S$ is a semiprime R-submodule of S by Lemma 2.2(a)(ii). The proof of (iii) is similar. ■

We need the following proposition proved in [9, Proposition 4.4].

PROPOSITION 3.3. *Let $S = (S_1 \to \bar{S} \leftarrow S_2)$ be any separated module over the pullback ring as in* (3.1). *Then* $\mathrm{seSpec}(S) = \emptyset$ *if and only if* $\mathrm{seSpec}(S_i) = \emptyset$ *for $i = 1, 2$.*

THEOREM 3.4. *Let R be a pullback ring as in* (3.1), *and let $S = (S_1 \to \bar{S} \leftarrow S_2)$ be any separated R-module. Then S is a semiprime comultiplication R-module if and only if S_i is a semiprime comultiplication R_i-module for $i = 1, 2$.*

Proof. By Proposition 3.3, we may assume that $\mathrm{seSpec}(S) \neq \emptyset$. Suppose that S is a semiprime comultiplication R-module, and let L be a non-zero semiprime submodule of S_1. By Proposition 3.2, there exists a submodule $T = (T_1 \to \bar{T} \leftarrow T_2)$ of S such that $L = T_1$ and $T' = T + (0 \oplus P_2)S$ is a semiprime submodule of S. Clearly, $\mathrm{ann}(T') = \mathrm{ann}(T) \cap \mathrm{ann}((0 \oplus P_2)S) = 0$ or $P_1^n \oplus 0$ for some positive integer n. Since $S = (0 :_S 0)$, S being a semiprime comultiplication module gives $T' = (0 :_S P_1^n \oplus 0)$. It suffices to show that $L = T_1 = (0 :_{S_1} p_1^n)$. Let $t \in T_1$. There exists $t_2 \in T_2$ such that (t_1, t_2) is in $T \subseteq T'$; so $(P_1^n \oplus 0)(t_1, t_2) = 0$. It then follows that $T_1 \subseteq (0 :_{S_1} p_1^n)$. For the reverse inclusion let $s_1 \in (0 :_{S_1} p_1^n)$. Then there is $s_2 \in S_2$ such that $(s_1, s_2) \in S$ and $(P_1^n \oplus 0)(s_1, s_2) = 0$; hence $(s_1, s_2) \in T'$. Thus $s_1 \in T_1$ and we have equality. Therefore S_1 is semiprime comultiplication. Similarly, S_2 is semiprime comultiplication.

Conversely, assume that S_1, S_2 are semiprime comultiplication and let T be a semiprime submodule of S. By Proposition 3.2, T_1, T_2 are semiprime submodules of S_1, S_2, respectively. By assumption, $T_1 = (0 :_{S_1} P_1^n)$ and $T_2 = (0 :_{S_2} P_2^m)$ for some integers n, m. An inspection will show that $T = (0 :_S P_1^n) \oplus P_2^m$. Thus S is semiprime comultiplication. ∎

Lemma 3.5. *Let R be a pullback ring as in* (3.1). *Then, up to isomorphism, the following separated R-modules are indecomposable and semiprime comultiplication:*

(i) $S = (E(R_1/P_1) \to 0 \leftarrow 0), (0 \to 0 \leftarrow E(R_2/P_2))$, *where $E(R_i/P_i)$ is the R_i-injective hull of R_i/P_i for $i = 1, 2$;*
(ii) $S = (Q(R_1) \to 0 \leftarrow 0), (0 \to 0 \leftarrow Q(R_2))$, *where $Q(R_i)$ is the field of fractions of R_i for $i = 1, 2$;*
(iii) $S = (R_1/P_1^n \to \bar{R} \leftarrow R_2/P_2^m)$ *for all positive integers m, n.*

Proof. By [10, Lemma 2.8] these modules are indecomposable. Their being semiprime comultiplication follows from Theorems 2.8 and 3.4. ∎

We refer to modules of type (i) in Lemma 3.5 as P_1-*Prüfer* and P_2-*Prüfer*, respectively.

