

## Multidimensional Riemann derivatives

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**Abstract.** The well-known concepts of  $n$ th Peano, Lipschitz, Riemann, Riemann Lipschitz, symmetric Riemann, and symmetric Riemann Lipschitz derivatives of real functions of a single variable have natural extensions to functions of several variables. We show that if a function has any of these  $n$ th derivatives at each point of a measurable subset of  $\mathbb{R}^d$  then it has all these derivatives at almost every point of that subset.

**Introduction.** Three well-known generalizations of higher order differentiation for real-valued functions of a real variable are Peano differentiation, Riemann differentiation, and symmetric Riemann differentiation. These generalizations are generically equivalent to one another (see [As1]) and to their quantum versions (see [ACR]). There are similar  $L^p$  results (see [AC]). Some of these generalized derivatives play a role in numerical differentiation (see [AJJ] and [AJ]). We assign to each of the four above-mentioned differentiations a corresponding higher differential. Fix a real number  $x$ , a function  $f$ , an order  $n$ , and a small second independent real variable  $h$ . The four  $n$ th order differentials are defined as

$$\begin{aligned} d^{(n)}f(x, h) &= f^{(n)}(x)h^n, & R_n f(x, h) &= R^n f(x)h^n, \\ P_n f(x, h) &= f_n(x)h^n, & SR_n f(x, h) &= SR^n f(x)h^n, \end{aligned}$$

where  $f^{(n)}(x)$ ,  $f_n(x)$ ,  $R^n f(x)$ , and  $SR^n f(x)$  are, respectively, the  $n$ th order ordinary, Peano, Riemann, and symmetric Riemann derivatives. These notions of differentials have natural analogues for real-valued functions of  $d$  real variables. For example, if  $d = 2$ , the second Peano differential at

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$x = (x_1, x_2)$  is

$$P_2 f(x, h) = f_{20}(x)h_1^2 + 2f_{11}(x)h_1h_2 + f_{02}(x)h_2^2,$$

a homogeneous polynomial of degree 2 in the variable  $h = (h_1, h_2)$ .

The  $n$ th order ordinary differential is well known. We will define the other three  $n$ th order differentials, and prove the elementary proposition that in every dimension  $d$ , the existence of the  $n$ th order Peano differential implies pointwise the existence of both the  $n$ th order Riemann differential and the  $n$ th order symmetric Riemann differential. The main result of this work is that these three generalized differentials are generically equivalent. The main result validates a conjecture made by H. W. Oliver in 1951 (see [O11, p. 15]). Most [O11] results also appear in [O12].

Throughout this article, all functions are taken to be real-valued. However, all results remain true when the real-valued functions considered here are replaced by complex-valued ones, and all of the proofs are unchanged.

**1. Some multidimensional derivatives and equivalences between them.** Let  $d$  be a positive integer, and  $\kappa = (\kappa_1, \dots, \kappa_d)$  be a  $d$ -multiindex, where each  $\kappa_i$  is a nonnegative integer. Then  $|\kappa| = \kappa_1 + \dots + \kappa_d$  and  $\kappa! = \kappa_1! \cdots \kappa_d!$ .

Let  $x = (x_1, \dots, x_d)$  and  $h = (h_1, \dots, h_d)$  be points in  $\mathbb{R}^d$ . We denote  $\|h\| = \sqrt{h_1^2 + \dots + h_d^2}$  and  $h^\nu = h_1^{\nu_1} \cdots h_d^{\nu_d}$ .

We start with a one-dimensional theorem of Peano.

**THEOREM 1.1.** *Fix  $x \in \mathbb{R}$  and a positive integer  $n$ . If a function  $f : \mathbb{R}^1 \rightarrow \mathbb{R}$  has the property that  $f^{(n)}(x)$  exists, then*

$$f(x+h) = \sum_{k=0}^n f^{(k)}(x) \frac{h^k}{k!} + o(|h|^n).$$

In dimension  $d$ , the standard meaning of a function  $g$  being *differentiable* at  $x$  is that there exist all  $d$  first order partial derivatives  $\{\frac{\partial}{\partial x_i} g(x)\}$  and, additionally, we have the approximation

$$g(x+h) = g(x) + \frac{\partial}{\partial x_1} g(x)h_1 + \dots + \frac{\partial}{\partial x_d} g(x)h_d + o(\|h\|).$$

To extend Theorem 1.1 to higher dimensions, we generalize not the one-dimensional condition that  $f^{(n)}(x)$  exists, but rather the obviously identical one-dimensional condition that  $f^{(n-1)}$  is differentiable at  $x$ . This leads to the following generalization of Peano's Theorem (see [AGV]).

