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A GENERALIZED DOUBLE CROSSPRODUCT FOR MONOIDAL HOM-HOPF ALGEBRAS AND THE DRINFELD DOUBLE

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Abstract. A twisted generalization of a double crossproduct called a generalized double Hom-crossproduct is introduced. We give conditions under which this new monoidal Hom-algebra is a monoidal Hom-Hopf algebra. Moreover, its coquasitriangular structure is described. Finally, we also construct a new braided monoidal Hom-category $\widetilde{\mathcal{H}}({}^J_J\mathrm{Mod}^Q_Q)$ obtained from the structure of the generalized double Hom-crossproduct, and establish a kind of new quantum Yang–Baxter Hom-operators.

Introduction. The first examples of Hom-type algebras were related to q-deformations of Witt algebras and Virasoro algebras (see [CPP, H, K, LS]), which play an important role in physics. Motivated by these examples, Hartwig, Larsson and Silvestrov [HLS] first introduced the Hom-structures for Lie algebras, where the Jacobi identity is twisted by a linear endomorphism. Hom-algebras have been introduced for the first time in [MS1]. Also definitions of Hom-bialgebras and Hom-Hopf algebras have been proposed (see [MS3, MS4]), which involve two different linear maps α and β, with α twisting the associativity condition and β the coassociativity condition. Afterwards, two directions of study were developed, one considering the class such that $\beta = \alpha$, which are still called Hom-bialgebras and Hom-Hopf algebras (cf. [MS2, MS3, MS4]). Another one, called monoidal Hom-bialgebras and monoidal Hom-Hopf algebras in monoidal Hom-category, initiated in [CG], where the map α is assumed to be invertible and $\beta = \alpha^{-1}$. Hom-smash product [CWZ1, MLY], Hom-smash coproduct [LSh], quasitriangular Hom-Hopf algebras [CWZ3], Hom-structures [CZ], the Yetter–Drinfeld Hom-category [LM], the Long category [CWZ2], Majid's double Hom-product [LW] and generalized Hom-smash product [ZZ] for monoidal Hom-bialgebras have been studied.

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In the theory of classical Hopf algebras, a celebrated notion is Radford's biproduct [R1]. As a generalization of Radford's biproducts, Majid's double crossproduct $B \bowtie H$, with a Hopf algebra B right acting on a Hopf algebra H and H left acting on B, generalizes a construction of the well-known Drinfeld double. Many researchers have investigated it from different points of view [B, CW, DNR, DT, R2, R3, Si, W]. As a generalization of Majid's double crossproduct, in this paper we study generalized double Hom-crossproducts (see Section 2).

Recently, Majid's double Hom-crossproduct $(B \bowtie H, \xi_B \otimes \xi_H)$ has been studied [LW]. Our motivation is the following: First, if (B, ξ_B) acts on (H, ξ_H) trivially, then the multiplication in $(B \bowtie H, \xi_B \otimes \xi_H)$ is just the multiplication in $(B \# H, \xi_B \otimes \xi_H)$, a left Hom-smash product. Similarly, when (H, ξ_H) acts on (B, ξ_B), trivially then $(B \bowtie H, \xi_B \otimes \xi_H)$ is a right Hom-smash product. Secondly, a left Hom-smash product and a right Hom-smash product are generalized to $B \#_J^L H$ (a generalized left Hom-smash product), where (H, ξ_H) has a left J-Hom-action and (J, ξ_J) a monoidal Hom-Hopf algebra acting on (B, ξ_B), and to $B \#_J^R H$ (a generalized right Hom-smash product), where (H, ξ_H) has a right J-Hom-action and (B, ξ_B) is a right J-Hom-module algebra [ZZ]. However, a generalized left (right) Hom-smash product is not a special case of Majid's double Hom-crossproduct $(B \bowtie H, \xi_B \otimes \xi_H)$. Therefore, we ask: Can one give an algebra construction such that Majid's double Hom-crossproduct and the generalized Hom-smash product can be obtained as particular cases?

In this paper, we give a positive answer to this question (see Sections 2 and 3). The background material is presented in Section 1. In Section 2, we define and study a generalized double Hom-crossproduct $(B \overset{(J,Q)}{\bowtie} H, \xi_B \otimes \xi_H)$ for two monoidal Hom-algebras (B, ξ_B) and (H, ξ_H), where (J, ξ_J) and (Q, ξ_Q) are monoidal Hom-bialgebras (or monoidal Hom-Hopf algebras), (B, ξ_B) is a left J-Hom-module and right Q-Hom-comodule algebra, (H, ξ_H) is a left J-Hom-comodule algebra and a right Q-Hom-module. We show that the left (right) generalized Hom-smash product, Majid's double Hom-crossproduct, in particular the Hom-smash product, the Drinfeld double $D(H)$ and the Doi–Takeuchi monoidal Hom-algebra $(B \bowtie_\tau H, \xi_B \otimes \xi_H)$ are all special cases of our monoidal Hom-algebra structure.

In Section 3, we give conditions under which this new monoidal Hom-algebra $(B \overset{(J,Q)}{\bowtie} H, \xi_B \otimes \xi_H)$ is a monoidal Hom-Hopf algebra, called a generalized double Hom-crossproduct Hopf algebra, and we describe its coquasitriangular structure. In Sections 4 and 5, we construct a new braided monoidal Hom-category $\widetilde{\mathcal{H}}({}_J^J\mathrm{Mod}_Q^Q)$ obtained from the structure of the generalized double Hom-crossproduct, and we establish a kind of new quantum Yang–Baxter Hom-operators (see Theorem 5.6 and Corollary 5.7).

1. Preliminaries. Throughout this paper, k is a fixed field. All algebras, coalgebras and Hopf algebras are defined over k unless otherwise specified. We refer the readers to the book of Sweedler [Sw] for the relevant concepts of the general theory of Hopf algebras. Let (C,Δ) be a coalgebra. We use the following notation:

$$\Delta(c)=\sum c_1\otimes c_2,\qquad \forall c\in C.$$

Let $\mathcal{M}_k=(\mathcal{M}_k,\otimes,k,a,l,r)$ denote the usual monoidal category of k-vector spaces and linear maps between them. Recall from [CG] that the *monoidal Hom-category* $\widetilde{\mathcal{H}}(\mathcal{M}_k)=(\mathcal{H}(\mathcal{M}_k),\otimes,(k,\mathrm{id}),\widetilde{a},\widetilde{l},\widetilde{r})$ is a monoidal category associated with $\mathcal{M}_k$ as follows:

- The objects of $\mathcal{H}(\mathcal{M}_k)$ are couples (M,ξ_M), where $M\in\mathcal{M}_k$ and $\xi_M\in\mathrm{Aut}_k(M)$, the set of all k-linear automorphisms of M.
- A morphism $f:(M,\xi_M)\to(N,\xi_N)$ in $\mathcal{H}(\mathcal{M}_k)$ is a k-linear map $f:M\to N$ in $\mathcal{M}_k$ satisfying $\xi_N\circ f=f\circ\xi_M$ for any two objects $(M,\xi_M),(N,\xi_N)\in\mathcal{H}(\mathcal{M}_k)$.
- The tensor product is given by

$$(M,\xi_M)\otimes(N,\xi_N)=(M\otimes N,\xi_M\otimes\xi_N)$$

for any $(M,\xi_M),(N,\xi_N)\in\mathcal{H}(\mathcal{M}_k)$.

- The tensor unit is (k,id).
- The associativity constraint $\widetilde{a}$ is given by

$$\widetilde{a}_{M,N,L}=a_{M,N,L}\circ((\xi_M\otimes\mathrm{id})\otimes\varsigma^{-1})=(\xi_M\otimes(\mathrm{id}\otimes\varsigma^{-1}))\circ a_{M,N,L}$$

for any objects $(M,\xi_M),(N,\xi_N),(L,\varsigma)\in\mathcal{H}(\mathcal{M}_k)$.

- The left and right unit constraints $\widetilde{l}$ and $\widetilde{r}$ are given by

$$\widetilde{l}_M=\xi_M\circ l_M=l_M\circ(\mathrm{id}\otimes\xi_M),\qquad \widetilde{r}_M=\xi_M\circ r_M=r_M\circ(\xi_M\otimes\mathrm{id}),$$

for all $(M,\xi_M)\in\mathcal{H}(\mathcal{M}_k)$.

We now recall the following notions used later.

Definition 1.1 ([CG]). A *unital monoidal Hom-associative algebra* is a vector space A together with an element $1_A\in A$ and linear maps

$$m:A\otimes A\to A,\ a\otimes b\mapsto ab,\qquad \xi_A\in\mathrm{Aut}_k(A)$$

such that

$$\xi_A(a)(bc)=(ab)\xi_A(c),\qquad \xi_A(ab)=\xi_A(a)\xi_A(b),$$
$$a1_A=1_Aa=\xi_A(a),\qquad \xi_A(1_A)=1_A,$$

for all $a,b,c\in A$.

Definition 1.2. Let (A,ξ_A) and $(A',\xi_{A'})$ be monoidal Hom-algebras. A *monoidal Hom-algebra map* $f:(A,\xi_A)\to(A',\xi_{A'})$ is a linear map such that $f\circ\xi_A=\xi_{A'}\circ f$, $f(ab)=f(a)f(b)$ and $f(1_A)=1_{A'}$.

DEFINITION 1.3 ([CG]). A *counital monoidal Hom-coassociative coalgebra* is an object (C, ξ_C) in the category $\tilde{\mathcal{H}}(\mathcal{M}_k)$ together with linear maps $\Delta : C \to C \otimes C$, $\Delta(c) = c_1 \otimes c_2$ and $\varepsilon : C \to k$ such that

$$\xi_C^{-1}(c_1) \otimes \Delta(c_2) = \Delta(c_1) \otimes \xi_C^{-1}(c_2), \quad \Delta(\xi_C(c)) = \xi_C(c_1) \otimes \xi_C(c_2),$$
$$c_1\varepsilon(c_2) = \xi_C^{-1}(c) = \varepsilon(c_1)c_2, \quad \varepsilon(\xi_C(c)) = \varepsilon(c),$$

for all $c \in C$.

DEFINITION 1.4. Let (C, ξ_C) and $(C', \xi_{C'})$ be monoidal Hom-coalgebras. A *monoidal Hom-coalgebra map* $f : (C, \xi_C) \to (C', \xi_{C'})$ is a linear map such that $f \circ \xi_C = \xi_{C'} \circ f$, $\Delta \circ f = (f \otimes f) \circ \Delta$ and $\varepsilon' \circ f = \varepsilon$.

DEFINITION 1.5 ([CG]). A *monoidal Hom-bialgebra* $H = (H, \xi_H, m, 1_H, \Delta, \varepsilon)$ is a bialgebra in the monoidal category $\tilde{\mathcal{H}}(\mathcal{M}_k)$. This means that $(H, \xi_H, m, 1_H)$ is a monoidal Hom-algebra and $(H, \xi_H, \Delta, \varepsilon)$ is a monoidal Hom-coalgebra such that Δ and ε are morphisms of algebras, that is, for all $h, g \in H$,

$$\Delta(hg) = \Delta(h)\Delta(g), \quad \Delta(1_H) = 1_H \otimes 1_H,$$
$$\varepsilon(hg) = \varepsilon(h)\varepsilon(g), \quad \varepsilon(1_H) = 1.$$

DEFINITION 1.6 ([CG]). A monoidal Hom-bialgebra (H, ξ_H) is called a *monoidal Hom-Hopf algebra* if there exists a morphism (called *antipode*) $S : H \to H$ in $\tilde{\mathcal{H}}(\mathcal{M}_k)$ (i.e., $S \circ \xi_H = \xi_H \circ S$), which is the convolution inverse of the identity morphism id_H (i.e., $S * \mathrm{id} = 1_H \circ \varepsilon = \mathrm{id} * S$). Explicitly, for all $h \in H$,

$$S(h_1)h_2 = \varepsilon(h)1_H = h_1S(h_2).$$

REMARK 1.7. The antipode of monoidal Hom-Hopf algebras has almost all the properties of the antipode of Hopf algebras such as

$$S(hg) = S(g)S(h), \quad S(1_H) = 1_H,$$
$$\Delta(S(h)) = S(h_2) \otimes S(h_1), \quad \varepsilon \circ S = \varepsilon,$$

for all $h, g \in H$. That is, S is a monoidal Hom-anti-(co)algebra homomorphism. Since ξ_H is bijective and commutes with S, its inverse ξ_H^{-1} also commutes with S.

Now, we recall the notions of actions on monoidal Hom-algebras and coactions on monoidal Hom-coalgebras.

DEFINITION 1.8 ([CG]). Let (A, ξ_A) be a monoidal Hom-algebra. A *left (A, ξ_A)-Hom-module* consists of an object (M, ξ_M) in $\tilde{\mathcal{H}}(\mathcal{M}_k)$ together with a morphism $\psi : A \otimes M \to M$, $\psi(a \otimes m) = a \cdot m$, such that

$$\xi_A(a)(b \cdot m) = (ab) \cdot \xi_M(m), \quad 1_A \cdot m = \xi_M(m),$$
$$\xi_M(a \cdot m) = \xi_A(a) \cdot \xi_M(m),$$

for all $a, b \in A$ and $m \in M$.

