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# Hereditarily non-weakly chainable continua in products of plane continua

by

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**Abstract.** We prove that the product of any two non-degenerate plane continua contains a non-degenerate hereditarily non-weakly chainable continuum. Consequently, in the product of two pseudo-arcs, there exists a non-degenerate subcontinuum containing no pseudo-arc. This answers a question by D. P. Bellamy.

**1. Introduction.** The class $\mathcal{K}$ of continua containing only singletons as continuous images of the pseudo-arc is much smaller than the class of continua containing no arcs. Nevertheless, we show that the product of any two non-degenerate plane continua contains a non-degenerate member of $\mathcal{K}$. The class $\mathcal{K}$ is also the class of continua containing only singletons as weakly chainable subcontinua (see [8] and [15]).

Throughout this paper the letter $P$ will denote the pseudo-arc.

The pseudo-arc is one of the most intriguing continua and is related to some of the most important problems in continuum theory. Very recently [11], it has been shown that there are only three non-degenerate homogeneous continua in the plane, namely: the circumference, the pseudo-arc and the circle of pseudo-arcs. Moreover, the pseudo-arc is hereditarily equivalent, that is: $P$ is homeomorphic to each of its non-degenerate subcontinua, and it is an open problem to determine if there exists another hereditarily equivalent continuum besides the arc and the pseudo-arc. For information about the pseudo-arc, the reader can see the excellent survey [16].

The product $P \times P$ also has several singular properties and there are a number of related interesting open problems:

(a) Each autohomeomorphism of $P \times P$ is a product of homeomorphisms or a product of homeomorphisms followed by a permutation of the coor-

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dinates [4]. This result has been extended in two directions: to any power (finite or countable) of $P$ [3], and to embeddings from $P \times P$ into $P \times P$ [6]. It is not known if the result for embeddings can be extended to $P^n$ for $n \geq 3$.

(b) It is not known if for each map $f : P \times P \to P$, either $f$ is constant on each fiber of the form $P \times \{x\}$, or $f$ is constant on each fiber of the form $\{x\} \times P$.

(c) The product $P \times P$ is aposyndetic but the intersection of each pair of subcontinua of $P \times P$, with non-empty interior, is non-empty [9].

(d) Each subcontinuum $M$ of $P \times P$ with onto projections on each of the factors has arbitrarily small open connected neighborhoods in $P \times P$. This means that, for each open subset $U$ of $P \times P$ with $M \subset U$, there is an open connected subset $V$ of $P \times P$ such that $M \subset V \subset U$ [5] (see [12] and [13] for more information about this topic).

(e) Does any retraction $r : P \times P \to \varDelta = \{(x, x) \in P \times P : x \in P\}$ have the form of either $r(x, y) = (x, x)$ for all $(x, y)$, or $r(x, y) = (y, y)$ for all $(x, y)$? [17, Question 20].

(f) Wayne I. Lewis has announced that any hereditarily indecomposable plane continuum can be embedded in $P \times P$, and in fact, any hereditarily indecomposable continuum in $\mathbb{R}^n$ can be embedded in $P^n$.

(g) The product $P \times P$ has been useful for constructing homogeneous continua with interesting decomposition properties [19], [20].

It is not easy to construct non-obvious continua inside $P \times P$. So, David P. Bellamy has asked if, for each $n \in \mathbb{N}$, every non-degenerate subcontinuum of $P^n$ contains a pseudo-arc [2, Question 18]. This question was the main motivation for this paper. We answer Bellamy's question by showing that each product of two non-degenerate plane continua contains a non-degenerate hereditarily non-weakly chainable subcontinuum. In particular, there exists a non-degenerate subcontinuum of $P \times P$ containing no homeomorphic copies of $P$.

In this paper, a *continuum* is a non-empty compact connected metric space. Recall that the pseudo-arc is the only non-degenerate chainable and hereditarily indecomposable continuum. By [8] and [15], we can define a continuum $X$ to be *weakly chainable* if it is the continuous image of $P$; note that being weakly chainable is a topological invariant property. A continuum $X$ is *hereditarily non-weakly chainable* if it belongs to the class $\mathcal{K}$, in other words: the only weakly chainable subcontinua of $X$ are the one-point sets.

