ANNALES
POLONICI MATHEMATICI
117.3 (2016)

Existence and uniqueness of solutions for a quasilinear evolution equation in an Orlicz space

ZHENG ZHOU (Xiamen) and FEI FANG (Beijing)

Abstract. We consider the following quasilinear evolution equation in an Orlicz space:

$$u_t = \operatorname{div}(a(|\nabla u|)\nabla u) + f(x,t,u),$$

where $a \in C^1(\mathbb{R})$ and $f \in C^1(\overline{\Omega} \times [0,T] \times \mathbb{R})$. We use the difference method to transform the evolution problem to a sequence of elliptic problems. Then by making some uniform estimates for these elliptic problems, we obtain the existence of global solutions for the evolution problem. Uniqueness is also proved.

1. Introduction. Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with Lipschitz continuous boundary $\partial\Omega$. Consider the following quasilinear evolution equation:

$$\begin{cases} u_t = \operatorname{div}(a(|\nabla u|)\nabla u) + f(x,t,u), & x \in \Omega,\ 0 < t < T, \\ u|_{\Gamma_T} = 0, \quad u|_{t=0} = u_0, \end{cases} \tag{1.1}$$

where $\Gamma_T = \partial\Omega \times [0,T]$, $a \in C^1(\mathbb{R})$ and $f \in C^1(\bar{\Omega} \times [0,T] \times \mathbb{R})$. When $a(t) = |t|^{p-2}$, problem (1.1) is the well known evolution p-Laplace equation. In recent years, there have been a large number of papers on the existence, uniqueness and regularity of solutions of the evolution p-Laplace equation (see [D, Z1, Z2] and the references therein). For the $p(x,t)$-Laplace equation

$$u_t = \operatorname{div}(|\nabla u|^{p(x,t)-2}\nabla u) + f(x,t,u)$$

the authors of [AS] established the existence and uniqueness results with the exponent $p(x,t)$ satisfying the so-called logarithmic Hölder continuity condition. In our main result, there is no need to assume logarithmic Hölder continuity. Recently, the authors of [LGYC] studied the $p(x,t)$-Laplace equation and adopted the difference method and some new techniques to obtain the existence and uniqueness of solutions.

2010 *Mathematics Subject Classification*: Primary 35K55; Secondary 35K65.
Key words and phrases: Orlicz–Sobolev space, evolution equation, difference method, weak solution.
Received 11 October 2015; revised 25 April 2016.
Published online 4 October 2016.

DOI: 10.4064/ap3861-4-2016

In this paper, problem (1.1) will be studied in an Orlicz–Sobolev space setting. The corresponding elliptic problem in an Orlicz–Sobolev space has been considered in recent years. The reader is referred to [BMR, FT, FZu, GLMS, RR2] and the references therein for more results on the existence and regularity of solutions.

The function P (see Section 2) is allowed to belong to a larger class, which includes the special cases appearing in physical models, for instance:

(1) nonlinear elasticity: $P(t) = (1+t^2)^\gamma - 1$, $\gamma > 1/2$;
(2) plasticity: $P(t) = t^\alpha(\log(1+t))^\beta$, $\alpha > 1$, $\beta > 0$;
(3) generalized Newtonian fluids: $P(t) = \int_0^t s^{1-\alpha}(\sinh^{-1} s)^\beta\, ds$, $0 \le \alpha \le 1$, $\beta > 0$.

For details, see [BAH, FL, FO, FN].

The outline of this paper is the following: In Section 2, we present some necessary preliminary knowledge on Orlicz–Sobolev spaces, and the main result. In Section 3, we prove the existence of weak solutions to some difference equations related to problem (1.1). Section 4 is devoted to proving the global existence and uniqueness of solutions to problem (1.1).