Any monoidal Hom-algebra (A, ξ_A) can be viewed as a Hom-module over itself via Hom-multiplication. Let (M, ξ_M) and (N, ξ_N) be two left (A, ξ_A)-Hom-modules. A morphism $f : M \to N$ is called *left* (A, ξ_A)*-linear* if $f(a \cdot m) = a \cdot f(m)$ and $f \circ \xi_M = \xi_N \circ f$. We denote the category of left (A, ξ_A)-Hom modules by $\tilde{\mathcal{H}}({}_A\mathcal{M}_k)$.

Definition 1.9 ([CG]). Let (C, ξ_C) be a monoidal Hom-coalgebra. Then a *right* (C, ξ_C)*-Hom-comodule* is an object (M, ξ_M) in $\tilde{\mathcal{H}}(\mathcal{M}_k)$ together with a k-linear map $\rho_M : M \to M \otimes C$, $\rho_M(m) = m_{(0)} \otimes m_{(1)}$, such that

$$\xi_M^{-1}(m_{(0)}) \otimes \Delta_C(m_{(1)}) = (m_{(0)(0)} \otimes m_{(0)(1)}) \otimes \xi_C^{-1}(m_{(1)}),$$
$$\rho_M(\xi_M(m)) = \xi_M(m_{(0)}) \otimes \xi_C(m_{(1)}), \qquad m_{(0)}\varepsilon(m_{(1)}) = \xi_M^{-1}(m),$$

for all $m \in M$.

(C, ξ_C) is a Hom-comodule over itself via Hom-comultiplication. Let (M, ξ_M) and (N, ξ_N) be two right (C, ξ_C)-Hom-comodules. A morphism $g : M \to N$ is called *right* (C, ξ_C)*-colinear* if $g \circ \xi_M = \xi_N \circ g$ and $g(m_{(0)}) \otimes m_{(1)} = g(m)_{(0)} \otimes g(m)_{(1)}$. The category of right (C, γ)-Hom-comodules is denoted by $\tilde{\mathcal{H}}(\mathcal{M}^C)$.

Remark 1.10. The reader is referred to [CZ, Section 5] for a useful interpretation of Hom-structures.

2. A generalized double Hom-crossproduct

Definition 2.1. Assume that (B, ξ_B) is a left (J, ξ_J)-Hom-module and right (Q, ξ_Q)-Hom-comodule algebra, and let (H, ξ_H) be a right (Q, ξ_Q)-Hom-module and left (J, ξ_J)-Hom-comodule algebra. Define $B \overset{(J,Q)}{\bowtie} H$ to be $B \otimes H$ as a vector space with (possibly nonassociative) multiplication

$$(a \bowtie h)(b \bowtie g) = \sum a(h_{(-1)} \to b_0) \otimes (h_0 \leftarrow b_{(1)})g \tag{2.1}$$

for all $a, b \in B$ and $h, g \in H$.

Proposition 2.2. *Let (B, ξ_B) be a left (J, ξ_J)-Hom-module and right (Q, ξ_Q)-Hom-comodule algebra, and let (H, ξ_H) be a right (Q, ξ_Q)-Hom-module and left (J, ξ_J)-Hom-comodule algebra. Then $(B \overset{(J,Q)}{\bowtie} H, \xi_B \otimes \xi_H)$ is an associative monoidal Hom-algebra with unit element $1_B \otimes 1_H$ if and only if the following three conditions hold:*

$$\text{(A1)} \quad \begin{aligned} b \otimes 1_H &= \xi_B(b_0) \otimes (1_H \leftarrow b_{(1)}), \\ 1_B \otimes h &= (h_{(-1)} \to 1_B) \otimes \xi_H(h_0), \end{aligned}$$

$$\text{(A2)} \quad \sum \xi_J(h_{(-1)}) \to (b_0 c_0) \otimes \xi_H(h_0) \leftarrow (b_{(1)} c_{(1)})$$
$$= \sum (h_{(-1)} \to b_0)\big((h_0 \leftarrow b_{(1)})_{(-1)} \to c_0\big) \otimes (\xi_H(h_0) \leftarrow \xi_Q(b_{(1)}))_0 \leftarrow \xi_Q(c_{(1)}),$$

(A3) $\sum(h_{(-1)}g_{(-1)}) \to \xi_B(c_0) \otimes (h_0 g_0) \leftarrow \xi_Q(c_{(1)})$

$= \sum \xi_J(h_{(-1)}) \to (\xi_J(g_{(-1)}) \to \xi_B(c_0))_0 \otimes \big(h_0 \leftarrow (g_{(-1)} \to c_0)_{(1)}\big)(g_0 \leftarrow c_{(1)}),$

for all $a, b \in B$ and $h, g \in H$.

Proof. It is easy to see that sufficiency is true, and it is obvious that $1_B \otimes 1_H$ is a unit element by (A1). For necessity, we calculate as follows:

$$
\begin{aligned}
&[(a \otimes h)(b \otimes g)](\xi_B(c) \otimes \xi_H(l)) \\
&= \sum \big(a(h_{(-1)} \to b_0) \otimes (h_0 \leftarrow b_{(1)})g\big)(\xi_B(c) \otimes \xi_H(l)) \\
&= \sum (a(h_{(-1)} \to b_0))\big(((h_0 \leftarrow b_{(1)})_{(-1)}g_{(-1)}) \to \xi_B(c_0)\big) \\
&\qquad\qquad \otimes \big(((h_0 \leftarrow b_{(1)})_0 g_0) \leftarrow \xi_Q(c_{(1)})\big)\xi_H(l) \\
&\overset{(A3)}{=} \sum (a(h_{(-1)} \to b_0))\big(\xi_J((h_0 \leftarrow b_{(1)})_{(-1)}) \to (\xi_J(g_{(-1)}) \to \xi_B(c_0))_0\big) \\
&\qquad\qquad \otimes \big(((h_0 \leftarrow b_{(1)})_0 \leftarrow (g_{(-1)} \to c_0)_{(1)})(g_0 \leftarrow c_{(1)})\big)\xi_H(l) \\
&= \sum \xi_B(a)\big((h_{(-1)} \to b_0)((h_0 \leftarrow b_{(1)})_{(-1)} \to (g_{(-1)} \to c_0)_0)\big) \\
&\qquad\qquad \otimes \big(\xi_H((h_0 \leftarrow b_{(1)})_0) \leftarrow \xi_Q((g_{(-1)} \to c_0)_{(1)})\big)((g_0 \leftarrow c_{(1)})l) \\
&\overset{(A2)}{=} \sum \xi_B(a)\big(\xi_J(h_{(-1)}) \to (b_0(g_{(-1)} \to c_0)_0)\big) \\
&\qquad\qquad \otimes \big(\xi_H(h_0) \leftarrow (b_{(1)}(g_{(-1)} \to c_0)_{(1)})\big)((g_0 \leftarrow c_{(1)})l) \\
&\overset{(2.1)}{=} (\xi_B(a) \otimes \xi_H(h))\big(b(g_{(-1)} \to c_0) \otimes (g_0 \leftarrow c_{(1)})l\big) \\
&\overset{(2.1)}{=} (\xi_B(a) \otimes \xi_H(h))[(b \otimes g)(c \otimes l)].
\end{aligned}
$$

This completes the proof. ■

Definition 2.3. We call $(B \overset{(J,Q)}{\bowtie} H, \xi_B \otimes \xi_H)$ a *generalized double Hom-crossproduct* if $(B \overset{(J,Q)}{\bowtie} H, \xi_B \otimes \xi_H)$ is an associative algebra.

Definition 2.4. Let $(B, \xi_B), (H, \xi_H)$ be two monoidal Hom-Hopf algebras. A *skew pairing* of (B, ξ_B) and (H, ξ_H) is a map $\tau : B \otimes H \to k$ satisfying

(SP1) $\tau(ab, h) = \sum \tau(a, h_1)\tau(b, h_2),$

(SP2) $\tau(a, hl) = \sum \tau(a_1, l)\tau(a_2, h),$

(SP3) $\tau(\xi_B(a), \xi_H(h)) = \tau(a, h),$

(SP4) $\tau(a, 1) = \varepsilon(a), \quad \tau(1, h) = \varepsilon(h),$

for all $a, b \in B$ and $h, l \in H$. In this case, we call $(B, H, \tau, \xi_B, \xi_H)$ a *τ-skew pair monoidal Hom-Hopf algebra.* If (H, ξ_H) has a bijective antipode S, then $\tau^{-1}(a, h) = \tau(a, S^{-1}(h))$, by the above.

EXAMPLE 2.5. Let τ be a skew pairing of (B,ξ_B) and (H,ξ_H). Set

$$h\to a=\sum\tau(a_2,\xi_H^{-1}(h))\xi_B^2(a_1),\qquad h\leftarrow a=\sum\tau(\xi_B^{-1}(a),h_1)\xi_H^2(h_2),$$

for any $a\in B$ and $h\in H$.

It is easy to check that (B,ξ_B) is a left (H^{op},ξ_H)-Hom-module with structure map $\to$, and (H,ξ_H) is a right (B^{cop},ξ_B)-Hom-module with structure map $\leftarrow$. Moreover, (B,ξ_B) is a right (B^{cop},ξ_B)-Hom-comodule algebra with structure map Δ_B^{cop}, and (H,ξ_H) is a left (H^{op},ξ_H)-Hom-comodule algebra with structure map $\rho_H^l : H\to H^{\mathrm{op}}\otimes H$ given by $\rho(h)=\sum S^{-1}(h_2)\otimes h_1$.

Now, we check that conditions (A1)–(A3) in Proposition 2.2 hold.

It is obvious that (A1) holds. We next prove (A2) as follows: for all $b,c\in B$ and $h\in H$,

$$\begin{aligned}
&\sum(h_{(-1)}\to b_0)\big((h_0\leftarrow b_{(1)})_{(-1)}\to c_0\big)\otimes(\xi_H(h_0)\leftarrow\xi_B(b_{(1)}))_0\leftarrow\xi_B(c_{(1)})\\
&=\sum(S^{-1}(h_2)\to b_2)\big(S^{-1}((h_1\leftarrow b_1)_2)\to c_2\big)\otimes\big(\xi_H((h_1\leftarrow b_1)_1)\leftarrow\xi_B(c_1)\big)\\
&=\sum\tau(b_{22},S^{-1}\xi_H^{-1}(h_2))\tau(c_{22},S^{-1}\xi_H(h_{122}))\xi_B^2(b_{21}c_{21})\\
&\qquad\qquad\otimes\tau(\xi_B^{-1}(b_1),h_{11})\tau(c_1,\xi_H^3(h_{1211}))\xi_H^5(h_{1212})\\
&=\sum\tau(b_{22},S^{-1}\xi_H(h_{222}))\tau(c_{22},S^{-1}\xi_H(h_{221}))\xi_B^2(b_{21}c_{21})\\
&\qquad\qquad\otimes\tau(b_1,\xi_H(h_{11}))\tau(c_1,\xi_H(h_{12}))\xi_H^3(h_{21})\\
&\overset{(SP1)}{=}\sum\tau(b_{22}c_{22},S^{-1}\xi_H(h_{22}))\xi_B^2(b_{21}c_{21})\otimes\tau(\xi_B^{-1}(b_1c_1),h_1)\xi_H^3(h_{21})\\
&=\sum\tau(b_{22}c_{22},S^{-1}(h_2))\xi_B^2(b_{21}c_{21})\otimes\tau(\xi_B^{-1}(b_1c_1),\xi_H(h_{11}))\xi_H^3(h_{12})\\
&=\sum S^{-1}\xi_H(h_2)\to(b_2c_2)\otimes\xi_H(h_1)\leftarrow(b_1c_1)\\
&=\sum\xi_H(h_{(-1)}\to(b_2c_2))\otimes\xi_H(h_0)\leftarrow(b_1c_1),
\end{aligned}$$

and (A2) is proved.