Theorem 3.6. *Let R be a pullback ring as in* (3.1), *and let $S = (S_1 \to \bar{S} \leftarrow S_2)$ be an indecomposable separated semiprime comultiplication R-module. Then S is isomorphic to one of the modules listed in Lemma* 3.5.

Proof. If $\text{seSpec}(S) = \emptyset$, then $\text{seSpec}(S_i) = \emptyset$ by Proposition 3.3, so $S_i = P_iS_i$ for each $i = 1, 2$ by Theorem 2.8; hence $S = PS = P_1S_1 \oplus P_2S_2 = S_1 \oplus S_2$. Therefore, $S = S_1$ or S_2, and so S is of type (i) in the list of Lemma 3.5 by Theorem 2.8. So we may assume that $\text{seSpec}(S) \neq \emptyset$.

If $S = PS$, then by [10, Lemma 2.7(i)], $S = S_1$ or $S = S_2$, and so S is an indecomposable semiprime comultiplication R_i-module for some i, and since $PS = S$, it is of type (ii) by Theorem 2.8. So we may assume that $S \neq PS$.

By Theorem 3.4, S_i is a semiprime comultiplication R_i-module for each $i = 1, 2$ (here for each i, S_i is torsion and it is not divisible in view of Theorem 2.8). Therefore by the structure of semiprime comultiplication modules over valuation domains (see Theorem 2.9), $S_i = M_i \oplus N_i$, where N_i is a direct sum of copies of R_i/P_i^n $(n \geq 1)$ and M_i is a direct sum of copies of $E(R_i/P_i)$ and $Q(R_i)$. Then $S = (N_1 \to \bar{S} \leftarrow N_2) \oplus (M_1 \to 0 \leftarrow 0) \oplus (0 \to 0 \leftarrow M_2)$. As S is indecomposable and $S \neq PS$, we get $S = (N_1 \to \bar{S} \leftarrow N_2)$. We will see that S_i $(= N_i)$ is indecomposable. Then there are positive integers u, v and w such that $P_1^uS_1 = 0$, $P_2^vS_2 = 0$ and $P^wS = 0$. For $s \in S$, let $o(s)$ denote the least positive integer m_1 such that $P^{m_1}s = 0$. Now choose $s \in S_1 \cup S_2$ with $\bar{s} \neq 0$ and such that $o(t)$ is maximal. There exists an

$s = (s_1, s_2)$ such that $o(s) = n_1$, $o(s_1) = m_2$ and $o(s_2) = k_1$. Then $R_i s_i$ is pure in S_i for $i = 1, 2$ (see [10, Theorem 2.9]). Therefore, $R_1 s_1 \cong R_1/P_1^{m_2}$ (resp. $R_2 s_2 \cong R_2/P_2^{k_1}$) is a direct summand of S_1 (resp. S_2) since for each i, $R_i s_i$ is pure-injective. Let $\bar{M}$ be the $\bar{R}$-subspace of $\bar{S}$ generated by $\bar{s}$. Then $\bar{M} \cong \bar{R}$. Let $M = (R_1 s_1 = M_1 \to \bar{M} \leftarrow M_2 = R_2 s_2)$. Then M is an R-submodule of S which is semiprime comultiplication by Lemma 3.5 and is a direct summand of S; this implies that $S = M$, and S is as in (iii) in the list of Lemma 3.5 (see [10, Theorem 2.9]). ∎

COROLLARY 3.7. *Let R be a pullback ring as in* (3.1).

(i) *Every separated semiprime comultiplication R-module S is of the form $S = M \oplus N$, where M is a direct sum of copies of modules as in* (iii)*, and N is a direct sum of copies of modules as in* (i)–(ii) *of Lemma* 3.5.
(ii) *Every separated semiprime comultiplication R-module is pure-injective.*

Proof. Apply Theorem 3.6 and [10, Theorem 2.9]. ∎

4. The non-separated case. We continue to use the notation already established, so R is a pullback ring as in (3.1). In this section we find indecomposable non-separated semiprime comultiplication modules with finite-dimensional top. It turns out that each can be obtained by amalgamating finitely many separated indecomposable semiprime comultiplication modules. We need the following proposition proved in [9, Propositions 5.1 and 5.2].