**THEOREM 1.2.** *Fix  $x \in \mathbb{R}^d$  and a positive integer  $n$ . If  $f : \mathbb{R}^d \rightarrow \mathbb{R}$  and for every multiindex  $\kappa = (\kappa_1, \dots, \kappa_d)$  with  $|\kappa| = n - 1 \geq 0$ , the  $\kappa$ th partial derivative*

$$\partial^\kappa f(x) = \frac{\partial^{n-1}}{\partial x_1^{\kappa_1} \dots \partial x_d^{\kappa_d}} f(x)$$

(in particular,  $\partial^0 f(x) = f(x)$ ) is differentiable at  $x$ , then

$$(1.1) \quad f(x+h) = \sum_{0 \leq |\kappa| \leq n} \partial^\kappa f(x) \frac{h^\kappa}{\kappa!} + o(\|h\|^n).$$

DEFINITION 1.3. We say  $f$  is  $n$  times differentiable at  $x$  if all partial derivatives of  $f$  of orders  $\leq n$  exist at  $x$  and

$$(1.2) \quad \begin{aligned} f(x+h) &= \sum_{0 \leq |\kappa| \leq n} \partial^\kappa f(x) \frac{h^\kappa}{\kappa!} + o(\|h\|^n) \\ &= \sum_{k=0}^n \frac{1}{k!} \sum_{|\kappa|=k} \frac{k!}{\kappa!} \partial^\kappa f(x) h^\kappa + o(\|h\|^n). \end{aligned}$$

In order to write this in a more compact way, one can employ the multinomial expansion

$$(x_1 + \dots + x_d)^k = \sum_{\kappa_1 + \dots + \kappa_d = k} \frac{k!}{\kappa_1! \dots \kappa_d!} x_1^{\kappa_1} \dots x_d^{\kappa_d} = k! \sum_{|\kappa|=k} \frac{x^\kappa}{\kappa!}.$$

This applied backwards to  $x = (x_1, \dots, x_d)$ , with  $x_i = h_i \partial_i$ , leads to

$$(1.3) \quad \sum_{|\kappa|=k} \partial^\kappa f(x) \frac{h^\kappa}{\kappa!} = \frac{(h_1 \partial_1 + \dots + h_d \partial_d)^k f(x)}{k!} = \frac{(h \cdot \nabla)^k f(x)}{k!},$$

where  $\nabla = (\partial_1, \dots, \partial_d)$  is the usual gradient operator. Equation (1.1) then has the equivalent compact form

$$(1.4) \quad f(x+h) = \sum_{k=0}^n \frac{(h \cdot \nabla)^k f(x)}{k!} + o(\|h\|^n).$$

See [F] to get a feel for this now standard material.

The following definitions all assume that the functions under consideration are Lebesgue measurable mappings from  $\mathbb{R}^d$  to  $\mathbb{C}$ . Suppose a function  $f : \mathbb{R}^d \rightarrow \mathbb{R}$  has no a priori smoothness other than having polynomial approximation of degree  $n$  near  $x$ . By grouping terms of equal homogeneity together, this may be written as

$$f(x+h) = \sum_{k=0}^n \frac{P_{nk}(x, h)}{k!} + o(\|h\|^n),$$

where  $P_{nk}$  is a homogeneous polynomial of degree  $k$  in the variable  $h$  with coefficients depending on  $x$ . Motivated by the expression appearing in formula (1.2), we will write  $P_{nk}$  as the homogeneous degree  $k$  polynomial

$$P_{nk}(x, h) = \sum_{|\kappa|=k} \frac{k!}{\kappa!} f_{\kappa}(x) h^{\kappa},$$

thereby defining  $f_{\kappa}(x)$  as the  $\kappa$ th partial Peano derivative of  $f$  at  $x$ . We further define  $P_n(x, h) := P_{nn}(x, h)$  to be the  $n$ th Peano differential of  $f$  at  $x$ . Because of Theorem 1.2 and calculation (1.3), when  $f$  is smooth enough, the  $n$ th Peano differential  $P_n$  exists and is equal to the ordinary  $n$ th differential  $(h \cdot \nabla)^k f(x)$ . Notice that from their definitions, mixed partial derivatives automatically commute. For example, the coefficient of the  $h_1 h_2$ -term of the polynomial  $P_{n2}(x, h)$  is  $\frac{2!}{1! \cdot 1!} f_{12}(x)$ , but since  $h_1 h_2 = h_2 h_1$ , the same coefficient is also  $\frac{2!}{1! \cdot 1!} f_{21}(x)$ .

DEFINITION 1.4. If  $f(x+h) = \sum_{|\kappa| \leq n-1} f_{\kappa}(x) h^{\kappa} / \kappa! + O(\|h\|^n)$  at  $x$ , say that  $f$  is Lipschitz of order  $n$  at  $x$ . This is denoted  $f \in T_n(x)$  in [CZ].

DEFINITION 1.5. If  $f(x+h) = \sum_{|\kappa| \leq n} f_{\kappa}(x) h^{\kappa} / \kappa! + o(\|h\|^n)$  at  $x$ , say that  $f$  has an  $n$ th Peano derivative at  $x$ . This is denoted  $f \in t_n(x)$  in [CZ].

If the dimension  $d$  is 1, there exists only one  $n$ th partial Peano derivative  $f_n(x)$ , and it coincides with the usual  $n$ th Peano derivative. In this case, the  $n$ th Peano differential is just  $f_n(x)h^n$ .

Notice that Theorem 1.2 can now be interpreted as saying that if all the order  $n-1$  partial derivatives of a function  $f$  are differentiable, then  $f$  has an  $n$ th order Peano differential, and all the  $n$ th order partial Peano derivatives exist and agree with their corresponding ordinary partial derivatives.

DEFINITION 1.6. If  $f : \mathbb{R}^d \rightarrow \mathbb{R}$  satisfies

$$\sum_{i=0}^n (-1)^{n-i} \binom{n}{i} f(x+ih) = O(\|h\|^n)$$

at  $x$ , say that  $f$  is Riemann Lipschitz of order  $n$  at  $x$ .

