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Additive problems with smooth integers

by

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1. Introduction and statement of results. We use the standard notation: for any real θ,

$$\mathbf{e}(\theta) := e^{2\pi i\theta}, \qquad \|\theta\| := \min_{n\in\mathbb{Z}} |\theta - n|.$$

We denote the largest prime factor of an integer n by $P^+(n)$ (with the convention that $P^+(1) = 0$). For fixed $Y > 0$, $X \geq 100$ we define the set of Y*-smooth integers* by

$$\mathcal{S}(X, Y) := \{n \leq X : P^+(n) \leq Y\}.$$

We start with a few old but important and interesting results in additive number theory closely connected to the questions of this article. For instance, Romanoff [18] showed that the positive integers which are representable as the sum of a prime and a kth power have positive density. Davenport and Heilbronn [6] proved that almost all positive integers are representable as $p + x^k$. From their proof it also follows that if k is odd, almost all positive integers are representable in one of the two forms $p + x^k, 2p + x^k$ with $p \equiv 1 \bmod 4$. From this it also follows that almost all positive integers are representable as $x^2 + y^2 + z^k$. Estermann [9] established that if $j(n)$ denotes the number of those even positive integers less than n which are not representable as sums of two primes, then as $n \to \infty$,

$$j(n) \ll n(\log n)^{-A}$$

for any positive number A.

Additive problems with smooth integers are of great interest and they have been investigated previously by several authors. For instance, Balog [1]

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showed that every positive integer N can be written as the sum of two $N^{0.2695}$-smooth integers. Balog and Sárközy [2] established that every positive integer N can be represented as the sum of three $L(N)^{3+o(1)}$-smooth integers, where $L(x) := \exp(((\log x)(\log\log x))^{1/2})$. In [3], Balog and Sárközy proved that if N is sufficiently large, then N can be written as $N = n_1 + n_2$ where $P^+(n_1 n_2) \leq 2N^{2/5}$. They also note that by using deep estimates for exponential sums, they are able to reduce the exponent $2/5$ to $0.392\ldots$. The corresponding almost-all result is also obtained.

Blomer, Brüdern and Dietmann [5] considered the representation of integers as sums of smooth squares. If we define

$$R(n,\theta) := \#\{n : n = m_1^2 + m_2^2 + m_3^2 + m_4^2 \quad \text{with each } m_i \text{ composed of primes } \leq n^{\theta/2}\},$$

then they proved that $R(n,\theta) > 0$ for sufficiently large n provided $\theta > \frac{365}{592}$ and that $R(n,\theta)$ satisfies a lower bound of the expected order of magnitude for sufficiently large n provided $\theta > e^{-1/3}$. In [16], the second author considered the representation of an arbitrary integer n as the sum of a Y-smooth number and a number whose g-ary expansion does not contain digits from a set $\mathcal{D}$. His result implies that if $Y = n^\theta$ (with $\theta > 0$ fixed) and $|\mathcal{D}| \geq 3$, then such a representation is possible for almost all n. For an overview on the theory of smooth integers, the readers may refer to Granville [11].

In this article, we consider the following two additive problems with smooth integers.

Problem 1. Find the number $R_s(n)$ of representations of a positive integer n $(\leq X)$ in the form

$$n = m^2 + l, \qquad m \in \mathbb{N},\ l \in \mathcal{S}(X,Y). \tag{1.1}$$

Problem 2. Find the number $R_p(n)$ of representations of n $(\leq X)$ in the form

$$n = p + l, \qquad p \text{ prime},\ l \in \mathcal{S}(X,Y). \tag{1.2}$$

Let the exceptional sets be defined by

$$E_s := \{1 \leq n \leq X : n \neq m^2 + l \text{ with } m \in \mathbb{N},\ l \in \mathcal{S}(X,Y)\},$$
$$E_p := \{1 \leq n \leq X : n \neq p + l \text{ with } p \text{ prime},\ l \in \mathcal{S}(X,Y)\}.$$

Either of these sets E_s and E_p may well be empty, although we have been unable to substantiate this statement.

Definition 1.1. For $1 \leq Y \leq X$, we write

$$\psi(X,Y) := |\{n \leq X : P^+(n) \leq Y\}|. \tag{1.3}$$

Dickmann [8] was the first to notice that the asymptotics of $\psi(X,Y)$ is ruled by the Dickmann function ρ, which is defined by

DEFINITION 1.2. We have

$$\rho(u) = 1 \qquad \text{for } 0 \le u \le 1, \tag{1.4}$$

$$\rho'(u) = -\frac{1}{u}\rho(u-1) \qquad \text{for } u > 1, \tag{1.5}$$

ρ is continuous at $u = 1$, where

$$u = \frac{\log X}{\log Y}. \tag{1.6}$$

He showed that

$$\psi(X,Y) \asymp X\rho(u). \tag{1.7}$$

We extend the definition of ρ by $\rho(u) = 0$ for $u < 0$.

DEFINITION 1.3. The function $\Lambda(X,Y)$ is defined by

$$\Lambda(X,Y) := \begin{cases} X\int_{-\infty}^{\infty}\rho(u-v)\,d([Y^v]Y^{-v}) & \text{if } X \notin \mathbb{N}, \\ \Lambda(X+0,Y) & \text{if } X \in \mathbb{N}. \end{cases}$$

LEMMA 1.1 (de Bruijn [7]). *We have*

(i) $\log\rho(u) \asymp -u\log u$;

(ii) $(-1)^j\rho^{(j)}(u) > 0$ $(u \ge 1)$;
$\rho^{(j)}(u) \asymp (-1)^j\rho(u)(\log u)^j$ $(u \to \infty)$;
$$\int_0^\infty \rho^{(j)}(u-v)Y^{-v}\,dv \ll_j \frac{\rho^{(j)}(u)}{\log Y}.$$

DEFINITION 1.4. For $Z > 1$, $k \ge 0$, let

$$\varepsilon_{k,j}(Z) := \min\left\{1, (k+1-j)\frac{\log Z}{Z}\right\} \qquad \text{for } 0 \le j \le k,$$

$$\rho_k(Z) := \Big(\bigcup_{j=0}^{k}[1+\varepsilon_{k,j}(Z), j+1]\Big) \cup [k+1,\infty).$$

DEFINITION 1.5. Let ζ be the Riemann zeta-function and let a_j be the jth coefficient of the Taylor expansion of $s\frac{\zeta(s+1)}{s+1}$ in the neighbourhood of $s = 0$.