2. Preliminaries and the main result. As in [CM, CGMS, FIN, TF], we can construct an Orlicz–Sobolev space setting for problem (1.1). Let the function

$$p(t) := \begin{cases} a(|t|)t, & t \neq 0, \\ 0, & t = 0, \end{cases} \tag{2.1}$$

be an increasing homeomorphism from $\mathbb{R}$ onto itself (such functions are called *Young* or *N-functions*). If we set

$$P(t) := \int_0^t p(s)\, ds, \qquad \tilde{P}(t) = \int_0^t p^{-1}(s)\, ds,$$

then P and $\tilde{P}$ are complementary N-functions (see [AF, RR1, RR2]).

In order to construct an Orlicz–Sobolev space setting for problem (1.1), we impose the following conditions on $p(t)$:

$$(\mathrm{p}_0) \qquad a \in C^1(0,\infty), \qquad a(t) > 0 \quad \text{for } t > 0,$$

$$(\mathrm{p}_1) \qquad 2 < p^- := \inf_{t>0} \frac{tp(t)}{P(t)} \le p^+ := \sup_{t>0} \frac{tp(t)}{P(t)} < \infty,$$

$$(\mathrm{p}_2) \qquad 1 < a^- := \inf_{t>0} \frac{tp'(t)}{p(t)} \le a^+ := \sup_{t>0} \frac{tp'(t)}{p(t)} < \infty.$$

Under condition (p_1), the function $P(t)$ satisfies the Δ_2-*condition*, i.e.

$$P(2t) \le kP(t), \qquad t > 0,$$

for some constant $k > 0$ (see [AF, p. 265]). Under conditions (p_0) and (p_1), the Orlicz space L^P coincides with the set of (equivalence of classes of) measurable functions $u : \Omega \to \mathbb{R}$ such that

$$\int_\Omega P(|u|)\,dx < \infty. \tag{2.2}$$

The space $L^P(\Omega)$ is a Banach space endowed with the Luxemburg norm

$$|u|_P := \inf\Big\{k > 0 : \int_\Omega P(|u|/k)\,dx < 1\Big\}.$$

We shall denote by $W^{1,P}(\Omega)$ the corresponding Orlicz–Sobolev space with the norm

$$\|u\|_{W^{1,P}(\Omega)} := |u|_P + |\nabla u|_P.$$

We denote by $W_0^{1,P}(\Omega)$ the closure of C_0^∞ in $W^{1,P}(\Omega)$.

Denote

$$p_*^- = \begin{cases} \dfrac{Np^-}{N-p^-} & \text{if } N > p^-, \\ \dfrac{N+p^-}{N}p^- & \text{if } N \le p^-. \end{cases} \tag{2.3}$$

In this paper, the following equivalent norm on $W_0^{1,P}(\Omega)$ will be used:

$$\|u\| := \inf\Big\{k > 0 : \int_\Omega P(|\nabla u|/k)\,dx < 1\Big\}.$$

The reader is referred to [AF, RR2] for more details on Orlicz–Sobolev spaces. In the proofs we shall use the following results.

Lemma 2.1 (see [AF, RR2]). *Under conditions* (p_0) *and* (p_1), *the spaces* $L^P(\Omega)$, $W_0^{1,P}(\Omega)$ *and* $W^{1,P}(\Omega)$ *are separable and reflexive Banach spaces.*

Lemma 2.2. *Under conditions* (p_0), (p_1) *and* (p_2):

(1) *if* $0 < t < 1$, *then* $P(1)t^{p^+} \le P(t) \le P(1)t^{p^-}$;
(2) *if* $t > 1$, *then* $P(1)t^{p^-} \le P(t) \le P(1)t^{p^+}$.

Lemma 2.3 (see [FIN]). *Let* $\rho(u) = \int_\Omega P(u)\,dx$. *Then:*

(1) *if* $|u|_P < 1$, *then* $|u|_P^{p^+} \le \rho(u) \le |u|_P^{p^-}$;
(2) *if* $|u|_P > 1$, *then* $|u|_P^{p^-} \le \rho(u) \le |u|_P^{p^+}$;
(3) *if* $0 < t < 1$, *then* $t^{p^+}P(u) \le P(tu) \le t^{p^-}P(u)$;
(4) *if* $t > 1$, *then* $t^{p^-}P(u) \le P(tu) \le t^{p^+}P(u)$.