Let $\cdot^{\mathrm{op}}$ be the multiplication of H^{op}, i.e. $h\cdot^{\mathrm{op}}l=lh$. Then we have

$$\begin{aligned}
&\sum\big(\xi_H(h_{(-1)})\to(\xi_H(g_{(-1)})\to\xi_B(c_0))_0\big)\otimes\big(h_0\leftarrow(g_{(-1)}\to c_0)_{(1)}\big)(g_0\leftarrow c_{(1)})\\
&=\sum\big(S^{-1}\xi_H(h_2)\to\xi_H((S^{-1}(g_2)\to c_2)_2)\big)\otimes\big(h_1\leftarrow(S^{-1}(g_2)\to c_2)_1\big)(g_1\leftarrow c_1)\\
&=\sum\tau(c_{22},S^{-1}\xi_H^{-1}(g_2))\tau\big(\xi_B^2(c_{2122}),S^{-1}\xi_H^{-1}(h_2)\big)\xi_B^5(c_{2121})\\
&\qquad\qquad\otimes\tau(\xi_B(c_{211}),h_{11})\tau(\xi_B^{-1}(c_1),g_{11})\xi_H^2(h_{12}g_{12})\\
&=\sum\tau(\xi_B(c_{221}),S^{-1}\xi_H^{-1}(h_2))\tau\big(\xi_B(c_{222}),S^{-1}\xi_H^{-1}(g_2)\big)\\
&\qquad\qquad\xi_B^3(c_{21})\otimes\tau(c_{11},g_{11})\tau(c_{12},h_{11})\xi_H^2(h_{12}g_{12})
\end{aligned}$$

$$\overset{(SP2)}{=} \sum \tau\big(\xi_B(c_{22}), S^{-1}\xi_H^{-1}(h_2g_2)\big)\xi_B^3(c_{21}) \otimes \tau(c_1, h_{11}g_{11})\xi_H^2(h_{12}g_{12})$$
$$= S^{-1}(h_2g_2) \rightarrow \xi_B(c_2) \otimes (h_1g_1) \leftarrow \xi_B(c_1)$$
$$= (S^{-1}(h_2) \cdot^{\mathrm{op}} S^{-1}(g_2)) \rightarrow \xi_B(c_2) \otimes (h_1g_1) \leftarrow \xi_B(c_1)$$
$$= (h_{(-1)} \cdot^{\mathrm{op}} g_{(-1)}) \rightarrow \xi_B(c_0) \otimes (h_1g_1) \leftarrow \xi_B(c_{(1)}),$$

and so (A3) holds.

Thus $(B \overset{(H^{\mathrm{op}},B^{\mathrm{cop}})}{\bowtie} H, \xi_B \otimes \xi_H)$ is an associative monoidal Hom-algebra with multiplication

$$\begin{aligned}(a \otimes h)(b \otimes g) &= \sum a(h_{(-1)} \rightarrow b_0) \otimes (h_0 \leftarrow b_{(1)})g \\ &= \sum a(S^{-1}(h_2) \rightarrow b_2) \otimes (h_1 \leftarrow b_1)g \\ &= \sum \tau(b_{22}, S^{-1}\xi_H^{-1}(h_2))a\xi_B^2(b_{21}) \otimes \tau(\xi_B^{-1}(b_1), h_{11})\xi_H^2(h_{12})g \\ &= \sum \tau\big(\xi_B^{-1}(b_1), \xi_H^{-1}(h_1)\big)a\xi_B^2(b_{21}) \otimes \xi_H^2(h_{12})g\tau(b_{22}, S^{-1}(h_{22})) \\ &= \sum \tau(b_1, h_1)a\xi_B^2(b_{21}) \otimes \xi_H^2(h_{21})g\tau^{-1}(b_{22}, h_{22}),\end{aligned}$$

and this proves that

$$(B \overset{(H^{\mathrm{op}},B^{\mathrm{cop}})}{\bowtie} H, \xi_B \otimes \xi_H) = (B \bowtie_\tau H, \xi_B \otimes \xi_H).$$

EXAMPLE 2.6. Let (H, ξ_H) be a finite-dimensional monoidal Hom-Hopf algebra, and let $B = H^{*\mathrm{cop}}$. Then $D(H) = B \bowtie_\tau H$, where $D(H)$ is the Drinfeld double. Thus by Example 2.5,

$$(H^{*\mathrm{cop}} \overset{(H^{\mathrm{op}},H^*)}{\bowtie} H, \xi_H^{*-1} \otimes \xi_H) = D(H).$$

By applying Proposition 2.2, we get the following corollaries.

COROLLARY 2.7. *Let (J, ξ_J) be a monoidal Hom-bialgebra. Let (B, ξ_B) be a left (J, ξ_J)-Hom-module and right (J, ξ_J)-Hom-comodule algebra, and let (H, ξ_H) be a right (J, ξ_J)-Hom-module and left (J, ξ_J)-Hom-comodule algebra. Then $(B \overset{(J,J)}{\bowtie} H, \xi_B \otimes \xi_H)$ is an associative monoidal Hom-algebra with unit element $1_B \otimes 1_H$ if and only if conditions* (A1)–(A3) *of Proposition* 2.2 *hold.*

COROLLARY 2.8 ([ZZ]). *Let (J, ξ_J) be a monoidal Hom-bialgebra. Let (B, ξ_B) be a left (J, ξ_J)-Hom-module algebra, and let (H, ξ_H) be a left (J, ξ_J)-Hom-comodule algebra. Then $(B \overset{(J,J)}{\bowtie}_L H, \xi_B \otimes \xi_H) = (B \#_L^J H, \xi_B \otimes \xi_H)$ is an associative monoidal Hom-algebra with unit element $1_B \otimes 1_H$.*

Proof. Assume that the right coaction (J, ξ_J) on (B, ξ_B) is trivial in Corollary 2.7 and the right action (J, ξ_J) on (H, ξ_H) is trivial. Then condition (A3) naturally holds, and it follows from the left (J, ξ_J)-Hom-module algebra condition that (A2) holds. Thus $(B \overset{(J,J)}{\bowtie}_L H, \xi_B \otimes \xi_H) = (B \#_L^J H, \xi_B \otimes \xi_H)$, a generalized left Hom-smash product. ■

Corollary 2.9. *Let* (J, ξ_J) *be a monoidal Hom-bialgebra. Let* (B, ξ_B) *be a right* (J, ξ_J)*-Hom-comodule algebra, and let* (H, ξ_H) *be a right* (J, ξ_J)*-Hom-module algebra. Then* $(B \overset{(J,J)}{\bowtie}_R H, \xi_B \otimes \xi_H) = (B \#_R^J H, \xi_B \otimes \xi_H)$ *is an associative monoidal Hom-algebra with unit element* $1_B \otimes 1_H$*.*

Proof. Apply the arguments used in the proof of Corollary 2.8. ■

By Corollary 2.8 with $H = J$, we get:

Corollary 2.10 ([CWZ1]). *If* (H, ξ_H) *is a monoidal Hom-bialgebra and* (A, ξ_A) *is a left* (H, ξ_H)*-Hom-module algebra, then* $(A \#_L H, \xi_B \otimes \xi_H)$ *is an associative monoidal Hom-algebra with unit element* $1_B \otimes 1_H$*.*

By Corollary 2.9 with $H = J$, we get:

Corollary 2.11 ([CWZ1]). *If* (H, ξ_H) *is a monoidal Hom-bialgebra and* (A, ξ_A) *is a right* (H, ξ_H)*-Hom-module algebra, then* $(A \#_R H, \xi_B \otimes \xi_H)$ *is an associative monoidal Hom-algebra with unit element* $1_B \otimes 1_H$*.*

Corollary 2.12 ([LW]). *Assume that* $(B, \xi_B), (H, \xi_H)$ *are monoidal Hom-bialgebras,* (B, ξ_B) *is a left* (H, ξ_H)*-Hom-module coalgebra, and* (H, ξ_H) *is a right* (B, ξ_B)*-Hom-module coalgebra. Then* $(B \bowtie H, \xi_B \otimes \xi_H)$ *is an associative monoidal Hom-algebra with unit element* $1_B \otimes 1_H$ *if and only if the following three conditions hold:*

$$(\mathrm{A}_1) \qquad 1_H \leftarrow b = \varepsilon_B(b) 1_H, \qquad h \to 1_B = \varepsilon_H(h) 1_B,$$

$$(\mathrm{A}_2) \qquad h \to (bc) = \sum (h_1 \to \xi_B(b_1))\big((\xi_H^{-1}(h_2) \leftarrow b_2) \to c\big),$$

$$(\mathrm{A}_3) \qquad (hg) \leftarrow c = \sum \big(h \to (g_1 \to \xi_B^{-1}(c_1))\big)(\xi_H(g_2) \leftarrow c_2),$$

for all $a, b \in B$ *and* $h, g \in H$*.*

Proof. In Proposition 2.2, let $J = H$, $Q = B$. Then (B, ξ_B) has a natural (B, ξ_B)-Hom-comodule algebra structure given by Δ_B, and (H, ξ_H) has a natural (H, ξ_H)-Hom-comodule algebra structure given by Δ_H. It is not hard to see that conditions (A1)–(A3) in Proposition 2.2 are equivalent to (A_1)–(A_3) respectively. ■

3. A generalized double crossproduct of monoidal Hom-Hopf algebras. In this section we give conditions under which the generalized double Hom-crossproduct $(B \overset{(J,Q)}{\bowtie} H, \xi_B \otimes \xi_H)$ equipped with the tensor product coalgebra structure (i.e. $\Delta(b \otimes h) = \sum (b_1 \otimes h_1) \otimes (b_2 \otimes h_2)$ for all $b \in B$ and $h \in H$) is a monoidal Hom-Hopf algebra.

DEFINITION 3.1. Let (B, ξ_B) and (H, ξ_H) be monoidal Hom-Hopf algebras. The associated *generalized left-left Long module Hom-category* is denoted by $\tilde{\mathcal{H}}({}^B_H LM)$; its objects (M, ξ_M) are objects in $\tilde{\mathcal{H}}({}^B\mathrm{Mod})$ and in $\tilde{\mathcal{H}}({}_H\mathrm{Mod})$ satisfying

$$\rho(h \rightarrow m) = \sum \xi_B(m_{(-1)}) \otimes \xi_H^{-1}(h) \rightarrow m_0, \quad \forall m \in M,\ h \in H, \tag{3.1}$$

and its morphisms are left B-Hom-comodule maps and H-Hom-module maps.

Similarly, a generalized left-right Long module Hom-category is denoted by $\tilde{\mathcal{H}}({}_H LM^B)$. Explicitly, an object (M, ξ_M) is both in $\tilde{\mathcal{H}}(\mathrm{Mod}^B)$ and in $\tilde{\mathcal{H}}({}_H\mathrm{Mod})$, and satisfies the compatibility condition

$$\rho(h \rightarrow m) = \sum \xi_H^{-1}(h) \rightarrow m_0 \otimes \xi_B(m_{(1)}), \quad \forall m \in M,\ h \in H.$$

Analogously, we define a generalized right-right Long module Hom-category $\tilde{\mathcal{H}}(LM_H^B)$ and a generalized right-left Long module Hom-category $\tilde{\mathcal{H}}({}^B LM_H)$.

EXAMPLE 3.2. Let $B = H$ be a commutative and cocommutative monoidal Hom-Hopf algebra. Then $\tilde{\mathcal{H}}({}^B_H LM)$ is the Long category $\tilde{\mathcal{H}}({}^H_H L)$ (see [CWZ2]).

THEOREM 3.3. *Assume that (Q, ξ_Q), (J, ξ_J), (B, ξ_B) and (H, ξ_H) are monoidal Hom-bialgebras. Assume also (B, ξ_B) is a left (J, ξ_J)-Hom-module such that $(B, \rightarrow, \Delta_B, \xi_B)$ is an object of $\tilde{\mathcal{H}}({}^B_J LM)$ and a right (Q, ξ_Q)-Hom-comodule algebra, and (H, ξ_H) is a left (J, ξ_J)-Hom-comodule algebra and a right (Q, ξ_Q)-Hom-module such that $(H, \leftarrow, \Delta_H, \xi_H)$ is in $\tilde{\mathcal{H}}(LM_Q^H)$. Then the generalized double crossproduct $(B \overset{(J,Q)}{\bowtie} H, \xi_B \otimes \xi_H)$ equipped with the tensor product monoidal coalgebra structure is a monoidal Hom-bialgebra whenever*

$$\begin{aligned} \sum \xi_B(b_{01}) \otimes h_{01} \leftarrow \xi_Q^{-1}(b_{(1)}) \otimes \xi_J^{-1}(h_{(-1)}) \rightarrow b_{02} \otimes \xi_H(h_{02}) \\ = \sum h_{1(-1)} \rightarrow b_{10} \otimes h_{10} \leftarrow b_{1(1)} \otimes h_{2(-1)} \rightarrow b_{20} \otimes h_{20} \leftarrow b_{2(1)} \end{aligned} \tag{3.2}$$

for all $b \in B$ and $h \in H$.