LEMMA 1.2 (Saias [19]). *For all* $\epsilon > 0$ *and* $k \ge 0$, *uniformly for* $(\log X)^{1+\epsilon} \le Y \le X$ *and* $u \in \rho_k(\log Y)$, *we have*

$$\Lambda(X,Y) = X\sum_{j=0}^{k} a_j\frac{\rho^{(j)}(u)}{(\log Y)^j} + O\Big(X\frac{\rho^{(k+1)}(u)}{(\log Y)^{k+1}}\Big),$$

$$\psi(X,Y) = \Lambda(X,Y)\big(1 + O(\exp(-(\log Y)^{3/5+\epsilon}))\big).$$

As in Fouvry and Tenenbaum [10], we also define

Definition 1.6. For $2 \le q \in \mathbb{N}$, let

$$S_q(t) := \frac{1}{t}\sum_{n\le t}\frac{\mu(q/(n,q))}{\varphi(q/(n,q))} \quad (t>0), \qquad H_q(s) := \frac{\zeta(s)}{\varphi(q)}q^{1-s}\prod_{p\mid q}(1-p^{s-1})$$

and

$$\frac{sH_q(s+1)}{s+1} =: \sum_{j=1}^{\infty} b_j(q)s^k \qquad (|s|<1).$$

Lemma 1.3 ([19, Theorem 9]). *We have*

$$b_k(q) \ll_k \frac{2^{\omega(q)}}{q}(\log q)^k.$$

Definition 1.7. For $q \in \mathbb{N}$, define

$$S(a,q) := \sum_{m=0}^{q-1}\mathbf{e}\left(\frac{m^2a}{q}\right), \qquad A_n(q) := \sum_{\substack{a=1\\(a,q)=1}}^{q}\frac{S(a,q)}{q}\,\mathbf{e}\left(-\frac{na}{q}\right).$$

Lemma 1.4. *We have*

$$S(a,q) \ll q^{1/2} \quad \textit{for } (a,q)=1, \qquad A_n(q)=O(1).$$

Proof. The following well-known result is due to Gauss (see [14], [4] or [15]):

$$S(a,q) = \begin{cases} 0 & \text{if } q \equiv 2 \pmod 4,\\ \varepsilon_q\sqrt{q}\left(\frac{a}{q}\right) & \text{if } q \text{ is odd},\\ (1+i)\varepsilon_q^{-1}\sqrt{q}\left(\frac{q}{a}\right) & \text{if } q\equiv 0 \pmod 4 \text{ and } a \text{ is odd}. \end{cases} \tag{1.8}$$

Here

$$\varepsilon_q := \begin{cases} 1 & \text{if } q\equiv 1 \pmod 4,\\ i & \text{if } q \equiv 3 \pmod 4, \end{cases}$$

and $\left(\frac{a}{q}\right)$ denotes the Jacobi symbol.

Let $(q,r)=1$. Then

$$\begin{aligned} A_n(qr) &= \sum_{\substack{l=1\\(l,q)=1}}^{q}\sum_{\substack{m=1\\(m,r)=1}}^{r}\frac{S(lr+mq,qr)}{qr}\,\mathbf{e}\left(-\frac{(lr+mq)}{qr}\right)\\ &= \sum_{\substack{l=1\\(l,q)=1}}^{q}\sum_{\substack{m=1\\(m,r)=1}}^{r}\frac{S(l,q)}{q}\frac{S(m,r)}{r}\,\mathbf{e}\left(-\frac{ln}{q}\right)\mathbf{e}\left(-\frac{mn}{q}\right) = A_n(q)A_n(r). \end{aligned}$$

We determine the values of $A_n(q)$ for prime powers. We distinguish three cases.

Case 1: $q=2$. By (1.7), we have $A_n(q)=0$.

Case 2: $q \equiv 0 \pmod 4$. By (1.7), we have

$$\sum_{\substack{l=1\\(l,q)=1}}^{q} S(l,q)\,\mathbf{e}\Big(-\frac{ln}{q}\Big) = (1+i)\varepsilon_q^{-1}\sqrt{q}\sum_{\substack{l=1\\(l,q)=1}}^{q}\Big(\frac{q}{l}\Big)\,\mathbf{e}\Big(-\frac{ln}{q}\Big).$$

By [15, Lemma 3.1] we have

$$\Big|\sum_{\substack{l=1\\(l,q)=1}}^{q}\Big(\frac{q}{l}\Big)\,\mathbf{e}\Big(-\frac{ln}{q}\Big)\Big| \le \sqrt{q},$$

and thus $|A_n(q)| \le \sqrt{2}$.

Case 3: q odd. By (1.7), we have

$$\sum_{\substack{l=1\\(l,q)=1}}^{q} S(l,q)\,\mathbf{e}\Big(-\frac{ln}{q}\Big) = \varepsilon_q\sqrt{q}\sum_{\substack{l=1\\(l,q)=1}}^{q}\Big(\frac{q}{l}\Big)\,\mathbf{e}\Big(-\frac{ln}{q}\Big).$$

By [15, Lemma 3.1] we obtain $|A_n(q)| \le 1$. This proves the lemma. ∎

Definition 1.8. We define

$$R_0^* := \frac{1}{2}\sum_{s<n,\, m<n,\, m+s=n} \rho\Big(\frac{\log s}{\log Y}\Big) m^{-1/2},$$

$$R_1^* := \sum_{s<n,\, m<n,\, m+s=n} \rho\Big(\frac{\log s}{\log Y}\Big)(\log m)^{-1}.$$

We prove:

Theorem 1.1. *Let $C_0 > 0$ be fixed and*

$$Y \ge \exp\Big(C_0\frac{(\log X)(\log\log\log X)}{\log\log X}\Big).$$

Let $L \in \mathbb{N}$ be arbitrarily large and $\varepsilon > 0$. Then there is a constant $C_1 = C_1(C_0, L, \varepsilon)$ such that for all $n \le X$ with at most

$$\ll_{C_0,L,\varepsilon} X(\log X)^{-L}$$

exceptions, we have

$$R_s(n) = R_0^*(1 + r_s(n)) \quad \textit{where} \quad |r_s(n)| \le C_1(\log Y)^{-(1-\varepsilon)}.$$

Theorem 1.2. *Let C_0 and Y be as in Theorem 1.1. Let $L \in \mathbb{N}$ be arbitrarily large and $\varepsilon > 0$. Then there is a constant $C_2 = C_2(C_0, L, \varepsilon)$ such that for all $n \le X$ with at most*

$$\ll_{C_0,L,\varepsilon} X(\log X)^{-L}$$

exceptions, we have

$$R_p(n) = R_1^*(1 + r_p(n)) \quad \textit{where} \quad |r_p(n)| \le C_2(\log Y)^{-(1-\varepsilon)}.$$

REMARK. We observe from Theorems 1.1 and 1.2 that the exceptional sets satisfy

$$\#E_s \ll X(\log X)^{-L}, \qquad \#E_p \ll X(\log X)^{-L}$$

for an arbitrarily large positive integer L. We prove Theorem 1.1 in detail and sketch the necessary changes to obtain Theorem 1.2.

Recently, Harper [12] has investigated exponential sums over those numbers $\le x$ all of whose prime factors are $\le y$ (i.e. over y-smooth numbers) and obtained a fairly good minor arc estimate, valid whenever $\log^3 x \le y \le x^{1/3}$. As an application, he could obtain an asymptotic result for the number of solutions of $a+b=c$ in y-smooth integers less than x whenever $(\log x)^C \le y \le x$. The methods of Harper [12] could possibly be used to improve quantitatively our results on the quality of the smoothness parameter to some extent; we leave it open for further research.

2. The circle method.

Here we closely follow [16]. Let $\theta \in \mathbb{R}$ and set

$$E(\theta) := \sum_{s\in S(X,Y)} \mathbf{e}(s\theta),$$

$$U(\theta) := \sum_{m\le X^{1/2}} \mathbf{e}(m^2\theta).$$

LEMMA 2.1. *For $n \le X$, we have*

$$R_s(n) = \int_0^1 E(\theta)U(\theta)\,\mathbf{e}(-n\theta)\,d\theta.$$

Proof. This follows by orthogonality. ∎

DEFINITION 2.1. Let $Z > Q > 1$ (to be determined later). We call

$$I_p := \left[0, \frac{1}{Z}\right] \cup \left[1-\frac{1}{Z}, 1\right]$$

the *principal major arc*. Let

$$I_{a,q} := \left[\frac{a}{q} - \frac{1}{qZ}, \frac{a}{q} + \frac{1}{qZ}\right].$$

Then

$$\mathfrak{M} := \bigcup_{1<q\le Q}\ \bigcup_{a \bmod q} I_{a,q}$$

is called the *non-principal major arcs*. The *minor arcs* $\mathfrak{m}$ is defined to be

$$\mathfrak{m} := \bigcup_{Q<q\le Z}\ \bigcup_{a \bmod q} I_{a,q}.$$

For $n \in \mathbb{N}$ with $n \le X$, we set

$$R_1(n) := \int_{I_p} E(\theta)U(\theta)\,\mathbf{e}(-n\theta)\,d\theta,$$

$$R_2(n) := \int_{\mathfrak{M}} E(\theta)U(\theta)\,\mathbf{e}(-n\theta)\,d\theta,$$

$$R_3(n) := \int_{\mathfrak{m}} E(\theta)U(\theta)\,\mathbf{e}(-n\theta)\,d\theta.$$

Lemma 2.2. *For $n \in \mathbb{N}$ with $n \le X$, we have*

$$R_s(n) = R_1(n) + R_2(n) + R_3(n).$$

Proof. It is an immediate consequence of Dirichlet's Theorem that the interval $[0,1)$ is the disjoint union of I_p, $\mathfrak{M}$ and $\mathfrak{m}$, and hence the assertion follows. ∎

3. Approximation of $E(\theta)$ and $U(\theta)$. We shall now replace the exponential sum $E(\theta)$ inside the principal major arc I_p and the non-principal major arcs by some approximations. We replace the sum over $s \in \mathcal{S}(X,Y)$ by a sum over all $s \le X$ with appropriate weights.

Definition 3.1. For $k \in \mathbb{N}$ (to be specified later) $s \le X$, we set

$$h_k(s) := \sum_{j=0}^{k} a_j (\log Y)^{-j} \Big\{ \rho^{(j)}\Big(\frac{\log s}{\log Y}\Big) + \frac{1}{\log Y}\rho^{(j+1)}\Big(\frac{\log s}{\log Y}\Big)\Big\}.$$

Lemma 3.1. *For $k \in \mathbb{N}$, we have*

$$E(\theta) = \sum_{s \le X} h_k(s)\,\mathbf{e}(s\theta) + O_k\big(|\theta| X^2 (\log X)^{-(k+1)}\big) + O_k\big(X(\log X)^{-(k+1)}\big).$$

Proof. Let

$$w(s) := \begin{cases} 1 - h_k(s) & \text{if } s \in \mathcal{S}(X,Y), \\ -h_k(s) & \text{otherwise.} \end{cases}$$

Then

$$E(\theta) = \sum_{s \in \mathcal{S}(X,Y)} \mathbf{e}(s\theta) = \sum_{s \le X} h_k(s)\,\mathbf{e}(s\theta) + \sum_{s \le X} w(s)\,\mathbf{e}(s\theta). \tag{3.1}$$

Let $W(t) = \sum_{s \le t} w(s)$. By partial summation, we find that

$$\sum_{s \le X} w(s)\,\mathbf{e}(s\theta) = W(X)\,\mathbf{e}(\theta X) - 2\pi i\theta \int_{1/2}^{X} W(t)\,\mathbf{e}(\theta t)dt. \tag{3.2}$$