Lemma 2.4 (see [GLMS, RR2]). *Assume that* $A(t)$ *and* $\tilde{A}(t)$ *are complementary* N*-functions. We have*

(1) *Young inequality:* $uv \le A(u) + \tilde{A}(v)$;
(2) *Hölder inequality:* $|\int_\Omega u(x)v(x)\,dx| \le 2|u|_A|u|_{\tilde{A}}$;

(3) $\tilde{A}(A(u)/u) \le A(u)$;
(4) $\tilde{A}_*(A_*(u)/u) \le A_*(u)$.

REMARK 2.1. Since problem (1.1) has inhomogeneous nonlinearities, Lemmas 2.1–2.4 will be used to overcome the nonhomogeneity.

DEFINITION 2.1. A function u is said to be a *weak solution* of (1.1) if

- $u \in L^2(Q_T)$, $f(\cdot,\cdot,u) \in L^1(Q_T)$, $D_i u \in L^P(Q_T)$,
- $u = 0$ on $\partial\Omega \times (0,T)$ in the sense of traces,

$$\iint_{Q_T} \left(u\frac{\partial \varphi}{\partial t} - a(|\nabla u|)\nabla u \cdot \nabla\varphi + f\varphi \right) dx\, dt = 0 \tag{2.4}$$

for all $\varphi \in C_0^\infty(Q_T)$ and

- $u = 0$ on $\partial\Omega \times (0,T)$ in the sense of traces,

$$\lim_{t\to 0} \int_\Omega (u(x,t) - u_0(x))\psi\, dx = 0 \tag{2.5}$$

for all $\psi \in C_0^\infty(Q_T)$, where $Q_T = \Omega \times (0,T)$.

Next we assume the following condition:

(A) $\quad f \in C^1(\overline{\Omega} \times [0,T] \times \mathbb{R}) \quad$ and $\quad |f(x,t,z)| \le C_0(\phi(x,t) + |z|^\alpha)$

where $\phi \ge 0$, $\phi \in L^r(\Omega \times (0,T))$, $r > (N + p^-)/p^-$, and $C_0 > 0$, $\alpha \ge 0$ are constants.

Our main result is the following.

THEOREM 2.1. *Let $u_0 \in L^\infty(\Omega) \cap W_0^{1,P}(\Omega)$, suppose* (A) *holds, and assume that*

(B) $\quad \alpha < p^- - 1 \quad$ (*or $\alpha = p^- - 1$ with Ω sufficiently small*),

where $|\Omega|$ denotes the Lebesgue measure of Ω. Then for any $T > 0$, there exists a unique weak solution u of (1.1) *in the Orlicz–Sobolev sense such that*

$$u \in L^\infty(Q_T) \cap L^\infty(0,T; W_0^{1,P}(\Omega)), \qquad u_t \in L^2(\Omega \times (0,T)).$$

REMARK 2.2. If the assumption in (B) that Ω is sufficiently small is replaced by the assumption that C_0 is sufficiently small in (A), then the conclusions of Theorem 2.1 still hold.

3. Difference equation. Consider the difference equation corresponding to problem (1.1):

$$\begin{cases} \dfrac{1}{h}(u_i - u_{i-1}) = \operatorname{div}(a(|\nabla u_i|)\nabla u_i) + \dfrac{1}{h}\displaystyle\int_{ih}^{(i+1)h} f(x,\tau,u_i)\, d\tau, & x \in \Omega, \\ u_i|_{\partial\Omega} = 0, \quad i = 1, 2, \ldots, & \end{cases} \tag{3.1}$$

Set

$$F^i(x,u)=\int\limits_{u_{i-1}}^{u}\left(\frac{1}{h}\int\limits_{ih}^{(i+1)h} f(x,\tau,s)\,d\tau\right)ds, \tag{3.2}$$

$$\mathcal{P}(u)=\int\limits_{\Omega} P(|\nabla u|)\,dx,\qquad \forall u\in W_0^{1,P}(\Omega), \tag{3.3}$$

and

$$\psi^i(u)=\int\limits_{\Omega} P(|\nabla u|)dx-\int\limits_{\Omega} F^i(x,u)\,dx + \frac{1}{2h}\int\limits_{\Omega}(u-u_{i-1})^2dx,\qquad i=1,2,\ldots, \tag{3.4}$$

which is the functional corresponding to (3.1), where $h>0$ is a constant.