## 2. An auxiliary continuum.

A *ray* is a space homeomorphic to $[0, \infty)$. A *map* is a continuous function. Let $S^1$ be the unit circle, centered at the origin, in the Euclidean plane $\mathbb{R}^2$. Let $e : \mathbb{R} \to S^1$ be the exponential map given by $e(t) = (\cos 2\pi t, \sin 2\pi t)$.

Let

$$
\begin{aligned}
R_1 &= \{(1+1/t)e(t) \in \mathbb{R}^2 : t \in [1,\infty)\},\\
R_2 &= \{(1+1/t)e(t+1/2) \in \mathbb{R}^2 : t \in [1,\infty)\},\\
Y &= R_1 \cup S^1 \cup R_2 \subset [-2,2]^2.
\end{aligned}
$$

Then $Y$ is the union of two compactifications of $[0,\infty)$ having $S^1$ as the remainder. Notice that $R_1 \cap R_2 = \emptyset$. The points $2e(0) = (2,0)$ and $2e(1/2) = (-2,0)$ will be called the *end points* of $Y$.

Given a continuum $W$, a map $f : W \to S^1$ *can be lifted* if there exists a map $h : W \to \mathbb{R}$ such that $f = e \circ h$. By [7, Théorème 1], $f$ can be lifted if and only if $f$ is homotopic to a constant map. The continuum $W$ has *property* (b) provided that each map $f : W \to S^1$ can be lifted (or $f$ is homotopic to a constant map). Recall that $W$ is *chainable* provided that for each $\varepsilon > 0$ there exists a finite open cover $\mathcal{U} = \{U_1, \ldots, U_n\}$ of $W$ such that $U_i \cap U_j \neq \emptyset$ if and only if $|i-j| \leq 1$ and $\operatorname{diam}(U_i) < \varepsilon$ for each $i \in \{1, \ldots, n\}$. It is easy to show that chainable continua have property (b).

Lemma 2.1. *If $W$ is a continuum with property* (b), *then for each map $f : W \to Y_1 = R_1 \cup S^1$, either $f(W) \subset R_1$ or $f(W) \subset S^1$.*

*Proof.* Suppose that $f(W) \cap R_1 \neq \emptyset$. It suffices to show $f(W) \subset R_1$. Fix an $x_0 \in W$ with $f(x_0) \in R_1$, and let $A = f^{-1}(R_1)$ and $B = f^{-1}(S^1)$. Define $\alpha(t) = 1 + 1/t$ for $t \geq 1$ and $\alpha(t) = 3 - t$ for $t < 1$, and note $\alpha(t) \to \infty$ as $t \to -\infty$, and $\alpha : \mathbb{R} \to (1,\infty)$ is a homeomorphism. Expand the ray $R_1$ to the double ray $R_0 = \{\alpha(t)e(t) \in \mathbb{R}^2 : t \in \mathbb{R}\}$. Note that the map $\beta : \mathbb{R} \to R_0$ given by $\beta(t) = \alpha(t)e(t)$ is a homeomorphism.

Let $r : Y_1 \to S^1$ be the natural retraction $r(z) = z/|z|$. Since $W$ has property (b), there exists a map $h : W \to \mathbb{R}$ such that $r \circ f = e \circ h$. Note that this equality holds for any map $h_k(t) = h(t) + k$, where $k$ is an integer.

Define $g : A \to \mathbb{R}$ by $g(x) = \beta^{-1}(f(x)) - h(x)$. Then $g$ is continuous. Given $x \in A$, let $t = \beta^{-1}(f(x))$. Then

$$e(g(x)) = \frac{e(t)}{e(h(x))} = \frac{r(\beta(t))}{e(h(x))} = 1.$$

Hence, $g$ takes only integer values. Thus, without loss of generality, we assume that $g(x_0) = 0$, or equivalently $f(x_0) = \beta(h(x_0))$.

Let $A_0 = g^{-1}(0)$ and $A_1 = A - A_0$. Then $A = A_0 \cup A_1$, $A_1 = g^{-1}(\{k : k \text{ is an integer and } k \neq 0\})$, and $A_0$ and $A_1$ are closed in $A$. Thus, $A_1 \cup B$ is closed in $W$.