PROPOSITION 4.1. *Let R be a pullback ring as in* (3.1).

(i) *$E(R/P)$ is a non-separated semiprime comultiplication R-module.*
(ii) *Assume that M is an R-module and let $0 \to K \xrightarrow{i} S \xrightarrow{\varphi} M \to 0$ be a separated representation of M. Then* $\mathrm{seSpec}_R(S) = \emptyset$ *if and only if* $\mathrm{seSpec}_R(M) = \emptyset$.

THEOREM 4.2. *Let R be a pullback ring as in* (3.1) *and let M be any non-separated R-module. Let $0 \to K \to S \to M \to 0$ be a separated representation of M. Then S is semiprime comultiplication if and only if M is.*

Proof. By Proposition 4.1, we may assume that $\mathrm{seSpec}(S) \neq \emptyset$. Suppose that M is a semiprime comultiplication R-module and let T be a non-zero semiprime submodule of S. Then by [7, Proposition 4.3], $K \subseteq T$, and so T/K is a semiprime submodule of $S/K \cong M$ by Lemma 2.2(a)(ii). By an argument as in [7, Theorem 4.4], we find that S is semiprime comultiplication.

Conversely, assume that S is a semiprime comultiplication R-module and let N be a non-separated semiprime submodule of M. Then $\varphi^{-1}(N) = U$ is a semiprime submodule of S. Indeed, if $r^n s \in U$ for some $s \in S$, $r \in R$ and a positive integer n, then $r^n \varphi(s) \in N$; hence N semiprime gives $\varphi(rs) \in N$. Thus $rs \in U$, so $U = (0 :_S P_1^n \oplus P_2^m)$ for some integers m, n. By [12, Lemma 3.1], $U/K \cong N$ is a semiprime submodule of $S/K \cong M$, so an inspection will show that $N = U/K = (0 :_{S/K} P_1^n \oplus P_2^m)$, as required. ∎

PROPOSITION 4.3. *Let R be a pullback ring as in* (3.1) *and let M be an indecomposable non-separated semiprime comultiplication R-module with finite-dimensional top over $\bar{R}$. Let $0 \to K \to S \to M \to 0$ be a separated representation of M. Then S is pure-injective.*

Proof. By [10, Proposition 2.6(i)], $S/PS \cong M/PM$, so S has finite-dimensional top. Now the assertion follows from Theorem 4.2 and Corollary 3.7. ∎

Let R be a pullback ring as in (3.1) and let M be an indecomposable non-separated semiprime comultiplication R-module with M/PM finite-dimensional over $\bar{R}$. Consider the separated representation $0 \to K \to S \to M \to 0$. By Proposition 4.3, S is pure-injective. So in the proofs of [10, Lemma 3.1, Propositions 3.2 and 3.4] (here the pure-injectivity of M implies the pure-injectivity of S by [10, Proposition 2.6(ii)]) we can replace the statement "M is an indecomposable pure-injective non-separated R-module" by "M is an indecomposable non-separated semiprime comultiplication R-module": because the main ingredients in those results are the pure-injectivity of S, the indecomposability and the non-separability of M. So we have the following results:

COROLLARY 4.4. *Let R be a pullback ring as in* (3.1)*, let M be an indecomposable non-separated semiprime comultiplication R-module with M/PM finite-dimensional over $\bar{R}$, and let $0 \to K \to S \to M \to 0$ be a separated representation of M. Then:*

(i) *The quotient fields $Q(R_1)$ and $Q(R_2)$ do not occur among the direct summands of S.*
(ii) *S is a direct sum of finitely many indecomposable semiprime comultiplication modules.*
(iii) *At most two copies of modules of infinite length can occur among the indecomposable summands of S.*