DEFINITION 1.7. If  $f : \mathbb{R}^d \rightarrow \mathbb{R}$  satisfies

$$(1.5) \quad \sum_{i=0}^n (-1)^{n-i} \binom{n}{i} f(x+ih) = \sum_{|\kappa|=n} \frac{n!}{\kappa!} f_{\kappa}(x) h^{\kappa} + o(\|h\|^n)$$

at  $x$ , say that  $f$  is Riemann differentiable of order  $n$  at  $x$ .

If, in the last two definitions, we replace

$$\sum_{i=0}^n (-1)^{n-i} \binom{n}{i} f(x+ih) \quad \text{by} \quad \sum_{i=0}^n (-1)^{n-i} \binom{n}{i} f\left(x + \left(i - \frac{n}{2}\right)h\right),$$

the function  $f$  is respectively called symmetric Riemann Lipschitz of order  $n$  at  $x$ , and symmetric Riemann differentiable of order  $n$  at  $x$ .

The Riemann and symmetric Riemann derivatives are special cases of the following generalized derivative.

DEFINITION 1.8. Let  $e$  be a nonnegative integer and let  $\mathbf{A} = \{A_i; a_i\}$  be a set of  $2(n + 1 + e)$  real numbers satisfying

$$(1.6) \quad \sum_{i=0}^{n+e} A_i a_i^s = n! \delta_{sn} = \begin{cases} 0 & \text{if } s < n, \\ n! & \text{if } s = n. \end{cases}$$

Then  $f : \mathbb{R}^d \rightarrow \mathbb{C}$  has an  $n$ th order  $\mathcal{A}$ -derivative  $D_{\mathbf{A}}f$  at  $x \in \mathbb{R}^d$  if

$$(1.7) \quad \begin{aligned} \sum_{i=0}^{n+e} A_i f(x + a_i h) &= D_{\mathbf{A}}f(x, h) + o(\|h\|^n) \\ &= \sum_{|\kappa|=n} \frac{n!}{\kappa!} (D_{\mathbf{A}}f)_{\kappa}(x) h^{\kappa} + o(\|h\|^n). \end{aligned}$$

When  $d = 1$ , the  $\mathcal{A}$ -derivative  $D_{\mathbf{A}}f$  is called a *generalized Riemann derivative* in [As1]. Since  $\sum_{i=0}^n (-1)^{n-i} \binom{n}{i} i^s = n! \delta_{sn}$  for  $0 \leq s \leq n$ , the Riemann derivative is a special case of generalized Riemann derivative. It is easy to see that a slide of a generalized Riemann derivative is also a generalized Riemann derivative. This means that if  $D_{\mathbf{A}}f(x)$  exists for  $\mathbf{A} = \{A_i; a_i\}$ , then for any real constant  $\tau$ ,  $D_{\mathbf{A}_{\tau}}f(x)$  exists, where  $\mathbf{A}_{\tau} = \{A_i; a_i - \tau\}$ . In particular, let  $\tau = n/2$  to see that the symmetric Riemann derivative is also a generalized Riemann derivative.

DEFINITION 1.9. Let  $\mathbf{A} = \{A_i; a_i\}$  be as in Definition 1.8. If

$$\sum_{i=0}^{n+e} A_i f(x + a_i h) = O(\|h\|^n) \quad \text{at } x,$$

say that  $f$  is  $\mathbf{A}$ -Lipschitz of order  $n$  at  $x$ .

PROPOSITION 1.10. Whenever the  $n$ th Peano derivative  $P_n(x, h)$  exists, the generalized  $n$ th  $\mathcal{A}$ -derivative  $D_{\mathbf{A}}f$  also exists and is equal to  $P_n(x, h)$ .

*Proof.* Assume  $P_n(x, h)$  exists. Expand each  $f(x + a_i h)$  and interchange the order of summation. Note that  $(ah)^{\kappa} = (ah_1)^{\kappa_1} \cdots (ah_d)^{\kappa_d} = a^{|\kappa|} h^{\kappa}$  for every real number  $a$ . We have

$$\begin{aligned} \sum_{i=0}^{n+e} A_i f(x + a_i h) &= \sum_{i=0}^{n+e} A_i \sum_{|\kappa| \leq n} f_{\kappa}(x) (a_i)^{|\kappa|} \frac{h^{\kappa}}{\kappa!} + o(\|h\|^n) \\ &= \sum_{|\kappa| \leq k-1} f_{\kappa}(x) \frac{h^{\kappa}}{\kappa!} \left\{ \sum_{i=0}^{n+e} A_i (a_i)^{|\kappa|} \right\} + \sum_{|\kappa|=n} f_{\kappa}(x) \frac{h^{\kappa}}{\kappa!} \left\{ \sum_{i=0}^{n+e} A_i (a_i)^n \right\} + o(\|h\|^n) \\ &= \sum_{|\kappa| \leq k-1} f_{\kappa}(x) \frac{h^{\kappa}}{\kappa!} \{0\} + \sum_{|\kappa|=n} f_{\kappa}(x) \frac{h^{\kappa}}{\kappa!} \{n!\} + o(\|h\|^n) \\ &= P_n(x, h) + o(\|h\|^n). \quad \blacksquare \end{aligned}$$

What we call a generalized Riemann derivative, or an  $\mathcal{A}$ -derivative, is fairly general if the underlying space is  $\mathbb{R}^1$  and if one wishes to obtain a derivative of any integer order by means of a finite number of function evaluations, followed by a single limiting process. Over  $\mathbb{R}^d$  for  $d \geq 2$ , the points  $x + a_i h$  all lie on the line passing through  $x$  and  $x + h$ , so there are many other potential possible generalized derivatives similar to, but different from, our generalized Riemann derivatives. For example, I. B. Zibman [Zi] has dealt with the case where the evaluation points lie on a  $(d - 1)$ -dimensional sphere centered at  $x$ .