Note that the set $\beta(h(W))$ is a subcontinuum of $R_0$ having positive distance from $S^1$, and $f(A_0) \subset \beta(h(W))$. Thus $A_0$ is closed in $A$ and has no limit points in $B$. Therefore, $A_0$ is closed in $W$. Thus $A_0$ and $A_1 \cup B$ are closed disjoint subsets of $W$ and $W = A_0 \cup (A_1 \cup B)$. We have $A_0 \neq \emptyset$

because $x_0 \in A_0$. Thus $A_1 \cup B = \emptyset$ by the connectedness of $W$. Hence $f(W) \subset R_1$. ∎

Lemma 2.2. *If $W$ is a continuum with property* (b), *then for each map $f : W \to Y$ there exists $i \in \{1, 2\}$ such that $f(W) \subset S^1 \cup R_i$.*

*Proof.* Consider the retraction $r : Y \to S^1 \cup R_1$ given by $r(z) = z/|z|$ if $z \in R_2$, and $r(z) = z$ if $z \notin R_2$. By Lemma 2.1, either $r(f(W)) \subset R_1$ or $r(f(W)) \subset S^1$. In the first case we are done. In the second case, $f(W) \cap R_1 = \emptyset$, so $f(W) \subset S^1 \cup R_2$. ∎

Now we will construct a continuum $Z$ in $[0, 1]^2$ such that each of its subcontinua contains subcontinua that "resemble" $Y$. We will use a technique of Z. Janiszewski who constructed a hereditarily decomposable plane continuum without arcs. Janiszewski's technique has also been used in [10], and it is different from the one in [21].

We start by taking the simple closed curve $Z_1 = S^1 \subset [-1, 1]^2$. Choose a countable dense subset $D_1 = \{a_1, a_2, \ldots\}$ of $Z_1$.

Construct $Z_2$ by blowing up each $a_i$ to a simple closed curve $S_i$. This can be done by induction as follows. Choose a subarc $J$ of $Z_1$ with end points $p$ and $q$ such that $a_1 \in J - \{p, q\}$. Replace $J$ by a copy $Y_1$ of $Y$ in such a way that the end points of $Y_1$ are the points $p$ and $q$, and $Y_1 \cap (Z_1 - J) = \emptyset$. Let $Q_1 = Y_1 \cup (Z_1 - J)$ and let $S_1$ be the simple closed curve contained in $Y_1$; we assume that $\operatorname{diam}(S_1) < 1/2$. We will say that $Q_1$ is obtained from $Z_1$ by blowing up the point $a_1$ to a simple closed curve $S_1$. Notice that the continuum obtained from $Q_1$ by shrinking $S_1$ to a point is a simple closed curve. Now construct the continuum $Q_2$ by blowing up the point in $Q_1$ that corresponds to $a_2$ to a simple closed curve $S_2$ with $\operatorname{diam}(S_2) < 1/2^2$. Continue inductively until all points $a_i$ have been blown up to simple closed curves $S_i$ with $\operatorname{diam}(S_i) < 1/2^i$, to finally obtain a continuum $Z_2$. Applying the theorem of Anderson–Choquet [1, Theorem 1] it can be shown that this can be done in such a way that $Z_2$ is embeddable in $[0, 1]^2$. We denote by $f_1 : Z_2 \to Z_1$ the map that shrinks each $S_i$ into the point $a_i$. Then $f_1$ is a monotone map.

In the second step construct a continuum $Z_3$ from $Z_2$ in such a way that each simple closed curve $S_i$ is replaced by a copy of $Z_2$. That is, in each $S_i$ choose a countable dense subset $D_i$ and blow up each of its points to a simple closed curve. In this way, $Z_3$ contains countably many disjoint copies of $Z_2$, and we denote by $f_2 : Z_3 \to Z_2$ the map that shrinks to a point each of the newly created simple closed curves. Notice that $f_2$ is a monotone map.

Inductively, suppose that $Z_n$ has been constructed. Construct $Z_{n+1}$ by replacing each simple closed curve in $Z_n$ by a topological copy of $Z_2$. Let $f_n : Z_{n+1} \to Z_n$ denote the monotone map that shrinks to a point each of the newly created simple closed curves.

Define

$$Z = \varprojlim\{Z_n, f_n\} = \Big\{(z_1, z_2, \ldots) \in \prod Z_n : z_i = f_i(z_{i+1}) \text{ for each } i \in \mathbb{N}\Big\}.$$

By the theorem of Anderson–Choquet, $Z$ can be embedded in $[0,1]^2$. We may assume that $\{(0,0),(1,1)\} \subset Z$. Thus, we will denote the points of $Z$ in two ways, namely, as points in $[0,1]^2$ and as sequences of points in $\prod Z_n$.