Proof. Since the proof is a simple checking, we leave it to the reader. ∎

EXAMPLE 3.4. In Example 2.5, it is easy to see that $(B, \rightarrow, \Delta_B, \xi_B)$ is an object in $\tilde{\mathcal{H}}({}^B_{H^{\mathrm{op}}} LM)$ and that $(H, \leftarrow, \Delta_H, \xi_H)$ is in $\tilde{\mathcal{H}}(LM^H_{B^{\mathrm{cop}}})$. Note that $(B \overset{(H^{\mathrm{op}}, B^{\mathrm{cop}})}{\bowtie} H, \xi_B \otimes \xi_H)$ is a monoidal Hom-bialgebra. To see this, it suffices to check that (3.2) is satisfied, as follows:

$$\begin{aligned} &\sum h_{1(-1)} \rightarrow b_{10} \otimes h_{10} \leftarrow b_{1(1)} \otimes h_{2(-1)} \rightarrow b_{20} \otimes h_{20} \leftarrow b_{2(1)} \\ &\quad = \sum S^{-1}(h_{12}) \rightarrow b_{12} \otimes h_{11} \leftarrow b_{11} \otimes S^{-1}(h_{22}) \rightarrow b_{22} \otimes h_{21} \leftarrow b_{21} \end{aligned}$$

$$= \sum \tau\big(b_{122}, S^{-1}\xi_H^{-1}(h_{12})\big)\xi_B^2(b_{121}) \otimes h_{11} \leftarrow b_{11}$$
$$\otimes S^{-1}(h_{22}) \to b_{22} \otimes \tau(\xi_B^{-1}(b_{21}), h_{211})\xi_H^2(h_{212})$$
$$= \sum \tau(b_{211}, S^{-1}(h_{121}))\xi_B(b_{12}) \otimes h_{11} \leftarrow b_{11}$$
$$\otimes S^{-1}(h_{22}) \to b_{22} \otimes \tau(b_{212}, h_{122})\xi_H(h_{21})$$
$$\overset{\text{(SP2)}}{=} \sum \tau\big(b_{21}, h_{122}S^{-1}(h_{121})\big)\xi_B(b_{12}) \otimes h_{11} \leftarrow b_{11} \otimes S^{-1}(h_{22}) \to b_{22} \otimes \xi_H(h_{21})$$
$$= \sum \xi_B(b_{21}) \otimes h_{11} \leftarrow \xi_B^{-1}(b_1) \otimes S^{-1}\xi_H^{-1}(h_2) \to b_{22} \otimes \xi_H(h_{12})$$
$$= \sum \xi_B(b_{01}) \otimes h_{01} \leftarrow \xi_Q^{-1}(b_{(1)}) \otimes \xi_J^{-1}(h_{(-1)}) \to b_{02} \otimes \xi_H(h_{02})$$

for all $b \in B$ and $h \in H$, and thus (3.2) is proved. Therefore $B \overset{(H^{\mathrm{op}}, B^{\mathrm{cop}})}{\bowtie} H = B \bowtie_\tau H$ is a monoidal Hom-bialgebra.

Analogously, we prove that $H^{*\mathrm{cop}} \overset{(H^{\mathrm{op}}, H^*)}{\bowtie} H = D(H)$ is a monoidal Hom-bialgebra.

THEOREM 3.5. *Assume that (Q, ξ_Q), (J, ξ_J), (B, ξ_B) and (H, ξ_H) are monoidal Hom-bialgebras. Assume also that (B, ξ_B) is a left (J, ξ_J)-Hom-module coalgebra and a right (Q, ξ_Q)-Hom-comodule algebra. Assume that (H, ξ_H) is a left (J, ξ_J)-Hom-comodule algebra and a right (Q, ξ_Q)-Hom-module coalgebra.*

(a) *$B \overset{(J,Q)}{\bowtie} H$ equipped with the tensor product coalgebra structure is a monoidal Hom-bialgebra if and only if the following conditions hold:*

(A1) $b \otimes 1_H = \xi_B(b_0) \otimes (1_H \leftarrow b_{(1)})$,
$1_B \otimes h = (h_{(-1)} \to 1_B) \otimes \xi_H(h_0)$,

(A2) $\sum \xi_J(h_{(-1)}) \to (b_0 c_0) \otimes \xi_H(h_0) \leftarrow (b_{(1)} c_{(1)})$
$= \sum (h_{(-1)} \to b_0)\big((h_0 \leftarrow b_{(1)})_{(-1)} \to c_0\big) \otimes (\xi_H(h_0) \leftarrow \xi_Q(b_{(1)}))_0 \leftarrow \xi_Q(c_{(1)})$,

(A3) $\sum (h_{(-1)} g_{(-1)}) \to \xi_B(c_0) \otimes (h_0 g_0) \leftarrow \xi_Q(c_{(1)})$
$= \sum \xi_J(h_{(-1)}) \to (\xi_J(g_{(-1)}) \to \xi_B(c_0))_0 \otimes \big(h_0 \leftarrow (g_{(-1)} \to c_0)_{(1)}\big)(g_0 \leftarrow c_{(1)})$,

(A4) $\sum h_{(-1)} \to b_0 \otimes h_0 \leftarrow b_{(1)}$
$$= \sum \varepsilon_H(h_{10})\xi_J(h_{1(-1)}) \to b_1 \otimes h_2 \leftarrow \xi_Q(b_{2(1)})\varepsilon_B(b_{20})$$
$$= \sum \varepsilon_H(h_{20})\xi_J(h_{2(-1)}) \to b_2 \otimes h_1 \leftarrow \xi_Q(b_{1(1)})\varepsilon_B(b_{10}),$$

(A5) $\sum \varepsilon_H(h_0)\Delta_B(h_{(-1)} \to b)$
$$= \sum \varepsilon_H(h_{10}) h_{1(-1)} \to b_1 \otimes h_{2(-1)} \to b_2 \varepsilon_H(h_{20}),$$

(A6) $\sum \varepsilon_B(b_0)\Delta_H(h \leftarrow b_{(1)})$

$$= \sum \varepsilon_B(b_{10})h_1 \leftarrow b_{1(1)} \otimes h_2 \leftarrow b_{2(1)}\varepsilon_B(b_{20}).$$

(b) *If in addition* $(B, \xi_B), (H, \xi_H)$ *are monoidal Hom-Hopf algebras, then* $B \overset{(J,Q)}{\bowtie} H$ *is also a monoidal Hom-Hopf algebra with an antipode defined by*

$$S(b \bowtie h) = (1_B \bowtie S_H\xi_H^{-1}(h))(S_B\xi_B^{-1}(b) \bowtie 1_H)$$

for all $b, c \in B$ *and* $h, g \in H$.

Proof. $(\Leftarrow)$ It is clear that ε is an algebra map, and $\Delta(1_B \otimes 1_H) = 1_B \otimes 1_H$. Second, we will check that Δ is multiplicative, i.e. we need to check

$$\sum a_1(h_{(-1)1} \to b_{01}) \otimes (h_{01} \leftarrow b_{(1)1})g_1 \otimes a_2(h_{(-1)2} \to b_{02}) \otimes (h_{02} \leftarrow b_{(1)2})g_2$$
$$= \sum a_1(h_{1(-1)} \to b_{10}) \otimes (h_{10} \leftarrow b_{1(1)})g_1 \otimes a_2(h_{2(-1)} \to b_{20}) \otimes (h_{20} \leftarrow b_{(12)})g_2,$$

which is equivalent to

$$\sum (h_{(-1)1} \to b_{01}) \otimes (h_{01} \leftarrow b_{(1)1}) \otimes (h_{(-1)2} \to b_{02}) \otimes (h_{02} \leftarrow b_{(1)2})$$
$$= \sum (h_{1(-1)} \to b_{10}) \otimes (h_{10} \leftarrow b_{1(1)}) \otimes (h_{2(-1)} \to b_{20}) \otimes (h_{20} \leftarrow b_{(12)}). \quad (3.3)$$

We calculate as follows:

$$\sum (h_{1(-1)} \to b_{10}) \otimes (h_{10} \leftarrow b_{1(1)}) \otimes (h_{2(-1)} \to b_{20}) \otimes (h_{20} \leftarrow b_{(12)})$$
$$\overset{(A4)}{=} \sum \varepsilon_H(h_{120})\xi_J(h_{12(-1)}) \to b_{12} \otimes h_{11} \leftarrow \xi_Q(b_{11(1)})\varepsilon_B(b_{110})$$
$$\otimes\, \varepsilon_H(h_{210})\xi_J(h_{21(-1)}) \to b_{21} \otimes h_{22} \leftarrow \xi_Q(b_{22(1)})\varepsilon_B(b_{220})$$
$$= \sum \varepsilon_H(h_{2110})\xi_J^2(h_{211(-1)}) \to \xi_B(b_{211}) \otimes \xi_H^{-1}(h_1)$$
$$\leftarrow b_{1(1)}\varepsilon_B(b_{10}) \otimes \varepsilon_H(h_{2120})\xi_J^2(h_{212(-1)}) \to \xi_B(b_{212}) \otimes h_{22} \leftarrow \xi_Q(b_{22(1)})\varepsilon_B(b_{220})$$
$$\overset{(A5)}{=} \sum \varepsilon_H(h_{210})\xi_J^2(h_{21(-1)1}) \to \xi_B(b_{211}) \otimes \xi_H^{-1}(h_1)$$
$$\leftarrow b_{1(1)}\varepsilon_B(b_{10}) \otimes \xi_J^2(h_{21(-1)2}) \to \xi_B(b_{212}) \otimes h_{22} \leftarrow \xi_Q(b_{22(1)})\varepsilon_B(b_{220})$$
$$\overset{(\Delta\otimes id)(A4)}{=} \sum \varepsilon_H(h_{220})\xi_J^2(h_{22(-1)1}) \to \xi_B(b_{221}) \otimes \xi_H^{-1}(h_1)$$
$$\leftarrow b_{1(1)}\varepsilon_B(b_{10}) \otimes \xi_J^2(h_{22(-1)2}) \to \xi_B(b_{222}) \otimes h_{21} \leftarrow \xi_Q(b_{21(1)})\varepsilon_B(b_{210})$$
$$= \sum \varepsilon_H(h_{20})\xi_J(h_{2(-1)1}) \to b_{21} \otimes h_{11} \leftarrow \xi_Q(b_{11(1)})$$
$$\varepsilon_B(b_{110}) \otimes \xi_J(h_{2(-1)2}) \to b_{22} \otimes h_{12} \leftarrow \xi_Q(b_{12(1)})\varepsilon_B(b_{120})$$
$$\overset{(A6)}{=} \sum \varepsilon_H(h_{20})\xi_J(h_{2(-1)1}) \to b_{21} \otimes \varepsilon_B(b_{10})h_{11}$$
$$\leftarrow \xi_Q(b_{1(1)1}) \otimes \xi_J(h_{2(-1)2}) \to b_{22} \otimes h_{12} \leftarrow \xi_Q(b_{1(1)2})$$

$$\overset{(A4)}{=} \sum (h_{(-1)1} \rightarrow b_{01}) \otimes (h_{01} \leftarrow b_{(1)1}) \otimes (h_{(-1)2} \rightarrow b_{02}) \otimes (h_{02} \leftarrow b_{(1)2}),$$

and so (3.3) is satisfied.

($\Rightarrow$) Applying ($\mathrm{id}\otimes\varepsilon\otimes\mathrm{id}\otimes\varepsilon$) to both sides of (3.3), we get (A5); applying ($\varepsilon\otimes\mathrm{id}\otimes\varepsilon\otimes\mathrm{id}$), we obtain (A6); applying ($\mathrm{id}\otimes\varepsilon\otimes\varepsilon\otimes\mathrm{id}$) and ($\varepsilon\otimes\mathrm{id}\otimes\mathrm{id}\otimes\varepsilon$), we obtain (A4).

Finally, for any $b \bowtie h \in B \bowtie H$, we have

$$\begin{aligned}(I * S)(b \bowtie h) &= \sum (b_1 \bowtie h_1)\big((1_B \bowtie S_H\xi_H^{-1}(h_2))(S_B\xi_B^{-1}(b_2) \bowtie 1_H)\big)\\ &= \sum \big((\xi_B^{-1}(b_1) \bowtie \xi_H^{-1}(h_1))(1_B \bowtie S_H\xi_H^{-1}(h_2))\big)(S_B(b_2) \bowtie 1_H)\\ &\overset{(2.1)+(A1)}{=} \sum \big(b_1 \bowtie \xi_H^{-1}(h_1)S_H\xi_H^{-1}(h_2)\big)(S_B(b_2) \bowtie 1_H)\\ &\overset{(2.1)+(A1)}{=} \sum \varepsilon_H(h)(b_1S_B(b_2) \bowtie 1_H) = \sum \varepsilon_H(h)\varepsilon_B(b)(1_B \bowtie 1_H).\end{aligned}$$

A similar calculation shows that S is an antipode in $B \overset{((J,J)}{\bowtie} H, \xi_J \otimes \xi_J)$ and completes the proof. ∎

COROLLARY 3.6 ([LW]). *Let (B, ξ_B) and (H, ξ_H) be monoidal Hom-bialgebras. Let (B, ξ_B) be a left (H, ξ_H)-Hom-module coalgebra and (H, ξ_H) a right (B, ξ_B)-Hom-module coalgebra.*

(a) *$(B \bowtie H, \xi_B\otimes\xi_H)$ equipped with the tensor product coalgebra structure is a monoidal Hom-bialgebra if and only if the following four conditions hold:*

(A_1) $\quad 1_H \leftarrow b = \varepsilon_B(b)1_H, \quad h \rightarrow 1_B = \varepsilon_H(h)1_B,$

(A_2) $\quad h \rightarrow (bc) = \sum \big(h_1 \rightarrow \xi_B(b_1)\big)\big((\xi_H^{-1}(h_2) \leftarrow b_2) \rightarrow c\big),$

(A_3) $\quad (hg) \leftarrow c = \sum \big(h \rightarrow (g_1 \rightarrow \xi_B^{-1}(c_1))\big)(\xi_H(g_2) \leftarrow c_2),$

(A_4) $\quad \sum h_1 \rightarrow b_1 \otimes h_2 \leftarrow b_2 = \sum h_2 \rightarrow b_2 \otimes h_1 \leftarrow b_1,$

for all $a, b \in B$ and $h, g \in H$.