Since

$$\frac{d}{dt}\Big(t\rho^{(j)}\Big(\frac{\log t}{\log Y}\Big)\Big) = \rho^{(j)}\Big(\frac{\log t}{\log Y}\Big) + \frac{1}{\log Y}\rho^{(j+1)}\Big(\frac{\log t}{\log Y}\Big),$$

from Euler's summation formula and Lemma 1.2 we obtain

$$W(t) \ll_k t(\log X)^{-(k+1)} \quad \text{if } t \geq X^{1/2}. \tag{3.3}$$

Now the lemma follows from (3.1)–(3.3). ∎

Definition 3.2. Let

$$V_q(X,Y) := \begin{cases} X\int_{-\infty}^{\infty} \rho(u-v)\,dS_q(Y^v) & \text{if } X \notin \mathbb{N}, \\ V_q(X+0,Y) & \text{if } X \in \mathbb{N}. \end{cases}$$

Lemma 3.2 ([19, Theorem 9]). *We have uniformly*

$$E(\theta) = V_q(X,Y) + O\big(\psi(X,Y)\exp(-C_2(\log Y)^{1/2})\big),$$

$$V_q(X,Y) = X\sum_{j=1}^{k} b_j(q)\frac{\rho^{(j)}(u)}{(\log Y)^j} + O_k\bigg(X\frac{2^{\omega(q)}}{\varphi(q)}\rho^{(k+1)}(u)\bigg(\frac{\log q}{\log Y}\bigg)^{k+1}\bigg).$$

Definition 3.3. For $k \in \mathbb{N}, s \in \mathbb{N}\cup\{0\}$ and $1 \leq q \leq Q$, we set

$$l_k(s;q) := \sum_{j=1}^{k}\bigg\{\frac{b_j(q)}{(\log Y)^j}\bigg(\rho^{(j)}\bigg(\frac{\log s}{\log Y}\bigg) + \frac{1}{\log Y}\rho^{(j+1)}\bigg(\frac{\log s}{\log Y}\bigg)\bigg)\bigg\}.$$

Lemma 3.3. *For $|\theta| \geq X^{-1}$, we have*

$$E\bigg(\frac{a}{q}+\theta\bigg) = \sum_{s\leq X} l_k(s;q)\,\mathbf{e}(s\theta) + O_k\bigg(|\theta|X^2\frac{2^{\omega(q)}}{\varphi(q)}\bigg(\frac{\log q}{\log Y}\bigg)^{k+1}\bigg) + O_k\bigg(X\frac{2^{\omega(q)}}{\varphi(q)}\bigg(\frac{\log q}{\log Y}\bigg)^{k+1}\bigg).$$

Proof. Let

$$z(s,q) := \begin{cases} \mathbf{e}\big(\frac{a}{q}s\big) - l_k(s;q) & \text{if } s \in \mathcal{S}(X,Y), \\ -l_k(s;q) & \text{otherwise.} \end{cases}$$

Let $Z(t) = \sum_{1\leq s\leq t} z(s,q)$. Then

$$E\bigg(\frac{a}{q}+\theta\bigg) = \sum_{s\leq X} l_k(s;q)\,\mathbf{e}(s\theta) + \sum_{s\leq X} z(s,q)\,\mathbf{e}(s\theta).$$

Partial summation leads to

$$\sum_{s\leq X} z(s,q)\,\mathbf{e}(s\theta) = Z(X)\,\mathbf{e}(\theta X) - 2\pi i\theta\int_{1/2}^{X} Z(t)\,\mathbf{e}(\theta t)\,dt. \tag{3.4}$$

From Euler's summation formula and Lemma 3.2, we obtain

$$Z(t) \ll_k t(\log X)^{-(k+1)}\frac{2^{\omega(q)}}{\varphi(q)}\bigg(\frac{\log q}{\log Y}\bigg)^{k+1}.$$

Now, the result follows from (3.4). ∎

Definition 3.4. For $\theta \in \mathbb{R}$, let

$$W(\theta) := \frac{1}{2}\sum_{m \le X} \mathbf{e}(m\theta)m^{-1/2}.$$

Lemma 3.4. *Let $(a,q) = 1$ and $|\beta| \le 1/q^2$. Then*

$$U\left(\frac{a}{q}+\beta\right) = \frac{S(a,q)}{q}W(\beta) + O(qX|\beta|) + O(q).$$

Proof. We observe that

$$U\left(\frac{a}{q}+\beta\right) = \sum_{l=0}^{q-1}\mathbf{e}\left(l^2\frac{a}{q}\right)\sum_{\substack{m\le X^{1/2}\\ m\equiv l \bmod q}}\mathbf{e}(m^2\beta). \tag{3.5}$$

By Euler's summation formula, we have

$$\begin{aligned}\sum_{\substack{m\le X^{1/2}\\ m\equiv l \bmod q}}\mathbf{e}(m^2\beta) &= \int_0^{(X^{1/2}-l)/q}\mathbf{e}((qu+l)^2\beta)\,du + O(X|\beta|) + O(1)\\ &= q^{-1}\int_0^{X^{1/2}}\mathbf{e}(w^2\beta)\,dw + O(X|\beta|) + O(1)\\ &= \frac{q^{-1}}{2}\int_0^{X}\mathbf{e}(y\beta)y^{-1/2}\,dy + O(X|\beta|) + O(1).\end{aligned} \tag{3.6}$$

On the other hand, by Euler's summation formula again,

$$\frac{1}{2}\sum_{m\le X}\mathbf{e}(m\beta)m^{-1/2} = \frac{1}{2}\int_0^X \mathbf{e}(y\beta)y^{-1/2}\,dy + O(1). \tag{3.7}$$

The claim now follows from (3.5)–(3.7). ∎

4. The basic integral

Definition 4.1. For $j \in \mathbb{N}$ and $\theta \in \mathbb{R}$, we set

$$G(\theta, j) := \sum_{s\le X}\rho^{(j)}\left(\frac{\log s}{\log Y}\right)\mathbf{e}(s\theta).$$