Now, we prove the most important property of $Z$.

Lemma 2.3. *The continuum $Z$ is hereditarily non-weakly chainable.*

*Proof.* Suppose to the contrary that there exists a non-degenerate subcontinuum $Q_0$ of $Z$ such that $Q_0$ is weakly chainable. Let $g : P \to Q_0$ be an onto mapping.

Recall that $Z = \varprojlim Z_n \subset \prod Z_n$. For each $m \in \mathbb{N}$, let $\rho_m : \prod Z_n \to Z_m$ be the projection on the $m$th coordinate. Since $Q_0$ is non-degenerate, we can define $m = \min\{i \in \mathbb{N} : \rho_i(Q_0) \text{ is non-degenerate}\}$. If $m > 1$, since $\rho_{m-1}(Q_0)$ is degenerate, $\rho_m(Q_0)$ is contained in one of the simple closed curves that are constructed in $Z_m$ by blowing up a point in $Z_{m-1}$.

Therefore, $\rho_m(Q_0)$ is contained in one simple closed curve $S \subset Z_m$. Notice that this is also true in the case $m = 1$.

Let $C(P)$ denote the hyperspace of subcontinua of $P$, endowed with the Hausdorff metric.

Fix a point $p_0 \in P$. Take a map $\sigma : [0,1] \to C(P)$ such that $\sigma(0) = \{p_0\}$, $\sigma(1) = P$ and if $0 \le s < t \le 1$, then $\sigma(s) \subset \sigma(t)$ [14, Theorem 14.6]. Since $\rho_m(g(\sigma(0)))$ is a one-point set and $\rho_m(g(\sigma(1))) = \rho_m(g(P)) = \rho_m(Q_0)$ is a non-degenerate subcontinuum of $S$, there exists $t_0 \in [0,1]$ such that $\rho_m(g(\sigma(t_0)))$ is a subarc of $S$. Let $Q_1 = \sigma(t_0)$. Then $\rho_m(g(Q_1))$ is a subarc of $S$, and $Q_1$ is a non-degenerate subcontinuum of $P$. Hence, $Q_1$ is also homeomorphic to $P$.

Let $J = \rho_m(g(Q_1))$. In the construction of $Z_{m+1}$ we replaced $S$ by a topological copy of $Z_2$. In particular, we took a countable dense subset $D = \{x_1, x_2, \ldots\}$ of $J$ and we blowed up each $x_i$ to a simple closed curve $R_i$, to obtain the continuum $f_m^{-1}(J)$. We may assume that $x_1$ is not an end point of $J$. Then $J - \{x_1\} = M_2 \cup M_3$, where $M_2$ and $M_3$ are rays.

Let $T$ be the quotient continuum obtained by shrinking to a point each of the simple closed curves $R_2, R_3, \ldots$. Let $\psi : f_m^{-1}(J) \to T$ be the quotient map. Notice that $T$ is homeomorphic to the continuum $Y$ of Lemma 2.2, where $R_1$ corresponds to $S^1$, and the rays $R_1^*$ and $R_2^*$ correspond to the rays $\psi(f_m^{-1}(M_1))$ and $\psi(f_m^{-1}(M_2))$.

Consider the map $\psi \circ \rho_{m+1} \circ g : Q_1 \to T$. Given a point $q \in Q_1$, let $u = (u_1, u_2, \ldots) = g(q)$. Then $u_m = \rho_m(g(q)) \in J$, so $u_{m+1} \in f_m^{-1}(u_m) \in f_m^{-1}(J)$

and $u_{m+1} = \rho_{m+1}(g(q))$. Hence, we can apply $\psi$ to the element $\rho_{m+1}(g(q))$. This proves that $\psi \circ \rho_{m+1} \circ g$ is well defined.