The following well-known result says that the property of being Lipschitz is generically equivalent to being differentiable in the sense that for every measurable function, the set  $E$  of points where the function  $f$  is Lipschitz (relative to  $E$ ) of order  $k$  contains a subset  $F$  of full measure (i.e.,  $|E \setminus F| = 0$ ), so that  $f$  is differentiable of order  $k$  (relative to  $F$ ) at every point of  $F$ . This theorem provides a powerful tool for proving that various conditions that are pointwise weaker than differentiability are generically equivalent to differentiability.

**THEOREM 1.11.** *If a function  $f$  on  $\mathbb{R}^d$  is Lipschitz of order  $n$  at each point of a (Lebesgue) measurable set  $E$  of  $\mathbb{R}^d$ , then there is a subset  $F$  of  $E$  such that  $|E \setminus F| = 0$  and  $f$  is Peano differentiable relative to  $F$  of order  $n$  at every point of  $F$ .*

Theorem 1.11 appeared in a more general form in [CZ], where the proof of the special case that is our Theorem 1.11 is correctly attributed to the 1951 University of Chicago thesis of H. William Oliver [Ol1]. A second proof of Theorem 1.11 that appeared in [Bu] had the additional assumption that  $E$  be a closed set. Actually, with very little additional work, this assumption may be lightened. Here is a simple meta-theorem that, when combined with the result proved in [Bu], yields a proof of Theorem 1.11.

**THEOREM 1.12** ([As2]). *Suppose  $p$  and  $q$  are two properties such that for any closed set  $C$ , if  $p$  is true at each point of  $C$ , then  $q$  holds at a.e. point of  $C$ . Then for any measurable set  $E$ , if  $p$  is true at each point of  $E$ , then  $q$  holds at a.e. point of  $E$ .*

Our main theorem asserts the equivalence of six different smoothness conditions. No pair of these are necessarily equivalent at a single point, but the six conditions are all equivalent if appropriate sets of measure zero are neglected.

**THEOREM 1.13** (Main theorem). *Let  $n$  be a positive integer, and suppose that, for every  $x$  in a measurable set  $E \subset \mathbb{R}^d$ , the measurable function  $f : \mathbb{R}^d \rightarrow \mathbb{R}$  satisfies one of the following six conditions.*

1.  $f$  is  $n$ th Peano differentiable at  $x$ ,
2.  $f$  is Lipschitz of order  $n$  at  $x$ ,
3.  $f$  is Riemann differentiable of order  $n$  at  $x$ ,
4.  $f$  is Riemann Lipschitz of order  $n$  at  $x$ ,
5.  $f$  is symmetric Riemann differentiable of order  $n$  at  $x$ ,
6.  $f$  is symmetric Riemann Lipschitz of order  $n$  at  $x$ .

Then  $f$  satisfies all six conditions at almost every point of  $E$ .

*Proof.* The simple fact that  $|h^\kappa| \leq C\|h\|^n$  for every  $\kappa$  with  $|\kappa| = n$  justifies the pointwise valid implications “1 implies 2”, “3 implies 4”, and “5 implies 6”. Proposition 1.10 justifies the pointwise valid implications “1 implies 3” and “1 implies 5”. The sliding Lemma 2.3 below shows that condition 6 on  $E$  implies condition 4 at a.e. point of  $E$ . Theorem 1.11 asserts that condition 2 on  $E$  implies condition 1 at a.e. point of  $E$ . So looking at a diagram of implications makes it clear that in order to have a fully closed chain of implications, it is sufficient to prove that condition 4 holding on  $E$  implies condition 2 holding at a.e. point of  $E$ . This is the result of the next theorem, whose proof is given in Section 3. ■

**THEOREM 1.14.** *Suppose  $f$  is Riemann Lipschitz of order  $n$  at every point of  $E$ . Then  $E$  has a subset  $F$  such that  $|E \setminus F| = 0$ , and  $f$  is Lipschitz of order  $n$  relative to  $F$  at every point of  $F$ .*

**2. Multidimensional sliding lemma.** Of the two results needed to complete the proof of our main Theorem 1.13, we first do the sliding lemma. The proof of its one-dimensional case appearing in [As1, pp. 183–185] uses two preliminary lemmas. Here is the  $d$ -dimensional version of the first result.