(b) *If (B, ξ_B) and (H, ξ_H) are monoidal Hom-Hopf algebras, then $(B \bowtie H, \xi_B \otimes \xi_H)$ is also a monoidal Hom-Hopf algebra with an antipode defined by*

$$S(b \bowtie h) = (1_B \bowtie S_H\xi_H^{-1}(h))(S_B\xi_B^{-1}(b) \bowtie 1_H).$$

Proof. Apply the arguments used in the proof of Theorem 3.5. ∎

COROLLARY 3.7. *Let (J, ξ_J), (B, ξ_B) and (H, ξ_H) be monoidal Hom-bialgebras. Assume that (B, ξ_B) is a left (J, ξ_J)-Hom-module algebra and (H, ξ_H) is a left (J, ξ_J)-Hom-comodule algebra.*

(a) *$(B\#_L^J H, \xi_B\otimes\xi_H)$ equipped with the tensor product coalgebra structure is a monoidal Hom-bialgebra if and only if*

$$(3.4)\qquad \sum h_{(-1)} \to b \otimes \xi_H(h_0) = \sum \varepsilon_H(h_{10})\xi_J(h_{1(-1)}) \to b \otimes \xi_H(h_2) = \sum \varepsilon_H(h_{20})\xi_J(h_{2(-1)}) \to b \otimes \xi_H(h_1).$$

(b) *If* (B, ξ_B) *and* (H, ξ_H) *are monoidal Hom-Hopf algebras, then* $(B \#_L^J H, \xi_B \otimes \xi_H)$ *is also a monoidal Hom-Hopf algebra with an antipode defined by*

$$S(b \bowtie h) = (1_B \bowtie S_H\xi_H^{-1}(h))(S_B\xi_B^{-1}(b) \bowtie 1_H).$$

Proof. By Corollary 3.6, conditions (A1)–(A3) of Theorem 3.5 hold. In this case, (A6) is also true, and (A4) becomes (3.4).

As for (A5), we have

$$\begin{aligned}
&\sum \varepsilon_H(h_{10})h_{1(-1)} \to b_1 \otimes h_{2(-1)} \to b_2\varepsilon_H(h_{20})\\
&\overset{(3.4)}{=} \sum \varepsilon_H(h_{20})h_{2(-1)} \to b_1 \otimes \varepsilon_H(h_{10})h_{1(-1)} \to b_2\\
&\overset{(3.4)}{=} \sum \xi_J^{-1}(h_{(-1)}) \to b_1 \otimes \varepsilon_H(h_{00})h_{0(-1)} \to b_2\\
&= \sum h_{(-1)1} \to b_1 \otimes \varepsilon_H(h_0)h_{(-1)2} \to b_2 = \sum \varepsilon_H(h_0)\Delta(h_{(-1)} \to b),
\end{aligned}$$

which completes the proof. ■

4. A coquasitriangular structure of $(B \overset{(J,Q)}{\bowtie} H, \xi_B \otimes \xi_H)$**.** In this section we investigate a relation between the coquasitriangular structure of $(B \overset{(J,Q)}{\bowtie} H, \xi_B \otimes \xi_H)$ and $(B, \xi_B), (H, \xi_H)$.

We recall from [YW] that a coquasitriangular monoidal Hom-Hopf algebra is a triple (A, ξ_A, σ), where (A, ξ_A) is a monoidal Hom-Hopf algebra over k and $\sigma : A \otimes A \to k$ is an invertible bilinear form on (A, ξ_A) satisfying the following four conditions: for all $x, y, z \in A$,

$$(\mathrm{BR}_1)\qquad \sigma(xy, z) = \sum \sigma(x, z_1)\sigma(y, z_2),$$

$$(\mathrm{BR}_2)\qquad \sigma(x, yz) = \sum \sigma(x_1, z)\sigma(x_2, y),$$

$$(\mathrm{BR}_3)\qquad \sum \sigma(x_1, y_1)x_2y_2 = \sum y_1x_1\sigma(x_2, y_2),$$

$$(\mathrm{BR}_4)\qquad \sigma(\xi_A(x), \xi_A(y)) = \sigma(x, y).$$

If σ^{-1} is the convolution inverse of σ, we have

$$(\mathrm{BR}'_1)\qquad \sigma^{-1}(xy, z) = \sum \sigma^{-1}(y, z_1)\sigma^{-1}(x, z_2),$$

$$(\mathrm{BR}'_2)\qquad \sigma^{-1}(x, yz) = \sum \sigma^{-1}(x_1, y)\sigma^{-1}(x_2, z),$$

$$(\mathrm{BR}'_3)\qquad \sum \sigma^{-1}(x_1, y_1)y_2x_2 = \sum x_1y_1\sigma^{-1}(x_2, y_2),$$

$$(\mathrm{BR}'_4)\qquad \sigma^{-1}(\xi_A(x), \xi_A(y)) = \sigma^{-1}(x, y).$$

Hence, $\sigma^{-1}(x,y) = \sigma(S(x),y) = \sigma(x,S^{-1}(y))$ and $\sigma(1_A,x) = \varepsilon_A(x) = \sigma(x,1_A)$. Moreover, if (A,ξ_A,σ) is coquasitriangular, then $(A^{\mathrm{op}},\xi_A,\sigma^{-1})$ and $(A^{\mathrm{cop}},\xi_A,\sigma^{-1})$ are also coquasitriangular.

LEMMA 4.1. *Let* $(B \overset{(J,Q)}{\bowtie} H,\xi_B\otimes\xi_H,\bar{\sigma})$ *be a coquasitriangular monoidal Hom-bialgebra. Then there exists a u-skew pair monoidal Hom-bialgebra* (B,H,u) *such that*

$$B \overset{(J,Q)}{\bowtie} H = B \bowtie_u H \tag{4.1}$$

with $u(b,h) = \bar{\sigma}(b\otimes 1_H, 1_B\otimes h)$.

Proof. Straightforward. ∎

THEOREM 4.2. *Let* $(B \overset{(J,Q)}{\bowtie} H,\xi_B\otimes\xi_H)$ *be a monoidal Hom-Hopf algebra. Let* (B,ξ_B,σ) *and* (H,ξ_H,τ) *be coquasitriangular monoidal Hom-Hopf algebras. Assume that* (B,H,u) *is a u-skew pair monoidal Hom-Hopf algebra and* (H,B,u) *is a u-skew pair monoidal Hom-Hopf algebra. Then* $(B \overset{(J,Q)}{\bowtie} H,\xi_B\otimes\xi_H,\bar{\sigma})$ *is a coquasitriangular monoidal Hom-Hopf algebra with*

$$\bar{\sigma}(a \overset{(J,Q)}{\bowtie} h, b \overset{(J,Q)}{\bowtie} g) = \sum u(a_1,g_1)\sigma(a_2,b_1)\tau(h_1,g_2)\upsilon(h_2,b_2),$$

if and only if the following four conditions hold:

$$\text{(a)}\qquad \sum \upsilon(h,c_1)\sigma(b,c_2) = \sum \upsilon(h_0\leftarrow b_{(1)},c_2)\sigma(h_{(-1)}\rightarrow b_0,c_1),$$

$$\text{(b)}\qquad \sum \tau(h,l_1)u(b,l_2) = \sum \tau(h_0\leftarrow b_{(1)},l_2)u(h_{(-1)}\rightarrow b_0,l_1),$$

$$\text{(c)}\qquad \sum \upsilon(h_1,c)\tau(h_2,g) = \sum \tau(h_1,g_0\leftarrow c_{(1)})\upsilon(h_2,g_{(-1)}\rightarrow c_0),$$

$$\text{(d)}\qquad \sum \sigma(a_1,c)u(a_2,g) = \sum u(a_1,g_0\leftarrow c_{(1)})\sigma(a_2,g_{(-1)}\rightarrow c_0),$$

for all $a,b,c\in B$ *and* $h,l,g\in H$.

Proof. First, we note that $\bar{\sigma}(1_B\otimes 1_H, b\otimes l) = \varepsilon_B(b)\varepsilon_H(l)$ and $\bar{\sigma}(a\otimes h, 1_B\otimes 1_H) = \varepsilon_B(a)\varepsilon_H(h)$. By the definition of $\bar{\sigma}$, (BR4) is straightforward. Next, for (BR1) we need to check that for all $a,b,c\in B$ and $h,g,l\in H$,

$$\begin{aligned}\bar{\sigma}((a \overset{(J,Q)}{\bowtie} h)(b \overset{(J,Q)}{\bowtie} g), c \overset{(J,Q)}{\bowtie} l)&\\ &= \sum \bar{\sigma}(a \overset{(J,Q)}{\bowtie} h, c_1 \overset{(J,Q)}{\bowtie} l_1)\bar{\sigma}(b \overset{(J,Q)}{\bowtie} g, c_2 \overset{(J,Q)}{\bowtie} l_2),\end{aligned}$$

that is,

$$\begin{aligned}&\sum u(a_1,l_{11})\sigma(a_2,c_{11})\tau(h_1,l_{12})\upsilon(h_2,c_{12})u(b_1,l_{21})\sigma(b_2,c_{21})\tau(g_1,l_{22})\upsilon(g_2,c_{22})\\ &\quad= \sum u(a_1,l_{11})u(h_{(-1)1}\rightarrow b_{01},l_{12})\sigma(a_2,c_{11})\sigma(h_{(-1)2}\rightarrow b_{02},c_{12})\\ &\qquad\qquad \tau(h_{01}\leftarrow b_{(1)1},l_{21})\tau(g_1,l_{22})\upsilon(h_{02}\leftarrow b_{(1)2},c_{21})\upsilon(g_2,c_{22}).\end{aligned}$$

This is equivalent to

$$(4.2)\qquad \sum u(h_{(-1)1} \rightarrow b_{01}, l_1)\sigma(h_{(-1)2} \rightarrow b_{02}, c_1)\tau(h_{01} \leftarrow b_{(1)1}, l_2) \upsilon(h_{02} \leftarrow b_{(1)2}, c_2) = \sum \tau(h_1, l_1)\upsilon(h_2, c_1)u(b_1, l_2)\sigma(b_2, c_2).$$

(a) and (b) follow from (4.2) by respectively letting $l = 1_H$ and $c = 1_B$. Conversely, we show that (4.2) holds as follows:

$$\begin{aligned}
&\sum \tau(h_1, l_1)\upsilon(h_2, c_1)u(b_1, l_2)\sigma(b_2, c_2)\\
&\overset{(a),(b)}{=} \sum \tau(h_{10} \leftarrow b_{1(1)}, l_2)u(h_{1(-1)} \rightarrow b_{10}, l_1)\upsilon(h_{20} \leftarrow b_{2(1)}, c_2) \sigma(h_{2(-1)} \rightarrow b_{20}, c_1)\\
&\overset{(A4)}{=} \sum \varepsilon_H(h_{110})u\big(\xi_J(h_{11(-1)}) \rightarrow b_{11}, l_1\big)\varepsilon_H(h_{210}) \sigma\big(\xi_J(h_{21(-1)}) \rightarrow b_{21}, c_1\big)\varepsilon_B(b_{120})\tau\big(h_{12} \leftarrow \xi_Q(b_{12(1)}), l_2\big) \varepsilon_B(b_{220})\tau\big(h_{22} \leftarrow \xi_Q(b_{22(1)}), c_2\big)\\
&\overset{(A4)}{=} \sum \varepsilon_H(h_{10})u(h_{1(-1)} \rightarrow \xi_B^{-1}(b_1), l_1)\varepsilon_H(h_{2110}) \sigma\big(\xi_J^2(h_{211(-1)}) \rightarrow \xi_B(b_{211}), c_1\big)\varepsilon_B(b_{2120})\tau(\xi_H(h_{212}) \leftarrow \xi_Q^2(b_{212(1)}), l_2)\varepsilon_B(b_{220})\tau\big(h_{22} \leftarrow \xi_Q(b_{22(1)}), c_2\big)\\
&= \sum \varepsilon_H(h_{110})u\big(\xi_H(h_{11(-1)}) \rightarrow b_{11}, l_1\big)\varepsilon_H(h_{120}) \sigma\big(\xi_J(h_{12(-1)}) \rightarrow b_{12}, c_1\big)\varepsilon_B(b_{210})\tau\big(h_{21} \leftarrow \xi_Q(b_{21(1)}), l_2\big) \varepsilon_B(b_{220})\tau\big(h_{22} \leftarrow \xi_Q(b_{22(1)}), c_2\big)\\
&\overset{(A5),(A6)}{=} \sum \varepsilon_H(h_{10})u\big(\xi_H(h_{1(-1)1}) \rightarrow b_{11}, l_1\big)\sigma\big(\xi_J(h_{1(-1)2}) \rightarrow b_{12}, c_1\big) \varepsilon_B(b_{20})\tau\big(h_{21} \leftarrow \xi_Q(b_{2(1)1}), l_2\big)\tau\big(h_{22} \leftarrow \xi_Q(b_{2(1)2}), c_2\big)\\
&= \sum \varepsilon_H(h_{10})u\big((\xi_H(h_{1(-1)}) \rightarrow b_1)_1, l_1\big)\sigma\big((\xi_J(h_{1(-1)}) \rightarrow b_1)_2, c_1\big) \varepsilon_B(b_{20})\tau\big((h_2 \leftarrow \xi_Q(b_{2(1)}))_1, l_2\big)\tau\big((h_2 \leftarrow \xi_Q(b_{2(1)}))_2, c_2\big)\\
&\overset{(A4)}{=} \sum u(h_{(-1)1} \rightarrow b_{01}, l_1)\sigma(h_{(-1)2} \rightarrow b_{02}, c_1) \tau(h_{01} \leftarrow b_{(1)1}), l_2)\tau(h_{02} \leftarrow b_{(1)2}, c_2).
\end{aligned}$$

Thus we have proved that (BR1) holds.