Take a point $v \in M_1 - D$. Since to construct $Z_{n+1}$, we did not blow up $v$, the set $f_m^{-1}(v)$ is a singleton, say $f_m^{-1}(v) = \{v_1\}$. Since $v \in J = \rho_m(g(Q_1))$, there exists a point $t = (t_1, t_2, \ldots) \in g(Q_1)$ such that $v = t_m$. Since $f_m(t_{m+1}) = t_m$, we have $t_{m+1} = v_1$. Let $t = g(q_v)$, where $q_v \in Q_1$. Thus, $(\psi \circ \rho_{m+1} \circ g)(q_v) = \psi(\rho_{m+1}(t)) = \psi(v_1) \in \psi(f_m^{-1}(M_1))$. This proves that $(\psi \circ \rho_{m+1} \circ g)(Q_1)$ intersects the ray $\psi(f_m^{-1}(M_1))$. Similarly, $(\psi \circ \rho_{m+1} \circ g)(Q_1)$ intersects the ray $\psi(f_m^{-1}(M_2))$.

Therefore, $Q_1$ has property (b) and $\psi(\rho_{m+1}(g(Q_1)))$ intersects both rays of the continuum $T$. This contradicts 2.2 and completes the proof of Lemma 2.3. ■

## 3. A general construction.

We will use the following version of the Mountain Climbing Theorem [18, Theorem 2].

Lemma 3.1. *Suppose that* $g, h : [0,1] \to [0,1]$ *are piecewise linear maps such that* $g(0) = 0 = h(0)$ *and* $g(1) = 1 = h(1)$. *Then there exist piecewise linear maps* $\alpha, \beta : [0,1] \to [0,1]$ *such that* $\alpha(0) = 0 = \beta(0)$, $\alpha(1) = 1 = \beta(1)$ *and* $g \circ \alpha = h \circ \beta$.

In this section we consider subcontinua $N_1$ and $N_2$ of $[0,1]^2$ such that $\{(0,0),(1,1)\} \subset N_1 \cap N_2$.

We consider an arbitrary continuum $E$ in $[0,1]^2$ such that $(0,0), (1,1) \in E$, although we will ultimately take $E$ homeomorphic to $Z$.

We construct a subcontinuum $X$ of $N_1 \times N_2$, depending on $E$, that will be a limit of continua.

Let $H$ denote the Hausdorff metric in the hyperspace of subcontinua of $[0,1]^2$.

For each $n \in \mathbb{N}$, take piecewise linear maps $g, h, k : [0,1] \to [0,1]^2$ such that $g(0) = h(0) = k(0) = (0,0)$, $g(1) = h(1) = k(1) = (1,1)$, $H(\operatorname{Im} g, E) < 1/n$, $H(\operatorname{Im} h, N_1) < 1/n$ and $H(\operatorname{Im} k, N_2) < 1/n$. We write $h = (h_1, h_2)$, $g = (g_1, g_2)$ and $k = (k_1, k_2)$.

Since $g_1(0) = 0 = h_1(0)$ and $g_1(1) = 1 = h_1(1)$, by Lemma 3.1, there exist piecewise linear maps $\alpha, \beta : [0,1] \to [0,1]$ such that $\alpha(0) = 0 = \beta(0)$, $\alpha(1) = 1 = \beta(1)$ and $h_1 \circ \alpha = g_1 \circ \beta$.

We can apply Lemma 3.1 to the maps $\beta$ and the identity in $[0,1]$, so there exist piecewise linear maps $\gamma, \lambda : [0,1] \to [0,1]$ such that $\gamma(0) = 0 = \lambda(0)$, $\gamma(1) = 1 = \lambda(1)$ and $\beta \circ \gamma = \lambda$.

We can also apply Lemma 3.1 to the maps $k_1$ and $g_2 \circ \lambda$, so there exist piecewise linear maps $\delta, \eta : [0,1] \to [0,1]$ such that $\delta(0) = 0 = \eta(0)$, $\delta(1) = 1 = \eta(1)$ and $k_1 \circ \delta = g_2 \circ \lambda \circ \eta$.

Let $\varphi : [0,1] \to [0,1]^4$ be given by

$$\begin{aligned}\varphi &= (h_1 \circ \alpha \circ \gamma \circ \eta, h_2 \circ \alpha \circ \gamma \circ \eta, k_1 \circ \delta, k_2 \circ \delta)\\ &= (g_1 \circ \beta \circ \gamma \circ \eta, h_2 \circ \alpha \circ \gamma \circ \eta, g_2 \circ \lambda \circ \eta, k_2 \circ \delta)\\ &= (g_1 \circ \beta \circ \gamma \circ \eta, h_2 \circ \alpha \circ \gamma \circ \eta, g_2 \circ \beta \circ \gamma \circ \eta, k_2 \circ \delta).\end{aligned}$$

Let $X_n = \operatorname{Im} \varphi$. Notice that $X_n$ is a subcontinuum of $[0,1]^4$.