**LEMMA 2.1.** *Let  $\mathbf{0} = (0, \dots, 0)$  be a point of outer density of a (possibly nonmeasurable) set  $\mathcal{E} \subset \mathbb{R}^d$ . Let  $\alpha, \beta$  be real numbers such that  $\beta \neq 0$ . Given  $\epsilon > 0$ , there is a  $\delta > 0$  such that for all  $u \neq \mathbf{0}$  with  $\|u\| < \delta$ ,*

$$(2.1) \quad m^*(\{v \in N : \alpha u + \beta v \in \mathcal{E}\}) > |N|(1 - \epsilon),$$

where  $N$  is the annulus  $\{v : \|u\| \leq \|v\| \leq 2\|u\|\}$ ,  $m^*$  denotes outer measure, and  $|F|$  denotes Lebesgue measure whenever  $F$  is a subset of  $\mathbb{R}^d$ .

*Proof.* Let  $\mathcal{G}$  be a cover of  $\mathcal{E}$ . This means that  $\mathcal{E}$  is contained in the Lebesgue measurable set  $\mathcal{G}$  and for every measurable set  $C$ ,  $m^*(\mathcal{E} \cap C) = |\mathcal{G} \cap C|$ . In particular,  $\mathbf{0}$  is a point of density of  $\mathcal{G}$ . For fixed  $\alpha, \beta$ , and  $u$ , let  $B$  be the ball of radius  $|\alpha|\|u\| + |\beta|2\|u\|$  centered at  $\mathbf{0}$  and consider the affine map  $\varphi : N \rightarrow B$  given by  $\varphi(v) = \alpha u + \beta v$ . The radius of  $B$  was chosen to guarantee that  $\varphi(N) \subset B$ . Let  $b$  be the volume of a ball of radius  $\|u\|$  so that the volume  $|N|$  satisfies  $|N| = |B_{2\|u\|}(0)| - |B_{\|u\|}(0)| = (2^d - 1)b$ ,

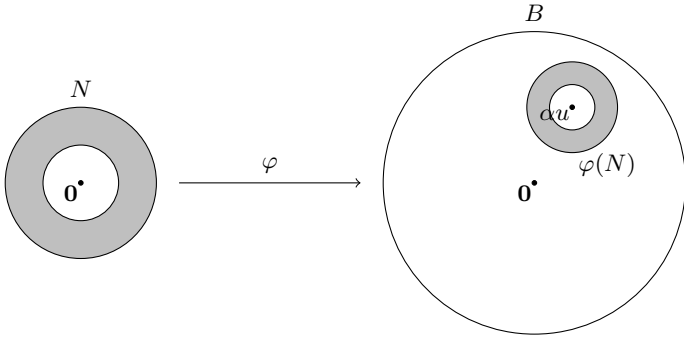


Fig. 1. Situation in the proof of Lemma 2.1

$|\varphi(N)| = |\beta|^d |N| = |\beta|^d (2^d - 1)b$ , and  $|B| = (|\alpha| + 2|\beta|)^d b$ . Since choosing  $\|u\|$  very small forces  $\mathcal{G}$  to nearly fill  $B$ , and since the ratio

$$\frac{|\varphi(N)|}{|B|} = \frac{|\beta|^d (2^d - 1)}{(|\alpha| + 2|\beta|)^d}$$

is positive and independent of  $\|u\|$ , choosing  $\|u\|$  very small also forces  $\mathcal{G}$  to nearly fill  $\varphi(N)$ . In other words, given  $\epsilon > 0$ , there is a  $\delta > 0$  such that for all  $u \neq \mathbf{0}$  with  $\|u\| < \delta$ ,

$$(2.2) \quad |\mathcal{G} \cap \varphi(N)| > |\varphi(N)|(1 - \epsilon).$$

Furthermore, the mapping  $\varphi^{-1}$  consists of a translation,  $y \mapsto -\alpha u + y$ , followed by a dilation,  $y \mapsto \beta^{-1}y$ . Applying either a translation or a dilation does not change the relative proportion of a subset's measure to a set's measure. Thus from estimate (2.2), it follows that also

$$|\varphi^{-1}(\mathcal{G} \cap \varphi(N))| > |\varphi^{-1}(\varphi(N))|(1 - \epsilon).$$

Since  $\varphi^{-1}(A \cap B) = \varphi^{-1}(A) \cap \varphi^{-1}(B)$  and  $\varphi^{-1}(\varphi(N)) = N$ , the last inequality may be written as

$$|\varphi^{-1}(\mathcal{G}) \cap N| > |N|(1 - \epsilon).$$

Since  $\mathcal{G}$  is a cover for  $\mathcal{E}$ , we see that  $\varphi^{-1}(\mathcal{G})$  is a cover for  $\varphi^{-1}(\mathcal{E})$ , and we finally arrive at

$$m^*(\varphi^{-1}(\mathcal{E}) \cap N) > |N|(1 - \epsilon),$$

which is a restatement of the desired inequality (2.1). ■

The second result needed in the proof of the sliding lemma gives a sufficient condition for local boundedness.

LEMMA 2.2. *Suppose that  $E$  is a measurable set,  $E \subset \mathbb{R}^d$ ,  $f$  is a measurable function, and*

$$\sum_{i=0}^m A_i f(x + a_i h) = O(1) \quad \text{for all } x \in E,$$



where the numbers  $A_i$  are nonzero and the numbers  $a_i$  are distinct. Then  $f$  is bounded in a neighborhood of almost every point of  $E$ .