To prove (BR2), we have to show that

$$\bar{\sigma}(a \overset{(J,Q)}{\bowtie} h, (b \overset{(J,Q)}{\bowtie} g)(c \overset{(J,Q)}{\bowtie} l)) = \sum \bar{\sigma}(a_1 \overset{(J,Q)}{\bowtie} h_1, c \overset{(J,Q)}{\bowtie} l)\bar{\sigma}(a_1 \overset{(J,Q)}{\bowtie} h_1, b \overset{(J,Q)}{\bowtie} g).$$

By the definition of $\bar{\sigma}$, we have

$$\sum u(a_{11}, l_1)u(a_{12}, g_{01} \leftarrow c_{(1)1})\sigma(a_{21}, g_{(-1)1} \rightarrow c_{01})\sigma(a_{22}, b_1)$$
$$\tau(h_{11}, l_2)\tau(h_{12}, g_{02} \leftarrow c_{(1)2})\upsilon(h_{21}, g_{(-1)2} \rightarrow c_{02})\upsilon(h_{22}, b_2)$$
$$= \sum u(a_{11}, l_1)\sigma(a_{12}, c_1)\tau(h_{11}, l_2)\upsilon(h_{12}, c_2)u(a_{21}, g_1)\sigma(a_{22}, b_1)$$
$$\tau(h_{21}, g_2)\upsilon(h_{22}, b_2).$$

This is equivalent to

$$(4.3) \qquad \sum u(a_1, g_{01} \leftarrow c_{(1)1})\sigma(a_2, g_{(-1)1} \rightarrow c_{01})$$
$$\tau(h_1, g_{02} \leftarrow c_{(1)2})\upsilon(h_2, g_{(-1)2} \rightarrow c_{02})$$
$$= \sum \sigma(a_1, c_1)\upsilon(h_1, c_2)u(a_2, g_1)\tau(h_2, g_2).$$

Letting $a = 1_B$ we obtain (c), and letting $h = 1_H$ we get (d). Conversely, we check that (4.3) holds as follows:

$$\sum \sigma(a_1, c_1)\upsilon(h_1, c_2)u(a_2, g_1)\tau(h_2, g_2)$$
$$\overset{(c),(d)}{=} \sum u(a_1, g_{10} \leftarrow c_{1(1)})\sigma(a_2, g_{1(-1)} \rightarrow c_{10})\tau(h_1, g_{20} \leftarrow c_{2(1)})$$
$$\upsilon(h_2, g_{2(-1)} \rightarrow c_{20})$$
$$\overset{(A4)}{=} \sum \varepsilon_B(c_{120})u\big(a_1, g_{12} \leftarrow \xi_Q(c_{12(1)})\big)\sigma(a_2, \xi_J(g_{11(-1)}) \rightarrow c_{11})$$
$$\varepsilon_H(g_{110})\varepsilon_B(c_{220})\tau\big(h_1, g_{22} \leftarrow \xi_Q(c_{22(1)})\big)$$
$$\upsilon\big(h_2, \xi_J(g_{21(-1)}) \rightarrow c_{21}\big)\varepsilon_H(g_{210})$$
$$\overset{(A4)}{=} \sum \varepsilon_B(c_{2120})u\big(a_1, \xi_H(g_{212}) \leftarrow \xi_Q^2(c_{212(1)})\big)\sigma(a_2, g_{1(-1)} \rightarrow \xi_B^{-1}(c_1))$$
$$\varepsilon_H(g_{10})\varepsilon_B(c_{220})\tau\big(h_1, g_{22} \leftarrow \xi_Q(c_{22(1)})\big)$$
$$\upsilon\big(h_2, \xi_J^2(g_{211(-1)}) \rightarrow \xi_B(c_{211})\big)\varepsilon_H(g_{2110})$$
$$= \sum \varepsilon_B(c_{210})u\big(a_1, g_{21} \leftarrow \xi_Q(c_{21(1)})\big)\sigma(a_2, \xi_J(g_{11(-1)}) \rightarrow c_{11})$$
$$\varepsilon_H(g_{110})\varepsilon_B(c_{220})\tau\big(h_1, g_{22} \leftarrow \xi_Q(c_{22(1)})\big)\upsilon(h_2, \xi_J(g_{12(-1)}) \rightarrow c_{12}))\varepsilon_H(g_{120})$$
$$\overset{(A5),(A6)}{=} \sum \varepsilon_B(c_{20})u\big(a_1, (g_2 \leftarrow \xi_Q(c_{2(1)}))_1\big)\sigma(a_2, (\xi_J(g_{1(-1)}) \rightarrow c_1)_1)$$
$$\varepsilon_H(g_{10})\tau\big(h_1, (g_2 \leftarrow \xi_Q(c_{2(1)}))_2\big)\upsilon(h_2, (\xi_J(g_{1(-1)}) \rightarrow c_1)_2)$$
$$\overset{(A4)}{=} \sum u(a_1, g_{21} \leftarrow c_{(1)1})\sigma(a_2, g_{(-1)1} \rightarrow c_{01})\tau(h_1, g_{22} \leftarrow c_{(1)2})$$
$$\upsilon(h_2, g_{(-1)2} \rightarrow c_{02}).$$

Thus we have shown that (BR2) holds.

Finally, we check (BR3):

$$\sum \bar{\sigma}(a_1 \overset{(J,Q)}{\bowtie} h_1, b_1 \overset{(J,Q)}{\bowtie} g_1)\big((a_2 \overset{(J,Q)}{\bowtie} h_2)(b_2 \overset{(J,Q)}{\bowtie} g_2)\big)$$

$$= \sum u(a_{11}, g_{11})\sigma(a_{12}, \xi_B^{-1}(b_1))\tau(\xi_H^{-1}(h_1), g_{12})\upsilon(h_{21}, b_{21})(a_2 \overset{(J,Q)}{\bowtie} 1_H)$$
$$\big(((1_B \overset{(J,Q)}{\bowtie} \xi_H^{-1}(h_{22}))(\xi_B^{-1}(b_{22}) \overset{(J,Q)}{\bowtie} 1_H))(1_B \overset{(J,Q)}{\bowtie} \xi_H^{-1}(g_2))\big)$$

$$\overset{(4.1)}{=} \sum u(a_{11}, g_{11})\sigma(a_{12}, \xi_B^{-1}(b_1))\tau(\xi_H^{-1}(h_1), g_{12})\upsilon(h_{22}, b_{22})$$
$$\big((\xi_B^{-1}(a_2) \overset{(J,Q)}{\bowtie} 1_H)(b_{21} \overset{(J,Q)}{\bowtie} 1_H)\big)((1_B \overset{(J,Q)}{\bowtie} h_{21})(1_B \overset{(J,Q)}{\bowtie} \xi_H^{-1}(g_2)))$$

$$\overset{(BR4)}{=} \sum u(a_1, g_1)\sigma(a_{21}, b_{11})\tau(h_{11}, g_{21})\upsilon(h_{22}, \xi_B^{-1}(b_2))$$
$$(\xi_B(a_{22}b_{12}) \overset{(J,Q)}{\bowtie} \xi_H(h_{12}g_{22}))$$

$$\overset{(BR3)}{=} \sum u(a_1, g_1)\sigma(a_{22}, b_{12})\tau(h_{12}, g_{22})\upsilon(h_{22}, \xi_B^{-1}(b_2))$$
$$(\xi_B(b_{11}a_{21}) \overset{(J,Q)}{\bowtie} \xi_H(g_{21}h_{11}))$$

$$\overset{(BR4)}{=} \sum u(\xi_B^{-1}(a_{11}), \xi_H^{-1}(g_{11}))\sigma(\xi_B^{-1}(a_2), b_{12})\tau(h_{12}, \xi_H^{-1}(g_2))\upsilon(h_{22}, \xi_B^{-1}(b_2))$$
$$\big((b_{11} \overset{(J,Q)}{\bowtie} 1_H)((\xi_B^{-1}(a_{12}) \overset{(J,Q)}{\bowtie} 1_H)(1_B \overset{(J,Q)}{\bowtie} \xi_H^{-1}(g_{12})))\big)(1_H \overset{(J,Q)}{\bowtie} \xi_H(h_{11}))$$

$$\overset{(4.1)}{=} \sum u(\xi_B^{-1}(a_{12}), \xi_H^{-1}(g_{12}))\sigma(\xi_B^{-1}(a_2), b_{12})\tau(h_{12}, \xi_H^{-1}(g_2))$$
$$\upsilon(h_{22}, \xi_B^{-1}(b_2))\big((b_{11} \overset{(J,Q)}{\bowtie} 1_H)((1_B \overset{(J,Q)}{\bowtie} \xi_H^{-1}(g_{11}))(\xi_B^{-1}(a_{11}) \overset{(J,Q)}{\bowtie} 1_H))\big)$$
$$(1_H \overset{(J,Q)}{\bowtie} \xi_H(h_{11}))$$

$$\overset{(BR4)}{=} \sum u(a_{21}, g_{21})\sigma(a_{22}, b_{21})\tau(h_{12}, g_{22})\upsilon(h_{22}, b_{22})(b_1 \overset{(J,Q)}{\bowtie} g_1)(a_1 \overset{(J,Q)}{\bowtie} h_1)$$
$$= \sum \bar{\sigma}(a_2 \overset{(J,Q)}{\bowtie} h_2, b_2 \overset{(J,Q)}{\bowtie} g_2)\big((b_1 \overset{(J,Q)}{\bowtie} g_1)(a_1 \overset{(J,Q)}{\bowtie} h_1)\big).$$

This completes the proof of Theorem 4.2. ∎

5. The braided monoidal Hom-category $\widetilde{\mathcal{H}}({}_J^J\mathrm{Mod}_Q^Q)$. The aim of this section is to construct a new braided monoidal Hom-category $\widetilde{\mathcal{H}}({}_J^J\mathrm{Mod}_Q^Q)$, obtained from the given braided monoidal Yetter–Drinfeld Hom-category $\widetilde{\mathcal{H}}({}_J^J\mathcal{YD})$, $\widetilde{\mathcal{H}}(\mathcal{YD}_Q^Q)$ and the structure of the generalized double Hom-cross-product, and establish a kind of new quantum Yang–Baxter Hom-operators. Furthermore, when (J, ξ_J) and (Q, ξ_Q) are arbitrary two monoidal Hom-Hopf algebras with bijective antipodes, we show that both $\widetilde{\mathcal{H}}({}_J^J\mathcal{YD})$ and $\widetilde{\mathcal{H}}(\mathcal{YD}_Q^Q)$ are subbraided monoidal Hom-categories of $\widetilde{\mathcal{H}}({}_J^J\mathrm{Mod}_Q^Q)$.

In this section we always suppose that (J,ξ_J) and (Q,ξ_Q) are monoidal Hom-Hopf algebras. If (M,ξ_M,ρ^J) is a left J-Hom-comodule and (M,ξ_M,ρ^Q) is a right Q-Hom-comodule, we formally define $\rho^J(m)=\sum m_{(-1)}\otimes m_0$ and $\rho^Q(m)=\sum m^0\otimes m^{(1)}$ for all $m\in M$. If $(M,\xi_M,\cdot_J)$ is a left J-Hom-module and $(M,\xi_M,\cdot_Q)$ is a right Q-Hom-module, we formally represent $\cdot_J: J\otimes M\to M$ by $\cdot_J(l\otimes m)=l\cdot_J m$ and $\cdot_Q: M\otimes Q\to M$ by $\cdot_Q(m\otimes p)=m\cdot_Q q$, for all $l\in J$, $q\in Q$ and $m\in M$.

A *left-left Yetter–Drinfeld Hom-category* $\widetilde{\mathcal{H}}({}^J_J\mathcal{YD})$ is the category of objects $(M,\xi_M,\rho^J,\cdot_J)$ such that $(M,\xi_M,\rho^J)\in\tilde{\mathcal{H}}({}^J\mathrm{Mod})$ and $(M,\xi_M,\cdot_J)\in\tilde{\mathcal{H}}({}_J\mathrm{Mod})$, and the compatibility condition

$$\rho^J(l\cdot_J m)=\sum\big(l_{11}\xi_J^{-1}(m_{(-1)})\big)S(l_2)\otimes\xi_H(l_{12})\cdot_J m_0 \tag{5.1}$$

is satisfied for all $l\in J$ and $m\in M$.