Given $i, j \in \{1,2,3,4\}$ with $i \neq j$, define $\pi_{i,j} : [0,1]^4 \to [0,1]^2$ by

$$\pi_{i,j}(x_1, x_2, x_3, x_4) = (x_i, x_j).$$

Then $H(\pi_{1,2}(X_n), N_1) = H(\operatorname{Im}(\pi_{1,2} \circ \varphi), N_1) = H(\operatorname{Im} h, N_1) < 1/n$. Thus, $H(\pi_{1,2}(X_n), N_1) < 1/n$. Similarly, $H(\pi_{3,4}(X_n), N_2) < 1/n$ and $H(\pi_{1,3}(X_n), E) < 1/n$.

By the compactness of the hyperspace of subcontinua of $[0,1]^4$, there exist a subcontinuum $X$ of $[0,1]^4$ and a subsequence $\{X_{n_l}\}_{l=1}^{\infty}$ such that

$$X = \lim X_{n_l}.$$

By the continuity of the projections, $\pi_{1,2}(X) = N_1$, $\pi_{3,4}(X) = N_2$ and $\pi_{1,3}(X) = E$. Thus, $X \subset N_1 \times N_2$.

Now, we show the most important property of $X$.

LEMMA 3.2. *Suppose for every $i \in \{1,2\}$ and $t \in [0,1]$, $(\{t\} \times [0,1]) \cap N_i$ is totally disconnected. Let $Q$ be a non-degenerate subcontinuum of $X$. Then $\pi_{1,3}(Q)$ is a non-degenerate subcontinuum of $E$.*

*Proof.* Suppose to the contrary that $\pi_{1,3}(Q) = \{(x_1, x_3)\}$ for some $(x_1, x_3) \in E$. Let $\pi_1 : [0,1]^2 \to [0,1]$ be the projection on the first coordinate.

We have $\pi_1(\pi_{3,4}(Q)) = \{x_3\}$. This yields $\pi_{3,4}(Q) \subset (\{x_3\} \times [0,1]) \cap N_2$. The hypothesis implies that $\pi_{3,4}(Q)$ is totally disconnected. Since $\pi_{3,4}(Q)$ is connected, we conclude that $\pi_{3,4}(Q)$ is degenerate. Similarly, $\pi_{1,2}(Q)$ is degenerate. This implies that $Q$ is degenerate, a contradiction. ∎

## 4. Main result.

Consider the continuum $X$ constructed by taking a continuum $E$ homeomorphic to $Z$ and satisfying $\{(0,0),(1,1)\} \subset E \subset [0,1]^2$.

THEOREM 4.1. *Let $M_1$ and $M_2$ be non-degenerate subcontinua of $[0,1]^2$. Then $M_1 \times M_2$ contains a non-degenerate hereditarily non-weakly chainable subcontinuum.*

*Proof.* For each $i \in \{1,2\}$, we choose a subcontinuum $N_i$ of $M_i$ as follows. If $M_i$ contains no arcs, set $N_i = M_i$, otherwise fix an arc $N_i$ contained in $M_i$. In order to prove the theorem, it is enough to construct the continuum $X$ in $N_1 \times N_2$.

For each $i \in \{1,2\}$, if $N_i$ is an arc, we assume that $N_i = \{(t,t) \in [0,1]^2 : t \in [0,1]\}$, and if $M_i$ does not contain an arc, we may assume that $\{(0,0),(1,1)\} \subset N_i$.

Then $N_1$ and $N_2$ satisfy the hypothesis of Lemma 3.2. Consider the continuum $X$ constructed for Lemma 3.2.

Suppose that $X$ contains a non-degenerate weakly chainable continuum $Q$. By Lemma 3.2, $\pi_{1,3}(Q)$ is non-degenerate, so $\pi_{1,3}(Q)$ is a weakly chainable subcontinuum of $Z$. This contradicts Lemma 2.3. ∎

Corollary 4.2. *There exists a non-degenerate subcontinuum of $P \times P$ that does not contain pseudo-arcs.*

Problem 4.3. *Let $M_1$ and $M_2$ be non-degenerate continua. Does the product $M_1 \times M_2$ contain a non-degenerate hereditarily non-weakly chainable subcontinuum?*

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