*Proof.* For each natural number  $j$ , let

$$E_j = \left\{ x \in E : \left| \sum_{i=0}^m A_i f(x + a_i h) \right| \leq j \text{ if } 0 < \|h\| < \frac{1}{j} \right\},$$

$$F_j = \{x \in R^d : |f(x)| \leq j\}.$$

Observe that the  $E_j$  may not be measurable. Since  $\bigcup_{j \in \mathbb{N}} E_j \cap F_j = E$ , it suffices to fix  $j$  and show that  $f$  is bounded in a neighborhood of every point of outer density of  $E_j \cap F_j$ . Assume  $\mathbf{0}$  is such a point. Let  $u \neq 0$ , let  $N$  be the annulus  $\{v : \|u\| \leq \|v\| \leq 2\|u\|\}$  and let  $B = \{v \in N : v \in E_j \cap F_j\}$ . By Lemma 2.1, there is a  $\delta_0 > 0$  such that  $\|u\| < \delta_0$  implies  $m^*(B) > \frac{1}{2}|N|$ . For  $i = 1, \dots, m$ , let  $C_i = \{v \in N : v + (a_i/a_0)(u - v) \in F_j\}$ . Since  $F_j \supset E_j \cap F_j$  and  $F_j$  is measurable,  $F_j$  has  $\mathbf{0}$  as a point of density. By Lemma 2.1 again, for each  $i = 1, \dots, m$ , there is a  $\delta_i > 0$  such that  $0 < \|u\| < \delta_i$  implies  $|C_i| > (1 - \frac{1}{2m})|N|$ . Set  $C = \bigcap_{i=1}^m C_i$ . Then if  $0 < \|u\| < \min\{\delta_i\}$ , we have  $|C| > \frac{1}{2}|N|$  and

$$\begin{aligned} \frac{1}{2}|N| < m^*(B) &\leq m^*(B \cap C) + m^*(B \setminus C) \leq m^*(B \cap C) + |N \setminus C| \\ &\leq m^*(B \cap C) + \frac{1}{2}|N|. \end{aligned}$$

Thus  $m^*(B \cap C) > 0$  and we may pick a  $v \in B \cap C$ . If additionally  $\|u\| < |a_0|/(3j)$ , then

$$\left\| \frac{1}{a_0}(u - v) \right\| \leq \frac{1}{|a_0|}(\|u\| + \|v\|) \leq \frac{1}{|a_0|}(\|u\| + 2\|u\|) < \frac{1}{j}.$$

Since  $v \in B$ ,

$$\left| \sum_{i=0}^m A_i f\left(v + a_i \left(\frac{1}{a_0}(u - v)\right)\right) \right| < j,$$

$$\left| A_0 f(u) + \sum_{i=1}^m A_i f\left(v + \frac{a_i}{a_0}(u - v)\right) \right| < j;$$

and since the  $v$  also belong to all  $C_i$ ,

$$|f(u)| < \frac{1}{|A_0|} \left( \sum_{i=1}^m \left| A_i f\left(v + \frac{a_i}{a_0}(u - v)\right) \right| + j \right) \leq \frac{1}{|A_0|} \left( \left\{ \sum_{i=1}^m |A_i| \right\} + 1 \right) j.$$

The right hand side provides the required uniform bound for  $f$  in a neighborhood of  $\mathbf{0}$ . ■

Here is the  $d$ -dimensional sliding lemma.

LEMMA 2.3 (Multi-dimensional sliding lemma). *Suppose that  $m \geq 1$ ,  $n \geq 0$ , and  $E$  is a measurable subset of  $\mathbb{R}^d$  such that*

$$\sum_{i=0}^m A_i f(x + a_i h) = O(\|h\|^n) \quad \text{for all } x \in E,$$

where the  $A_i$ 's are nonzero and the  $a_i$ 's are distinct. Then for any real  $a$ ,

$$\sum_{i=0}^m A_i f(x + (a_i - a)h) = O(\|h\|^n) \quad \text{for almost every } x \in E.$$

If “ $O$ ” is replaced by “ $o$ ” in the hypothesis, then the conclusion also holds with “ $o$ ” in place of “ $O$ ”.

*Proof.* We may assume  $a_0 \neq 0$  by rearranging the terms if necessary. We may also assume that  $0 < |E| < \infty$ . Let

$$E_j = \left\{ x \in E : \left| \sum_{i=0}^m A_i f(x + a_i h) \right| \leq j \|h\|^n \text{ if } \|h\| < \frac{1}{j} \right\}.$$

Observe that the  $E_j$  may not be measurable. Since  $\bigcup E_j = E$ , it suffices to fix  $j$  and prove the lemma at every point of  $E_j$  that is a point of outer density of  $E_j$ . To simplify notation, let  $x = \mathbf{0}$  be such a point. For each  $u \neq \mathbf{0}$ , let  $N$  be the annulus  $\{v : \|u\| \leq \|v\| \leq 2\|u\|\}$ . By Lemma 2.2, if  $\|u\|$  is sufficiently small, then all of the  $m + 1$  sets

$$B_i^* = \{v \in N : (a_i - a)u - a_0 v \in E_j\}, \quad i = 0, 1, \dots, m,$$

and all of the  $m$  sets

$$C_i^* = \{v \in N : -au + (a_i - a_0)v \in E_j\}, \quad i = 1, \dots, m,$$

have outer measure greater than  $|N|(1 - \frac{1}{2^{m+1}})$ . Although  $B_i^*$  and  $C_i^*$  may not be measurable, the sets