Let $C_3 > 0$ be a fixed constant and Z_0 be a positive quantity satisfying

$$X(\log X)^{-2C_3} < Z_0 \le X(\log X)^{-C_3}. \tag{4.1}$$

For $n \in \mathbb{N}$, the basic integral $\int := \int(j,n,Z_0)$ is defined as

$$\int := \int(j,n,Z_0) = \int_{-1/Z_0}^{1/Z_0} G(\theta,j)W(\theta)\,\mathbf{e}(-n\theta)\,d\theta.$$

Definition 4.2. Let Z_0 be as in Definition 4.1. Let $H = Z_0(\log X)^2$ and $\|\theta\| := \min_{z\in\mathbb{Z}} |z-\theta|$. We define a periodic function $\chi_0(\theta)$ of period 1 as

$$\chi_0(\theta) := \begin{cases} 1 & \text{if } \|\theta\| \le \frac{1}{Z_0 \log X}, \\ 0 & \text{otherwise.} \end{cases}$$

Let

$$\chi(\theta) := \frac{H}{\sqrt{\pi}} \int_{-\infty}^{\infty} \chi_0(\theta+v) \exp(-H^2 v^2)\, dv.$$

Definition 4.3. For $j \in \mathbb{N} \cup \{0\}$, we write

$$r_j := \frac{1}{4} \sideset{}{'}\sum_{(s_1,s_2,m_1,m_2)} \rho^{(j)}\left(\frac{\log s_1}{\log Y}\right) \rho^{(j)}\left(\frac{\log s_2}{\log Y}\right) m_1^{-1/2} m_2^{-1/2},$$

where the prime means that the sum is extended over all (s_1, s_2, m_1, m_2) with $s_1, s_2, m_1, m_2 \le X$ and $s_1 + m_1 = s_2 + m_2$.

Lemma 4.1. *The Fourier expansion*

$$\chi(\theta) = \sum_{h=-\infty}^{\infty} \hat{\chi}(h)\, \mathbf{e}(h\theta)$$

converges absolutely and uniformly. Furthermore, there are absolute constants C_4, C_5, C_6, C_7 (> 0) such that

$$|\hat{\chi}(h)| \le C_4 \exp(-C_5 h^2 H^{-2}). \tag{4.2}$$

Moreover, we have $0 \le \chi(\theta) \le 1, \chi(\theta) \ll \exp(-C_6(\log X)^2)$ for $\theta \in (-1/2, 1/2) \setminus (-1/Z_0, 1/Z_0)$, and

$$\chi(0) = 1 + O\big(\exp(-C_7(\log X)^2)\big).$$

Proof. We follow certain arguments of [17]. We start with the representation

$$\chi_0(\theta) = \sum_{m=-\infty}^{\infty} u_m\, \mathbf{e}(m\theta)$$

with

$$u_m := \int_{-(Z_0 \log X)^{-1}}^{(Z_0 \log X)^{-1}} \mathbf{e}(-m\theta)\, d\theta \ll \min\big((Z_0 \log X)^{-1}, |m|^{-1}\big).$$

This series converges uniformly on all compact sets not containing integers. We obtain

$$\begin{aligned}\hat{\chi}(h) &= \frac{H}{\sqrt{\pi}} \int\limits_{-1/2}^{1/2} \mathbf{e}(-h\theta) \int\limits_{-\infty}^{\infty} u_m\, \mathbf{e}(m(\theta+v)) \exp(-H^2v^2)\, dv\, d\theta \\ &= u_h \frac{H}{\sqrt{\pi}} \int\limits_{-\infty}^{\infty} \mathbf{e}(hv) \exp(-H^2v^2)\, dv = u_h \exp(-\pi^2 h^2 H^{-2}),\end{aligned}$$

which gives (4.2) on using the previous estimate $u_h \ll |h|^{-1}$.

We also have

$$\begin{aligned}\chi(0) &= \frac{H}{\sqrt{\pi}} \int\limits_{-(Z_0\log X)^{-1}}^{(Z_0\log X)^{-1}} \exp(-H^2v^2)\, dv \\ &= 1 + O\Big(H \int\limits_{|v|\ge H^{-1}\log X} \exp(-H^2v^2)\, dv\Big) \\ &= 1 + O\big(\exp(-C_7(\log X)^2)\big).\end{aligned}$$

Let $\theta \in (-1/2,1/2)\backslash(-1/Z_0,1/Z_0)$. Then $\chi_0(\theta+v) = 0$ unless $|v| \ge 1/(2Z_0)$. Therefore,

$$\chi(\theta) \ll H \int\limits_{(2Z_0)^{-1}}^{\infty} \exp(-H^2v^2)dv \ll \exp(-C_0(\log X)^2). \blacksquare$$

LEMMA 4.2. *Let C_0 be as in Theorem 1.1, and $j \le k$, $k \in \mathbb{N}$. Let $C_3 \ge C_8(C_0)$ where $C_8(C_0)$ is fixed and sufficiently large. Then, for Z_0 satisfying (4.1), and χ given by Definition 4.2, we have*

$$\int\limits_{-1/2}^{1/2} |G(\theta,j)|^2 |W(\theta)|^2 \chi(\theta)\, d\theta = r_j\big(1 + O_k((\log X)^{-C_3/3})\big).$$