Recall that every indecomposable R-module of finite length is semiprime comultiplication (see Lemma 3.5 and Theorem 4.2). So by Corollary 4.4, the infinite length non-separated indecomposable semiprime comultiplication modules are obtained in the same way as the deleted cycle type indecomposable ones are, except that at least one of the two "end" modules must be a separated indecomposable semiprime comultiplication module of infinite length (that is, P_1-Prüfer and P_2-Prüfer). Note that one cannot have, for instance, a P_1-Prüfer module at each end (consider the alternation of primes P_1, P_2 along the amalgamation chain). So, apart from any finite length modules, we have amalgamations involving two Prüfer modules as well as modules of finite length (the injective hull $E(R/P)$ is the simplest module of this type), a P_1-Prüfer module and a P_2-Prüfer module. If the P_1-Prüfer and the P_2-Prüfer modules are direct summands of S then we will refer to these modules as *doubly infinite*. Those where S has just one infinite length summand will be called *singly infinite* (the reader is referred to [10, 12, 15, 8] for more details). It remains to show that modules obtained by these amalgamations are, indeed, indecomposable semiprime comultiplication modules.

THEOREM 4.5. *Let $R = (R_1 \to \bar{R} \leftarrow R_2)$ be the pullback of two valuation domains R_1, R_2 with common factor field $\bar{R}$. Then, up to isomorphism, the indecomposable non-separated semiprime comultiplication modules with finite-dimensional top are the following:*

(i) *the indecomposable modules of finite length (apart from R/P which is separated);*
(ii) *the doubly infinite semiprime comultiplication modules as described above;*
(iii) *the singly infinite semiprime comultiplication modules as described above, apart from the two Prüfer modules* (i) *in Lemma* 3.5.

Proof. Let M be an indecomposable non-separated semiprime comultiplication R-module with finite-dimensional top, and let $0 \to K \xrightarrow{i} S \xrightarrow{\varphi} M \to 0$ be a separated representation of M.

(i) Clearly, M is a semiprime comultiplication R-module. The indecomposability follows from [26, 1.9].

(ii) & (iii) (involving one or two Prüfer modules). We observe that M is semiprime comultiplication (by Corollary 3.7 and Proposition 4.1). Finally, the indecomposability follows from [10, Theorem 3.5]. ∎

COROLLARY 4.6. *Let R be a pullback ring as in Theorem* 4.5. *Then every indecomposable semiprime comultiplication R-module with finite-dimensional top is pure-injective.*

Proof. Apply [10, Theorem 3.5] and Theorem 4.5. ∎

REMARK 4.7. Let R be a pullback of two valuation domains with common field k. As in [2], define the associated graded ring $G(R)$ to be the additive group $\bigoplus_{n\geq 0} P^i/P^{i+1}$ equipped with a ring structure by defining the multiplication as in [2]. Similarly, the associated graded module of an R-module M is $\bigoplus_{n\geq>0} P^iM/P^{i+1}M$, equipped with a $G(R)$-module structure by defining the scalar multiplication as in [2]. Arnold and Laubenbacher [2, Section 6] showed that $G(R)$ is the algebra $k[x, y : xy = 0]_{(x,y)}$, which is the pullback $(k[x]_{(x)} \to k \leftarrow k[y]_{(y)})$ of two valuation domains $k[x]_{(x)}, k[y]_{(y)}$. The R-modules of deleted and block cycle types correspond exactly to the $G(R)$-modules of string and band types. Furthermore, the classification of arbitrary indecomposable pure-injective (even indecomposable semiprime comultiplication) modules over (finite-dimensional quotients of) $k[x, y : xy = 0]_{(x,y)}$ appears to be a difficult problem. Therefore, notably, the classification of the subclass of pure-injective modules over the pullback of two valuation domains over a common factor field is important. One point of this paper is to introduce a subclass of pure-injective modules over such rings. Indeed, this paper includes the classification of those indecomposable semiprime comultiplication modules over $k[x, y : xy = 0]_{(x,y)}$ which have finite-dimensional top.

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