$$B_i = \left\{ v \in N : \left| \sum_{s=0}^m A_s f([(a_i - a)u - a_0 v] + a_s v) \right| \leq j 2^n \|u\|^n \right\}$$

for  $i = 0, 1, \dots, m$ , and

$$C_i = \left\{ v \in N : \left| \sum_{s=0}^m A_s ([-au + (a_i - a_0)v] + a_s u) \right| \leq j \|u\|^n \right\}$$

for  $i = 1, \dots, m$ , are measurable. Furthermore, if we additionally require that  $\|u\| < \frac{1}{2j}$ , then  $\|v\| < \frac{1}{j}$  and  $B_i \supset B_i^*$  for all  $i$ , so that  $|B_i| \geq m^*(B_i^*)$ . Also  $C_i \supset C_i^*$  for all  $i$  so that  $|C_i| \geq m^*(C_i^*)$ . Thus  $(\bigcap_{i=0}^m B_i) \cap (\bigcap_{i=1}^m C_i)$  has positive measure and we may choose  $v$  to belong to this set.

Since  $(a_i - a)u - a_0v \in B_i$  for all  $i$ , we have

$$\sum_{s=0}^m A_s f[\{(a_i - a)u - a_0v\} + a_s v] = O(\|v\|^n) = O(\|u\|^n), \quad i = 0, 1, \dots, m.$$

Multiplying the  $i$ th equation by  $A_i$ , and summing over  $i$ , we get

$$\sum_{i=0}^m A_i \left[ \sum_{s=0}^m A_s f[\{(a_i - a)u - a_0v\} + a_s v] \right] = O(\|u\|^n).$$

Rearranging the order of summation, we obtain

$$\sum_{s=0}^m A_s \left[ \sum_{i=0}^m A_i f[\{(a_i - a)u - a_0v\} + a_s v] \right] = O(\|u\|^n).$$

Since the term in curly brackets is in  $C_s$  for every  $s > 0$ , each term of the outer sum except the  $s = 0$  term is  $O(\|u\|^n)$ . Hence the latter term is also  $O(\|u\|^n)$ , i.e.,

$$A_0 \left[ \sum_{i=0}^m A_i f[\{-au + (a_0 - a_0)v\} + a_i u] \right] = O(\|u\|^n).$$

Dividing by  $A_0$  and simplifying yields

$$\sum_{i=0}^m A_i f[(a_i - a)u] = O(\|u\|^n) \quad \text{as } \|u\| \rightarrow 0,$$

which is the desired result. The “ $o$ ” case is proved in a very similar manner. ■

**3. Proof of Theorem 1.14.** Our assumption is that  $f$  is Riemann Lipschitz of order  $n$  at every  $x \in E$ . In other words, we assume that

$$(3.1) \quad \sum_{i=0}^n \binom{n}{i} (-1)^{n-i} f(x + ih) = O(\|h\|^n)$$

Denote the left side of (3.1) as  $\Delta_n(x, h)$  and, following [MZ], define another  $n$ th difference  $\tilde{\Delta}_n(x, h)$  by

$$\tilde{\Delta}_1(x, h) = f(x + h) - f(x)$$

and

$$(3.2) \quad \tilde{\Delta}_i(x, h) = \tilde{\Delta}_{i-1}(x, 2h) - 2^{i-1} \tilde{\Delta}_{i-1}(x, h) \quad \text{for } i = 2, 3, \dots$$

It is shown in [MZ] that this construction leads to another generalized derivative in the sense that there are constants  $\alpha_0, \alpha_1, \dots, \alpha_n$  and  $\lambda_n$  such that

$$(3.3) \quad \tilde{\Delta}_n(x, h) = \alpha_0 f(x) + \alpha_1 f(x+h) + \alpha_2 f(x+2h) + \dots + \alpha_n f(x+2^{n-1}h),$$

where

$$(3.4) \quad \sum_{i=0}^n \alpha_i = 0, \quad \sum_{i=1}^n \alpha_i 2^{is} = 0 \quad \text{for } s = 1, \dots, n-1, \quad \lambda_n \sum_{i=1}^n \alpha_i 2^{in} = n!.$$

Furthermore, there are constants  $C_j$  such that

$$\tilde{\Delta}_n(x, h) = \sum_{j=0}^{2^{n-1}-n} C_j \Delta_n(x + jh, h).$$

Because of this identity and Lemma 2.3 (the sliding lemma), we have

$$(3.5) \quad \tilde{\Delta}_n(x, h) = O(\|h\|^n) \quad \text{a.e. on } E.$$

Fix an  $x$  where this is true. There is a  $\delta = \delta(x) > 0$  and an  $M = M(x)$  such that whenever  $\|h\| < \delta$  then

$$|\tilde{\Delta}_n(x, h)| \leq M\|h\|^n.$$

From (3.2), for each  $i = 1, \dots, k$ , we have

$$2^{(n-1)(i-1)} \left\{ \left| \tilde{\Delta}_{n-1} \left( 2 \frac{h}{2^i} \right) - 2^{n-1} \tilde{\Delta}_{n-1} \left( \frac{h}{2^i} \right) \right| \right\} \leq 2^{(n-1)(i-1)} \left\{ M \left\| \frac{h}{2^i} \right\|^n \right\}.$$