Suppose that (M,ξ_M) and (N,ξ_N) are objects of $\widetilde{\mathcal{H}}({}^J_J\mathcal{YD})$. Then their tensor product is an object of $\widetilde{\mathcal{H}}({}^J_J\mathcal{YD})$ with the tensor product Hom-module and Hom-comodule structures given by $l\cdot_J(m\otimes n)=\sum l_1\cdot_J m\otimes l_2\cdot_J n$ and $\rho^J(m\otimes n)=\sum m_{(-1)}n_{(-1)}\otimes(m_0\otimes n_0)$, for all $m,n\in M$ and $l\in J$.

First, we note that the "prebraided" map $\tau_{M,N}: M\otimes N\to N\otimes M$ defined by

$$\tau_{M,N}(m\otimes n)=\sum m_{(-1)}\cdot_J\xi_N^{-1}(n)\otimes\xi_M(m_0)$$

for all $m\in M$ and $n\in N$ is a morphism of $\widetilde{\mathcal{H}}({}^J_J\mathcal{YD})$. The category $\widetilde{\mathcal{H}}({}^J_J\mathcal{YD})$ with the morphisms $\tau_{M,N}$ is a prebraided monoidal Hom-category (see [LSh]); if S_J is bijective then $\tau_{M,N}$ is an isomorphism and then $\widetilde{\mathcal{H}}({}^J_J\mathcal{YD})$ is a braided monoidal Hom-category.

Similarly, a *right-right Yetter–Drinfeld Hom-category* $\widetilde{\mathcal{H}}(\mathcal{YD}^Q_Q)$ is the category of objects $(M,\xi_M,\rho^Q,\cdot_Q)$ such that $(M,\xi_M,\rho^Q)\in\tilde{\mathcal{H}}(\mathrm{Mod}^Q)$ and $(M,\xi_M,\cdot_Q)\in\tilde{\mathcal{H}}(\mathrm{Mod}_Q)$, and the compatibility condition

$$\rho^Q(m\cdot_Q q)=\sum m_0\cdot_Q\xi_Q(q_{12})\otimes\big(S(q_{11})\xi_Q^{-1}(m_{(1)})\big)q_2 \tag{5.2}$$

is satisfied for all $q\in Q$ and $m\in M$.

Suppose that (M,ξ_M) and (N,ξ_N) are objects of $\widetilde{\mathcal{H}}(\mathcal{YD}^Q_Q)$. Then their tensor product is an object of $\widetilde{\mathcal{H}}(\mathcal{YD}^Q_Q)$ with the tensor product Hom-module and Hom-comodule structures given by $(m\otimes n)\cdot_Q q=\sum m\cdot_Q q_1\otimes n\cdot_Q q_2$ and $\rho^Q(m\otimes n)=\sum(m^0\otimes n^0)\otimes m^{(1)}n^{(1)}$ for all $m,n\in M$ and $q\in Q$.

Next, we note that the "prebraided" map $\tilde{\tau}_{M,N}: M\otimes N\to N\otimes M$ defined by

$$\tilde{\tau}_{M,N}(m\otimes n)=\sum\xi_N(n^0)\otimes\xi_M^{-1}(m)\cdot_J n^{(1)}$$

for all $m\in M$ and $n\in N$ is a morphism of $\widetilde{\mathcal{H}}(\mathcal{YD}^Q_Q)$. The category $\widetilde{\mathcal{H}}(\mathcal{YD}^Q_Q)$ with the morphisms $\tau_{M,N}$ is a prebraided monoidal Hom-category (see [LSh]); if S_Q is bijective then $\tilde{\tau}_{M,N}$ is an isomorphism and thus $\widetilde{\mathcal{H}}(\mathcal{YD}_Q^Q)$ is a braided monoidal Hom-category.

Definition 5.1. Let (M, ξ_M) be an object of $\tilde{\mathcal{H}}({}_J\mathrm{Mod})$ and of $\tilde{\mathcal{H}}(\mathrm{Mod}_Q)$. We call (M, ξ_M) a *J-Q-Hom-bimodule* if $(M, \xi_M, \cdot_Q) \in \tilde{\mathcal{H}}(\mathrm{Mod}_Q)$ and the compatibility condition

$$(l \cdot_J m) \cdot_Q \xi_Q(q) = \xi_J(l) \cdot_J (m \cdot_Q q) \tag{5.3}$$

is satisfied for all $l \in J$, $m \in M$ and $q \in Q$.

Definition 5.2. Let (M, ξ_M) be an object of $\tilde{\mathcal{H}}({}_J\mathrm{Mod})$ and of $\tilde{\mathcal{H}}(\mathrm{Mod}^Q)$. We say (M, ξ_M) is a *J-Q-Hom-bicomodule* if

$$\sum \xi_J^{-1}(m_{(-1)}) \otimes (m_0^0 \otimes m_0^{(1)}) = \sum (m^0_{(-1)} \otimes m^0_0) \otimes \xi_Q^{-1}(m^{(1)}) \tag{5.4}$$

for all $m \in M$.

Definition 5.3. Let $(M, \xi_M, \cdot_J, \cdot_Q)$ be a J-Q-Hom-bimodule, and $(M, \xi_M, \rho^J, \rho^Q)$ be a J-Q-Hom-bicomodule. We call $(M, \xi_M, \rho^J, \cdot_J, \rho^Q, \cdot_Q)$ a *generalized Yetter–Drinfeld–Long Hom-module* over (J, ξ_J) and (Q, ξ_Q) if $(M, \xi_M, \rho^J, \cdot_J)$ is an object of $\widetilde{\mathcal{H}}({}_J^J\mathcal{YD})$, $(M, \xi_M, \rho^Q, \cdot_Q)$ an object of $\widetilde{\mathcal{H}}(\mathcal{YD}_Q^Q)$, $(M, \xi_M, \rho^J, \cdot_Q)$ an object of $\tilde{\mathcal{H}}({}^JLM_Q)$ and $(M, \xi_M, \rho^Q, \cdot_J)$ an object of $\tilde{\mathcal{H}}({}_JLM^Q)$.

Remark 5.4. (a) $(M, \xi_M, \rho^J, \cdot_Q)$ is an object of $\tilde{\mathcal{H}}({}^JLM_Q)$, i.e.

$$\rho^J(m \cdot_Q q) = \sum \xi_J(m_{(-1)}) \otimes m_0 \cdot_Q \xi_Q^{-1}(q), \quad \forall q \in Q,\, m \in M. \tag{5.5}$$

(b) $(M, \xi_M, \rho^Q, \cdot_J)$ is an object of $\tilde{\mathcal{H}}({}_JLM^Q)$, i.e.

$$\rho^Q(l \cdot_J m) = \sum \xi_J^{(-1)}(l) \cdot_J m^0 \otimes \xi_Q(m^{(1)}), \quad \forall l \in J,\, m \in M. \tag{5.6}$$

For any monoidal Hom-Hopf algebras (J, ξ_J) and (Q, ξ_Q), $\widetilde{\mathcal{H}}({}_J^J\mathrm{Mod}_Q^Q)$ is the category of generalized Yetter–Drinfeld–Long Hom-modules. The morphisms in $\widetilde{\mathcal{H}}({}_J^J\mathrm{Mod}_Q^Q)$ are left J-Hom-module, left J-Hom-comodule, right Q-Hom-module and right Q-Hom-comodule maps.

Now we are able to prove the following useful result.

Proposition 5.5. (a) *The tensor product of two generalized Yetter–Drinfeld–Long Hom-modules (M, ξ_M) and (N, ξ_N) is again a generalized Yetter–Drinfeld–Long Hom-module.*

(b) *The category $\widetilde{\mathcal{H}}({}_J^J\mathrm{Mod}_Q^Q)$ is a monoidal Hom-category.*

Proof. We have to show that $(M \otimes N, \xi_M \otimes \xi_N)$ is a generalized Yetter–Drinfeld–Long Hom-module. For all $l \in J$, $q \in Q$, $m \in M$ and $n \in N$, we have

$$\begin{aligned}\rho^Q(l\cdot_J(m\otimes n)) &= \sum \rho^Q(l_1\cdot_J m\otimes l_2\cdot_J n)\\ &= \sum\big((l_1\cdot_J m)^0\otimes(l_2\cdot_J n)^0\big)\otimes(l_1\cdot_J m)^{(1)}(l_2\cdot_J n)^{(1)}\\ &\overset{(5.6)}{=} \sum\big(\xi_J^{-1}(l_1)\cdot_J m^0\otimes\xi_J^{-1}(l_2)\cdot_J n^0\big)\otimes\xi_Q(m^{(1)})\xi_Q(n^{(1)})\\ &= \sum\xi_J^{-1}(l)\cdot_J(m^0\otimes n^0)\otimes\xi_Q(m^{(1)}n^{(1)}),\end{aligned}$$

and (5.6) is proven. Also

$$\begin{aligned}\rho^J((m\otimes n)\cdot_Q q) &= \sum\rho^J(m\cdot_Q q_1\otimes n\cdot_Q q_2)\\ &= \sum(m\cdot_Q q_1)_{(-1)}(n\cdot_Q q_2)_{(-1)}\otimes\big((m\cdot_Q q_1)_0\otimes(n\cdot_Q q_2)_0\big)\\ &\overset{(5.5)}{=}\sum\xi_J(m_{(-1)})\xi_J(n_{(-1)})\otimes\big(m_0\cdot_Q\xi_Q^{-1}(q_1)\otimes n_0\cdot_Q\xi_Q^{-1}(q_2)\big)\\ &= \sum\xi_J(m_{(-1)}n_{(-1)})\otimes(m\otimes n)_0\cdot_Q\xi_Q^{-1}(q),\end{aligned}$$

as needed. It is straightforward to check that $(M\otimes N,\xi_M\otimes\xi_N)$ is both a J-Q-Hom-bimodule and a J-Q-Hom-bicomodule. Thus $(M\otimes N,\xi_M\otimes\xi_N)$ is a generalized Yetter–Drinfeld–Long Hom-module.

It is now clear that the tensor product defines a functor $\widetilde{\mathcal{H}}({}_J^J\mathrm{Mod}_Q^Q)\times\widetilde{\mathcal{H}}({}_J^J\mathrm{Mod}_Q^Q)\to\widetilde{\mathcal{H}}({}_J^J\mathrm{Mod}_Q^Q)$. It is obvious that the trivial J-Hom-action, Q-Hom-action, J-Hom-coaction and Q-Hom-coaction given by

$$1_M\cdot_Q q=\varepsilon(q)1_M,\qquad l\cdot_J 1_M=\varepsilon(l)1_M,$$
$$\rho^Q(1_M)=1_M\otimes 1_Q,\qquad \rho^J(1_M)=1_J\otimes 1_M$$

make k into an object of $\widetilde{\mathcal{H}}({}_J^J\mathrm{Mod}_Q^Q)$. It is clear that k is a unit object of $\widetilde{\mathcal{H}}({}_J^J\mathrm{Mod}_Q^Q)$. Let (U,ξ_U), (V,ξ_V) and (W,ξ_W) be generalized Yetter–Drinfeld–Long Hom-modules. The isomorphisms

$$\begin{aligned}&\tilde a_{U,V,W}:(U\otimes V)\otimes W\to U\otimes(V\otimes W),\\ &\qquad\qquad (u\otimes v)\otimes W\mapsto\xi_U(u)\otimes(v\otimes\xi_W(w)),\\ &\tilde r_U:U\otimes k\to U,\quad u\otimes 1\mapsto\xi_U(u),\\ &\tilde l_U:k\otimes U\to U,\quad 1\otimes u\mapsto\xi_U(u),\end{aligned}$$

obviously satisfy the pentagon axiom and the triangle axiom [CG]. Hence $\widetilde{\mathcal{H}}({}_J^J\mathrm{Mod}_Q^Q)$ is a monoidal Hom-category. ■

In what follows, let $(M,\xi_M),(N,\xi_N)\in\widetilde{\mathcal{H}}({}_J^J\mathrm{Mod}_Q^Q)$. Set

$$\Psi_{M,N}(m\otimes n)=\sum m_{(-1)}\cdot_J n^0\otimes m_0\cdot_Q n^{(1)}\tag{5.7}$$

for all $m\in M$ and $n\in N$.

The following theorem is one of the main results of this section.

THEOREM 5.6. *Assume that $\Psi_{M,N}$ is as in* (5.7).

(a) $\Psi_{M,N} : M \otimes N \to N \otimes M$ *is a morphism in* $\widetilde{\mathcal{H}}({}_J^J\mathrm{Mod}_Q^Q)$.