Proof. By Lemma 4.1 and orthogonality, we have

$$\int\limits_{-1/2}^{1/2} |G(\theta,j)|^2 |W(\theta)|^2 \chi(\theta)\, d\theta = \frac{1}{4} \sum_{l=-\infty}^{\infty} \hat{\chi}(l) \sum\nolimits_{(l)}$$

where

$$\sum\nolimits_{(l)} := \sum_{(s_1,s_2,m_1,m_2)}'' \tau(s_1,s_2,j,Y) m_1^{-1/2} m_2^{-1/2} \tag{4.3}$$

with

$$\tau(s_1,s_2,j,Y) := \rho^{(j)}\Big(\frac{\log s_1}{\log Y}\Big)\rho^{(j)}\Big(\frac{\log s_2}{\log Y}\Big)$$

and the sum $\sum''$ being extended over all (s_1, s_2, m_1, m_2) with $s_j \le X$, $m_j \le X$, and $s_1 + m_1 + l = s_2 + m_2$. We partition $\sum_{(l)}$ into

$$\sum_{(l)} = \sum_{(m_1,m_2)} \sum(l, m_1, m_2) \tag{4.4}$$

where

$$\sum(l, m_1, m_2) = m_1^{-1/2} m_2^{-1/2} \sum_{(s_1,s_2)}^{\star} \tau(s_1, s_2, j, Y),$$

where in $\sum^{\star}_{(s_1,s_2)}$, the sum is extended over all (s_1, s_2) with $s_1 + m_1 + l = s_2 + m_2$. In the following, we only discuss the case $m_1 \ge m_2$ (the case $m_1 \le m_2$ is similar).

By Lemma 4.1, inequality (4.2) and Definition 4.2, we have

$$\sum_{|l|>Z_0 \log X} \Big| \sum_{(l)} \Big| = O\bigg(X \exp\Big(-\frac{C_5}{2} (\log X)^2 \Big) \bigg) = O(X^{-1}). \tag{4.5}$$

In what follows, we assume that

$$|l| \le Z_0 \log X. \tag{4.6}$$

By Lemma 1.1,

$$(\log X)^{-C_0{}^{-1}(1-\varepsilon)} \gg_{k,\varepsilon} \rho^{(j)} \left(\frac{\log s_i}{\log Y} \right) \gg_{k,\varepsilon} (\log X)^{-C_0{}^{-1}(1+\varepsilon)}, \tag{4.7}$$

$C_0 > 0$ being the constant of Theorem 1.1 and $\varepsilon > 0$ being arbitrarily small. We set

$$s^{(0)} = X (\log X)^{-C_3/2} \tag{4.8}$$

and write

$$\sum(l, m_1, m_2) = \sum^{(1)}(l, m_1, m_2) + \sum^{(2)}(l, m_1, m_2)$$

with

$$\sum^{(i)}(l, m_1, m_2) = m_1^{-1/2} m_2^{-1/2} \sum_{(s_1,s_2)}^{(i)} \tau(s_1, s_2, j, Y),$$

where in $\sum^{(i)}_{(s_1,s_2)}$, the summation is extended over all (s_1, s_2) with $s_1+m_1+l = s_2 + m_2$, and $s^{(0)} \le s_1 \le X$ for $i = 1$, and $s_1 < s^{(0)}$ for $i = 2$. By the substitution $s_1' = s_1 + l$, we obtain

$$\sum^{(1)}(l, m_1, m_2) = m_1^{-1/2} m_2^{-1/2} \sum_{(s_1',s_2)}^{\prime\prime\prime} \tau(s_1' - l, s_2, j, Y),$$

where the summation is over all (s_1', s_2) with $s_1' + m_1 = s_2 + m_2$, $s^{(0)} + l \le$

$s_1' \le X + l$ and $s_2 \le X$. From the estimate $|\rho^{(j)}(u)| \le 1$, we get (for $l \ge 0$)

$$\sum_{\substack{s^{(0)} \le s_1 \le s^{(0)}+l \\ (s_1', s_2)}} \tau(s_1' - l, s_2, j, Y) \ll l \quad \text{and} \quad \sum_{X < s_1' \le X + l} \tau(s_1' - l, s_2, j, Y) \ll l.$$

These estimates together with analogous ones for $l < 0$ and with (4.7) yield

$$\begin{aligned} \sum\nolimits_{(l)} = \sideset{}{''}\sum_{(s_1', s_2, m_1, m_2)} \tau(s_1' - l, s_2, j, Y) m_1^{-1/2} m_2^{-1/2} \\ + O\Big(|l| \sum_{(m_1, m_2)} m_1^{-1/2} m_2^{-1/2}\Big). \end{aligned}$$

A trivial estimate gives

$$\sum\nolimits^{(2)}(l, m_1, m_2) \ll m_1^{-1/2} m_2^{-1/2} (\log X)^{-C_3/2}.$$

By the mean-value theorem, we have

$$\begin{aligned} \left|\rho^{(j)}\left(\frac{\log s_1}{\log Y}\right) - \rho^{(j)}\left(\frac{\log (s_1 + l)}{\log Y}\right)\right| &\le |l| \Big(\sup_{t \in J_l(s_1)} \left|\rho^{(j+1)}\left(\frac{\log t}{\log Y}\right)\right|\Big) \frac{1}{t \log Y} \\ &\le \frac{|l|}{s^{(0)} \log Y}, \end{aligned}$$

where

$$J_l(s_1) = \begin{cases} (s_1 + l, s_1) & \text{for } l \le 0, \\ (s_1, s_1 + l) & \text{for } l > 0. \end{cases}$$

From $|l| \le X(\log X)^{-C_3+1}$ and (4.7), we obtain

$$\sum\nolimits_{(l)} = \sum\nolimits_{(0)} \big(1 + O_j((\log X)^{-C_3/3})\big). \tag{4.9}$$

From (4.5) and (4.9), we have

$$\int_{-1/2}^{1/2} |G(\theta, j)|^2 |W(\theta)|^2 \chi(\theta)\, d\theta = \frac{1}{4} \sum_{l=-\infty}^{\infty} \hat{\chi}(l) \sum\nolimits_{(0)} \big(1 + O_j((\log X)^{-C_3/3})\big).$$

The claim of Lemma 4.2 follows because by Lemma 4.1 we have

$$\sum_{l=-\infty}^{\infty} \hat{\chi}(l) = \chi(0) = 1 + O\big(\exp(-C_7 (\log X)^2)\big). \ \blacksquare$$

LEMMA 4.3. *Let C_3 and Z_0 be as in Lemma 4.2. Then*

$$\int_{(-1/2, 1/2) \setminus (-Z_0^{-1}, Z_0^{-1})} |G(\theta, j)|^2 |W(\theta)|^2\, d\theta \ll X^2 (\log X)^{-C_3/3}.$$