By addition we obtain

$$(3.6) \quad \left| \tilde{\Delta}_n(h) - 2^{(n-1)k} \tilde{\Delta}_{n-1} \left( \frac{h}{2^k} \right) \right| \leq \left( 2^{1-n} \sum_{i=1}^{\infty} 2^{-i} \right) M\|h\|^n \leq M\|h\|^n.$$

LEMMA 3.1. *If  $f$  satisfies (3.5) at every  $x \in E$ , then  $f$  satisfies*

$$\tilde{\Delta}_{n-1}(x, h) = O(\|h\|^{n-1}) \quad \text{at a.e. } x \in E.$$

*Proof.* The above calculation shows that (3.6) holds at all points of  $E$  for appropriate  $M$  and  $\delta$ . By Lemma 2.2, we may also assume that  $f$  is uniformly bounded on the ball of radius  $\delta$  about  $x$ . Divide inequality (3.6) by  $\|h\|^{n-1}$  to get

$$(3.7) \quad \left| \frac{\tilde{\Delta}_{n-1}(h)}{\|h\|^{n-1}} - \frac{\tilde{\Delta}_{n-1}(h/2^k)}{\|h/2^k\|^{n-1}} \right| \leq M\|h\|$$

or, setting  $u = h/2^k$ ,

$$\left| \frac{\tilde{\Delta}_{n-1}(u)}{\|u\|^{n-1}} \right| \leq M\|h\| + \frac{|\tilde{\Delta}_{n-1}(h)|}{\|h\|^{n-1}}.$$

If  $h$  is constrained to the annulus  $\{x : \delta/2 \leq \|x\| \leq \delta\}$ , the right hand side is bounded. As  $k$  takes all positive integer values,  $u$  takes all values in the punctured sphere  $\{x : 0 < \|x\| < \delta/2\}$ . ■

LEMMA 3.2. *If  $f(x)$  is Peano differentiable of order  $n - 1$  at  $x$  and if (3.5) holds at  $x$ , then  $f$  is Lipschitz of order  $n$  at  $x$ .*

*Proof.* By subtracting and adding the approximating polynomial of degree  $n - 1$  to  $f$ , we see that without loss of generality we may assume that at  $x$  the Peano differentials of order  $0, 1, \dots, n - 1$  are all 0. Expand each term of formula (3.3) and use the identities (3.4) to see that  $\tilde{\Delta}_{n-1}(u) = o(\|u\|^{n-1})$ . Now inequality (3.7) holds here, and letting  $k$  tend to infinity makes the second term on the left hand side tend to 0. Thus  $\tilde{\Delta}_{n-1}(h)/\|h\|^{n-1} = O(\|h\|)$ , so that  $\tilde{\Delta}_{n-1}(h) = O(\|h\|^n)$ . Similarly,  $\tilde{\Delta}_{n-2}(h) = O(\|h\|^n), \dots$ , and finally  $\tilde{\Delta}_1(h) = O(\|h\|^n)$ , i.e.,  $f$  is Lipschitz of order  $n$  at  $x$ . ■

We have assembled all the parts required for a very quick inductive proof of Theorem 1.14. We have already deduced from the original hypothesis that condition (3.5) holds a.e. on  $E$ . So it suffices to inductively prove a different theorem.

**THEOREM 3.3.** *If condition (3.5) holds on  $E$ , then  $f$  is Lipschitz of order  $n$  at a.e.  $x \in E$ .*

*Proof.* If  $n = 1$ , then (3.5) coincides with the condition of being Lipschitz of order 1. Now assume that the theorem holds for  $n - 1$  and also that (3.5) holds on  $E$ . By Lemma 3.1, condition (3.5) with  $n$  replaced by  $n - 1$  holds a.e. on  $E$ . By the inductive hypothesis,  $f(x)$  is Lipschitz of order  $n - 1$  at a.e.  $x \in E$ . By Theorem 1.11,  $f$  has  $n - 1$  Peano differentials at a.e.  $x \in E$ . By Lemma 3.2,  $f$  is Lipschitz of order  $n$  at a.e.  $x \in E$ . ■

This finishes the proof of Theorem 1.14, which in turn completes the proof of the main Theorem 1.13. The proof depends on the fact that the generalized Riemann differentiation corresponding to the difference  $\tilde{\Delta}_n$  is also almost everywhere equivalent to  $n$ th Peano differentiation. Any  $n$ th order generalized Riemann derivative or generalized Lipschitz condition which can be slid ( $h \mapsto h - \tau$ ) and/or dilated ( $h \mapsto \alpha h$ ) into either Riemann differentiation or the generalized Riemann differentiation corresponding to the difference  $\tilde{\Delta}_n$  is also equivalent almost everywhere to  $n$ th order Peano differentiation.

The following conjecture is probably true, even in  $d$  dimensions.

**CONJECTURE 3.4.** Let  $n$  and  $e$  be nonnegative integers, and suppose that for every  $x$  in a measurable set  $E \subset \mathbb{R}^2$ , the measurable function  $f(x) : \mathbb{R}^2 \rightarrow \mathbb{R}$  satisfies

$$\sum_{i=0}^{n+e} A_i f(x + a_i h) = O(\|h\|^n),$$

where  $\sum_{i=0}^{n+e} A_i a_i^j = n! \delta_{jn}$  for  $j = 0, 1, \dots, n$ . Then  $f(x)$  is  $n$  times Peano differentiable at a.e.  $x \in E$ .

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