(b) *For any objects* $(M, \xi_M), (N, \xi_N)$ *and* (P, ξ_P) *in* $\widetilde{\mathcal{H}}({}_J^J\mathrm{Mod}_Q^Q)$,

$$\tilde{a}_{N,P,M} \circ \Psi_{M,N\otimes P} \circ \tilde{a}_{M,N,P} = (\mathrm{id}_N \otimes \Psi_{M,P}) \circ \tilde{a}_{N,M,P} \circ (\Psi_{M,N} \otimes \mathrm{id}_P),$$
$$\tilde{a}^{-1}_{P,M,N} \circ \Psi_{M\otimes N,P} \circ \tilde{a}^{-1}_{M,N,P} = (\Psi_{M,P} \otimes \mathrm{id}_N) \circ \tilde{a}^{-1}_{M,P,N} \circ (\mathrm{id}_M \otimes \Psi_{N,P}).$$

(c) *If the antipodes S_J and S_Q are bijective then $\Psi_{M,N}$ is an isomorphism. Therefore, the family of maps*

$$\Psi_{M,N} : M \otimes N \to N \otimes M, \qquad m \otimes n \mapsto \sum m_{(-1)} \cdot_J n^0 \otimes m_0 \cdot_Q n^{(1)},$$

defines a braiding on $\widetilde{\mathcal{H}}({}_J^J\mathrm{Mod}_Q^Q)$, *the category of generalized Yetter–Drinfeld–Long Hom-modules.*

Proof. (a) It is sufficient to show that $\Psi_{M,N}$ are left J-Hom-module, left J-Hom-comodule, right Q-Hom-module and right Q-Hom-comodule morphisms. For all $l \in J$, $m \in M$ and $n \in N$, we calculate:

$$\begin{aligned}
\Psi_{M,N}(l \cdot_J (m \otimes n)) &= \sum (l_1 \cdot_J m)_{(-1)} \cdot_J (l_2 \cdot_J n)^0 \otimes (l_1 \cdot_J m)_0 \cdot_Q (l_2 \cdot_J n)^{(1)} \\
&\overset{(5.6)}{=} \sum (l_1 \cdot_J m)_{(-1)} \cdot_J (\xi_J^{-1}(l_2) \cdot_J n^0) \otimes (l_1 \cdot_J m)_0 \cdot_Q \xi_Q(n^{(1)}) \\
&\overset{(5.1)}{=} \sum \xi_J^{-1}\big(((l_{111}\xi_J^{-1}(m_{(-1)}))S(l_{12}))l_2\big) \cdot_J \xi_N(n^0) \\
&\qquad\qquad \otimes (\xi_J(l_{112}) \cdot_J m_0) \cdot_Q \xi_Q(n^{(1)}) \\
&= \sum \xi_J^{-1}((l_{11}m_{(-1)})(S(l_{21})l_{22})) \cdot_J \xi_N(n^0) \otimes (l_{12} \cdot_J m_0) \cdot_Q \xi_Q(n^{(1)}) \\
&= \sum (\xi_J^{-1}(l_1)m_{(-1)}) \cdot_J \xi_N(n^0) \otimes (\xi_J^{-1}(l_2) \cdot_J m_0) \cdot_Q \xi_Q(n^{(1)}) \\
&\overset{(5.3)}{=} \sum l_1 \cdot_J (m_{(-1)} \cdot_J n^0) \otimes l_2 \cdot_J (m_0 \cdot_Q n^{(1)}) = l \cdot_J \Psi_{M,N}(m \otimes n),
\end{aligned}$$

and so $\Psi_{M,N}$ is a left J-Hom-module map.

Similarly, by using (5.2), (5.3) and (5.5) one can show that $\Psi_{M,N}$ is a right Q-Hom-module morphism. Also

$$\begin{aligned}
&\rho^Q \circ \Psi_{M,N}(m \otimes n) \\
&= \sum \big((m_{(-1)} \cdot_J n^0)^0 \otimes (m_0 \cdot_Q n^{(1)})^0\big) \otimes (m_{(-1)} \cdot_J n^0)^{(1)}(m_0 \cdot_Q n^{(1)})^{(1)} \\
&\overset{(5.6)}{=} \sum \big((\xi_J^{-1}(m_{(-1)}) \cdot_J n^{00}) \otimes (m_0 \cdot_Q n^{(1)})^0\big) \otimes \xi_Q(n^{0(1)})(m_0 \cdot_Q n^{(1)})^{(1)} \\
&\overset{(5.2)}{=} \sum \big((\xi_J^{-1}(m_{(-1)}) \cdot_J n^{00}) \otimes m_0^0 \cdot_Q \xi_Q(n^{(1)}_{12})\big) \otimes \xi_Q(n^{0(1)}) \\
&\qquad\qquad \big((S(n^{(1)}_{11})\xi_Q^{-1}(m_0^{(1)}))n^{(1)}_2\big)
\end{aligned}$$

$$= \sum \big((\xi_J^{-1}(m_{(-1)}) \cdot_J \xi_N^{-1}(n^0)) \otimes m_0^0 \cdot_Q \xi_Q(n_{21}^{(1)}) \big) \otimes (\xi_Q(n_{11}^{(1)}) S\xi_Q(n_{12}^{(1)}))(m_0^{(1)} \xi_Q(n_{22}^{(1)}))$$

$$= \sum \big(\xi_J^{-1}(m_{(-1)}) \cdot_J n^{00} \otimes m_0^0 \cdot_Q n^{0(1)} \big) \otimes \xi_Q(m_0^{(1)}) n^{(1)}$$

$$\overset{(5.4)}{=} \sum (m_{(-1)}^0 \cdot_J n^{00} \otimes m_0^0 \cdot_Q n^{0(1)}) \otimes m^{(1)} n^{(1)} = (\Psi_{M,N} \otimes \mathrm{id}_Q) \circ \rho^Q(m \otimes n),$$

and this proves that $\Psi_{M,N}$ is a right Q-Hom-comodule map.

A similar check shows that $\Psi_{M,N}$ is a left J-Hom-comodule morphism. Hence $\Psi_{M,N}$ is a morphism in $\widetilde{\mathcal{H}}({}_J^J\mathrm{Mod}_Q^Q)$.

(b) For any $m \in M$ and $n \in N$, and $p \in P$, one obtains

$$\begin{aligned}
&\tilde{a}_{N,P,M} \circ \Psi_{M,N\otimes P} \circ \tilde{a}_{M,N,P}((m \otimes n) \otimes p) \\
&\quad = \tilde{a}_{N,P,M}\big(\xi_M(m)_{(-1)} \cdot_J (n^0 \otimes \xi_P^{-1}(p)^0) \otimes \xi_M(m)_0 \cdot_Q (n^{(1)} \xi_P^{-1}(p)^{(1)})\big) \\
&\quad = \sum \xi_N(\xi_J(m_{(-1)1}) \cdot_J n^0) \otimes (\xi_J(m_{(-1)2}) \cdot_J \xi_P^{-1}(p^0) \\
&\qquad\qquad \otimes m_0 \cdot_Q (\xi_Q^{-1}(n^{(1)}) \xi_Q^{-2}(p^{(1)})\big) \\
&\quad = \sum \xi_N(m_{(-1)} \cdot_J n^0) \otimes (\xi_J(m_{0(-1)}) \cdot_J \xi_P^{-1}(p^0) \\
&\qquad\qquad \otimes (m_{00} \cdot_Q \xi_Q^{-1}(n^{(1)})) \cdot_Q \xi_Q^{-1}(p^{(1)})) \\
&\quad \overset{(5.5)}{=} \sum \xi_N(m_{(-1)} \cdot_J n^0) \otimes \big((m_0 \cdot_Q n^{(1)})_{(-1)} \cdot_J \xi_P^{-1}(p^0) \\
&\qquad\qquad \otimes (m_0 \cdot_Q n^{(1)})_0 \cdot_Q \xi_Q^{-1}(p^{(1)})\big) \\
&\quad = (\mathrm{id}_N \otimes \Psi_{M,P}) \Big(\sum \xi_N(m_{(-1)} \cdot_J n^0) \otimes (m_0 \cdot_Q n^{(1)} \otimes \xi_P^{-1}(p)) \Big) \\
&\quad = (\mathrm{id}_N \otimes \Psi_{M,P}) \circ \tilde{a}_{N,M,P} \circ (\Psi_{M,N} \otimes \mathrm{id}_P)((m \otimes n) \otimes p),
\end{aligned}$$

and so the equatlity $\tilde{a}_{N,P,M} \circ \Psi_{M,N\otimes P} \circ \tilde{a}_{M,N,P} = (\mathrm{id}_N \otimes \Psi_{M,P}) \circ \tilde{a}_{N,M,P} \circ (\Psi_{M,N} \otimes \mathrm{id}_P)$ is proven.

Similarly, we prove

$$\tilde{a}_{P,M,N}^{-1} \circ \Psi_{M\otimes N,P} \circ \tilde{a}_{M,N,P}^{-1} = (\Psi_{M,P} \otimes \mathrm{id}_N) \circ \tilde{a}_{M,P,N}^{-1} \circ (\mathrm{id}_M \otimes \Psi_{N,P}).$$

(c) We set

$$\Psi'_{M,N}(m \otimes n) = \sum n_0 \cdot_Q S_Q(m^{(1)}) \otimes S_J^{-1}(n_{(-1)}) \cdot_J m^0.$$

We check that $\Psi_{M,N}$ is bijective as follows:

$$\begin{aligned}
\Psi_{N,M}\Psi'_{M,N}(m \otimes n) &= \Psi_{N,M}\Big(\sum n_0 \cdot_Q S_Q(m^{(1)}) \otimes S_J^{-1}(n_{(-1)}) \cdot_J m^0 \Big) \\
&= \sum (n_0 \cdot_Q S_Q(m^{(1)}))_{(-1)} \cdot_J (S_J^{-1}(n_{(-1)}) \cdot_J m^0)^0 \\
&\qquad \otimes (n_0 \cdot_Q S_Q(m^{(1)}))_0 \cdot_Q (S_J^{-1}(n_{(-1)}) \cdot_J m^0)^{(1)}
\end{aligned}$$

$$\overset{(5.5),(5.6)}{=} \sum \xi_J(n_{0(-1)}) \cdot_J (S_J^{-1}\xi_J^{-1}(n_{(-1)}) \cdot_J m^{00}) \\ \otimes (n_{00} \cdot_Q S_Q\xi_Q^{-1}(m^{(1)})) \cdot_Q \xi_Q(m^{0(1)})$$

$$= \sum (n_{(-1)2}S_J^{-1}(n_{(-1)1})) \cdot_J m^0 \otimes n_0 \cdot_Q (S_Q(m_2^{(1)})m_1^{(1)}) = \mathrm{id}(m \otimes n),$$

and so $\Psi_{N,M}\Psi'_{M,N} = \mathrm{id}_{M\otimes N}$. Similarly, $\Psi'_{N,M}\Psi_{M,N} = \mathrm{id}_{M\otimes N}$. It follows that $\Psi_{M,N}^{-1} = \Psi'_{N,M}$.

Therefore, by Proposition 5.5, $\widetilde{\mathcal{H}}({}_J^J\mathrm{Mod}_Q^Q)$ is a braided monoidal Hom-category. ∎

COROLLARY 5.7. *Let (M, ξ_M) be an object in $\widetilde{\mathcal{H}}({}_J^J\mathrm{Mod}_Q^Q)$. Then $\Psi_{M,M}$ satisfies the following Yang–Baxter Hom-condition:*

$$(\Psi_{M,M} \otimes \mathrm{id}_M) \circ \tilde{a}^{-1} \circ (\mathrm{id}_M \otimes \Psi_{M,M}) \circ \tilde{a} \circ (\Psi_{M,M} \otimes \mathrm{id}_M) \\ = (\mathrm{id}_M \otimes \Psi_{M,M}) \circ \tilde{a} \circ (\Psi_{M,M} \otimes \mathrm{id}_M) \circ \tilde{a}^{-1} \circ (\mathrm{id}_M \otimes \Psi_{M,M}) \circ \tilde{a}.$$

REMARK 5.8. In general, assume that $\widetilde{\mathcal{H}}(C)$ is a braided monoidal Hom-category. A subcategory $\widetilde{\mathcal{H}}(C_1)$ of $\widetilde{\mathcal{H}}(C)$ is called a *subbraided monoidal Hom-category* of $\widetilde{\mathcal{H}}(C)$ if $\widetilde{\mathcal{H}}(C_1)$ itself is a braided monoidal Hom-category according to the braided monoidal structure over $\widetilde{\mathcal{H}}(C)$.

A straightforward checking yields:

THEOREM 5.9. *Under the notation of Remark* 5.8*, the categories $\widetilde{\mathcal{H}}({}_J^J\mathcal{YD})$ and $\widetilde{\mathcal{H}}(\mathcal{YD}_Q^Q)$ are the subbraided monoidal Hom-categories of $\widetilde{\mathcal{H}}({}_J^J\mathrm{Mod}_Q^Q)$.*

REMARK 5.10. Equality (2.1) can be obtained from the braided $\Psi_{H,B}$ as follows:

$$(a \bowtie h)(b \bowtie g) = (m_B \otimes m_H) \circ \tilde{a} \circ (\mathrm{id}_B \otimes \tilde{a}) \circ (\mathrm{id}_B \otimes \Psi_{H,B} \otimes \mathrm{id}_H) \\ \circ (\mathrm{id}_B \otimes \tilde{a}) \circ \tilde{a}((a \otimes h) \otimes (b \otimes g)),$$

where m_B and m_H are multiplications of B and H respectively.

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