Proof. We apply Lemma 4.2. From the properties $0 \le \chi(\theta) \le 1$ and $\chi(0) = 1 + O(\exp(-C_7(\log X)^2))$ we conclude that

$$\begin{aligned}
&\int\limits_{(-1/2,1/2)\setminus(-1/Z_0,1/Z_0)} |G(\theta,j)|^2|W(\theta)|^2\,d\theta \\
&\quad\le \int\limits_{(-1/2,1/2)} (1-\chi(\theta))|G(\theta,j)|^2|W(\theta)|^2\,d\theta \\
&\quad\quad + O\Big(\exp(-C_7(\log X)^2) \int\limits_{(-1/2,1/2)\setminus(-1/Z_0,1/Z_0)} |G(\theta,j)|^2|W(\theta)|^2\,d\theta\Big).
\end{aligned}$$

Now the result follows from Lemmas 4.1 and 4.2. ∎

Lemma 4.4. *Let C_3 and Z_0 be as in Lemma 4.2. Then for all $n \le X$ with at most $X(\log X)^{-C_3/4}$ exceptions, we have*

$$\int(j,n,Z_0) = \frac{1}{2}\sum_{\substack{s<n,\,m<n\\ m+s=n}} \rho^{(j)}\Big(\frac{\log s}{\log Y}\Big)m^{-1/2}(1+r(j,n))$$

where $|r(j,n)| \le (\log X)^{-2}$.

Proof. By orthogonality,

$$\int\limits_{(-1/2,1/2)} G(\theta,j)W(\theta)\,\mathbf{e}(-n\theta)\,d\theta = \frac{1}{2}\sum_{\substack{s<n,\,m<n\\ m+s=n}} \rho^{(j)}\Big(\frac{\log s}{\log Y}\Big)m^{-1/2}.$$

By Parseval's equation and Lemma 4.3 we have

$$\begin{aligned}
&\sum_{n\le X}\Big|\int\limits_{(-1/2,1/2)\setminus(-1/Z_0,1/Z_0)} G(\theta,j)W(\theta)\,\mathbf{e}(-n\theta)\,d\theta\Big|^2 \\
&\quad= \int\limits_{(-1/2,1/2)\setminus(-1/Z_0,1/Z_0)} |G(\theta,j)|^2|W(\theta)|^2\,d\theta \ll X^2(\log X)^{-C_3/3}. \ \blacksquare
\end{aligned}$$

5. The principal major arc

Definition 5.1. With C_3 as in Definition 4.1, we set

$$Z := X(\log X)^{-C_3}, \qquad Q := (\log X)^{C_3/10}.$$

Lemma 5.1. *Let $k \in \mathbb{N}$. For all $n \le X$ with at most $O_k(X(\log X)^{-C_3/4})$ exceptions we then have*

$$R_1(n) = R_0^*(1 + r(n))$$

with $r(n) = O(1/\log Y)$.

Proof. By Lemmas 3.1 and 3.4 (applied with $q = 1$) and Definition 4.1 for $Z_0 = Z$, for all $n \le X$ with at most $O_k(X(\log X)^{-C_3/4})$ exceptions we have

$$\begin{aligned}
R_1(n) &= \int_{I_p} E(\theta)U(\theta)\,\mathbf{e}(-n\theta)\,d\theta \\
&= \int_{I_p} \Big\{ \sum_{s\le X} h_k(s)\,\mathbf{e}(-s\theta) + O(|\theta| X^2 (\log X)^{-k+1}) \\
&\qquad\qquad\qquad + O(X(\log X)^{-(k+1)}) \Big\} W(\theta)\,\mathbf{e}(-n\theta)\,d\theta \\
&\quad + O\Big(\int_{I_p} |E(\theta)|(X|\theta| + O(1))\,d\theta \Big) \\
&= \sum_{j=0}^{k} a_j (\log Y)^{-j} \Big(\int (j,n,Z) + \frac{1}{\log Y} \int (j+1,n,Z) \Big) \\
&\quad + O\Big(X(\log X)^{-k} \int_{I_p} |W(\theta)|\,d\theta \Big) + O\Big(X^2 (\log X)^{-k} \int_{I_p} |\theta|\,|W(\theta)|\,d\theta \Big) \\
&= \sum_{j=0}^{k} a_j (\log Y)^{-j} \Big(\int (j,n,Z) + \frac{1}{\log Y} \int (j+1,n,Z) \Big) \\
&\quad + O(X^{1/2} (\log X)^{2C_3 - k}) = R_0^*(1 + r(n)),
\end{aligned}$$

by choosing k sufficiently large. ■

6. The non-principal major arcs

LEMMA 6.1. *For all $n \le X$ with at most $X(\log X)^{-C_3/6}$ exceptions, we have*

$$R_2(n) \le C_2 R_0^* (\log Y)^{-(1-\varepsilon)}$$

where $C_2 > 0$ is a fixed constant.

Proof. By Lemmas 3.3 and 3.4,

$$R_2(n) = \sum_{1<q\le Q} \sum_{\substack{a \bmod q \\ (a,q)=1}} \int_{I_{a,q}} E(\theta)U(\theta)\,\mathbf{e}(-n\theta)\,d\theta$$

where

$$\begin{aligned}
&\int_{I_{a,q}} E(\theta)U(\theta)\,\mathbf{e}(-n\theta)\,d\theta \\
&\quad = \int_{-1/(qZ)}^{1/(qZ)} \Big\{ \sum_{s\le X} l_k(s;q)\,\mathbf{e}(s\beta) + O_k\Big(|\beta| X^2 \frac{2^{\omega(q)}}{\varphi(q)} \Big(\frac{\log q}{\log Y} \Big)^{k+1} \Big) \\
&\qquad\qquad\qquad + O_k\Big(X \frac{2^{\omega(q)}}{\varphi(q)} \Big(\frac{\log q}{\log Y} \Big)^{k+1} \Big) \Big\}
\end{aligned}$$

$$\times \left\{ \frac{S(a,q)}{q} W(\beta) + O(qX|\beta|) + O(q) \right\} \mathbf{e}\Big(-n\Big(\frac{a}{q}+\beta\Big)\Big)\, d\beta$$

$$= \sum_{j=1}^{k} \frac{b_j(q)}{(\log Y)^j} \frac{S(a,q)}{q} \mathbf{e}\Big(-\frac{na}{q}\Big)\Big(\textstyle\int(j,n,qZ) + \frac{1}{\log Y}\int(j+1,n,qZ)\Big)$$

$$+ \sum_{j=1}^{k} O_k\Big(\frac{|b_j(q)|}{(\log Y)^j} Z^{-2} q^{-2} X^2 (\log X) \max_{\theta\in(0,1)} |W(\theta)|\Big)$$

$$+ \sum_{j=1}^{k} O_k\Big(\frac{|b_j(q)|}{(\log Y)^j} Z^{-1} q^{-3/2} X (\log X)^{-(k+1)} \max_{\theta\in(0,1)} |W(\theta)|\Big)$$

$$+ \sum_{j=1}^{k} O_k\Big(\frac{|b_j(q)|}{(\log Y)^j} Z^{-2} q^{-5/2} \max_{\theta\in(0,1)} |W(\theta)|\Big)$$

$$+ \sum_{j=1}^{k} O_k\Big(\frac{|b_j(q)|}{(\log Y)^j} Z^{-1} q^{-3/2} X \max_{\theta\in(0,1)} |W(\theta)|\Big).$$

By Lemma 1.3, the contribution of the O_k-terms is

$$\ll X(\log X)^{2C_3-(k+1)} |b_1(q)|/q.$$

Thus,

$$(6.1)\qquad \int_{I_{a,q}} E(\theta)U(\theta)\,\mathbf{e}(-n\theta)\,d\theta$$

$$= \sum_{j=1}^{k} \frac{b_j(q)}{(\log Y)^j} \frac{S(a,q)}{q} \mathbf{e}\Big(-\frac{na}{q}\Big)\Big(\textstyle\int(j,n,qZ) + \frac{1}{\log Y}\int(j+1,n,qZ)\Big)$$

$$+ O_k(X(\log X)^{2C_3-(k+1)}|b_1(q)|/q).$$

From (6.1) and from Definition 6.1, we obtain

$$R_2(n) = \sum_{1<q\le Q} \sum_{\substack{a \bmod q\\ (a,q)=1}} \int_{I_{a,q}} E(\theta)U(\theta)\,\mathbf{e}(-n\theta)\,d\theta$$

$$= \sum_{1<q\le Q} A_n(q) \sum_{j=1}^{k} \frac{b_j(q)}{(\log Y)^j} \Big(\textstyle\int(j,n,qZ) + \frac{1}{\log Y}\int(j+1,n,qZ)\Big)$$

$$+ O_k(X(\log X)^{2C_3-(k+1)}).$$

Now, we apply Lemmas 1.3, 1.4 and 4.4 to deduce that for all $n \le X$ with at most $X(\log X)^{-C_3/6}$ exceptions, we have

$$R_2(n) \le C_2 R_0^*(\log Y)^{-(1-\varepsilon)}$$

where $C_2 > 0$ is a fixed constant. ∎

7. The minor arcs

Lemma 7.1. *For all $n \le X$ with at most $X(\log X)^{-C_3/20}$ exceptions, we have*

$$R_3(n) \le C_8(\log Y)^{-(1-\varepsilon)}.$$

Proof. Let $\theta \in \mathfrak{m}$. By Dirichlet's approximation theorem, there are $a, q \in \mathbb{N}$ with $(a, q) = 1$ and $Q < q \le Z$ such that

$$\left|\theta - \frac{a}{q}\right| \le \frac{1}{q^2}.$$

By Weyl's inequality (see for instance [15, p. 200]),

$$|U(\theta)| \ll X^{1/2}(\log X)(X^{-1/2} + q^{-1} + X^{-1}q)^{1/2} \ll X^{1/2}(\log X)^{-C_3/20}.$$

From Parseval's equation, we thus have

$$\begin{aligned}\sum_{n\le X}(R_3(n))^2 &= \int_{\mathfrak{m}} |E(\theta)|^2|U(\theta)|^2\,d\theta \le \Big(\max_{\theta\in\mathfrak{m}}|U(\theta)|^2\Big)\Big(\int_{\mathfrak{m}}|E(\theta)|^2\,d\theta\Big)\\ &\ll X^2(\log X)^{-C_3/10}.\end{aligned}$$

Thus, the lemma follows. ■

8. Proofs of Theorems 1.1 and 1.2. Theorem 1.1 follows from Sections 5–7 and Lemma 2.2.

Sketch of proof of Theorem 1.2. For $n \le X$, we have

$$R_p(n) = \int_0^1 E(\theta)V(\theta)\,\mathbf{e}(-n\theta)\,d\theta, \quad \text{where} \quad V(\theta) := \sum_{p\le X}\mathbf{e}(p\theta).$$

Define

$$T(\theta) := \sum_{m\le X}(\log m)^{-1}\,\mathbf{e}(m\theta) \quad \text{and} \quad T(\theta, q) := \sum_{\substack{a \bmod q\\ (a,q)=1}}\frac{\mathbf{e}(aq)}{\varphi(q)}T(\theta).$$

As before, we make the same decomposition of the domain of integration into principal and non-principal major arcs and minor arcs. Inside the principal (respectively non-principal) major arcs, we use the approximation $V(\theta) \sim T(\theta)$ (respectively $V(\theta) \sim T(\theta, q)$). We estimate the error terms by using the Prime Number Theorem (respectively the Page–Walfisz Prime Number Theorem) (see [15]). The basic integral becomes

$$\int^{(p)}(j, n, Z_0) = \int_{-1/Z_0}^{1/Z_0} G(\theta, j)T(\theta)\,\mathbf{e}(-n\theta)\,d\theta.$$

Inside the minor arcs, we use the well-known estimate of Vinogradov [20] for $V(\theta)$. Thus the proof is complete. ■

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