LEMMA 3.1 (see [FIN, GLMS]). *The functional* $\mathcal{P}\in C^1(W_0^{1,P}(\Omega),\mathbb{R})$ *is convex and sequentially weakly lower semicontinuous, and*

$$\mathcal{P}'(u)\phi=\int\limits_{\Omega} p(\nabla u)\nabla\phi\,dx,\qquad \forall u,\phi\in W_0^{1,P}(\Omega).$$

Moreover, the mapping $\mathcal{P}':W_0^{1,P}(\Omega)\to W_0^{1,P}(\Omega)^*$ *is a bounded homeomorphism, and is of type* (S^+), *that is,*

$$u_n\rightharpoonup u \quad\text{and}\quad \limsup_{n\to\infty}\mathcal{P}'(u_n)(u_n-u)\le 0 \quad \text{imply that}\quad u_n\to u \text{ in } W_0^{1,P}(\Omega). \tag{3.5}$$

LEMMA 3.2. *Assume* (A) *and* (B) *hold, and* $u_{i-1}\in L^{p_*^-}(\Omega)$. *Then the functional* $\psi^i(u)$ *achieves its minimum on the set*

$$S=W_0^{1,P}(\Omega). \tag{3.6}$$

Proof. We will show that $\psi^i(u)$ satisfies the conditions which ensure the existence of a minimum on S.

STEP 1. *S is weakly closed.*

By Lemma 2.1, we know that $W_0^{1,P}(\Omega)$ is a reflexive Banach space, and thus by Mazur's theorem it is weakly closed.

STEP 2. *$\psi^i(u)$ satisfies the coerciveness conditions.*

By condition (A) we have

$$\psi^i(u)\ge\int\limits_{\Omega} P(|\nabla u|)\,dx - C_0\int\limits_{\Omega}\left(\frac{1}{h}\int\limits_{ih}^{(i+1)h}\phi(x,\tau)\,d\tau\right)|u-u_{i-1}|\,dx - C_1\int\limits_{\Omega}(|u|^{\alpha+1}+|u_{i-1}|^{\alpha+1})\,dx+\frac{1}{2h}\int\limits_{\Omega}(u-u_{i-1})^2\,dx. \tag{3.7}$$

We first estimate the second term on the right-hand side. Denote $r_1 = (N+p^-)/p^- < r$ and $r_2 = (N+p^-)/N$. By condition (A) and Hölder's inequality, we obtain

$$
\begin{aligned}
(3.8)\qquad I_1 &= C_0 \int_\Omega \biggl(\frac{1}{h}\int_{ih}^{(i+1)h} \phi(x,\tau)\,d\tau\biggr)|u-u_{i-1}|\,dx \\
&\le C_0\biggl(\int_\Omega\biggl(\frac{1}{h}\int_{ih}^{(i+1)h}\phi(x,\tau)\,d\tau\biggr)^{r_1}dx\biggr)^{1/r_1}\biggl(\int_\Omega |u-u_{i-1}|^{r_2}\biggr)^{1/r_2} \\
&\le C\biggl(\frac{1}{h}\int_\Omega\int_{ih}^{(i+1)h}\phi^{r_1}(x,\tau)\,d\tau\,dx\biggr)^{1/r_1}\biggl(\int_\Omega |u-u_{i-1}|^{r_2}\biggr)^{1/r_2} \\
&\le C\|u-u_{i-1}\|_{L^{r_2}(\Omega)} \le C(\|u\|_{L^{r_2}(\Omega)} + \|u_{i-1}\|_{L^{r_2}(\Omega)})
\end{aligned}
$$

Notice that $r_2 < p_*^-$ for $N > p^-$, so by Young's inequality we get

$$
I_1 \le \varepsilon\|u\|_{L^{p_*^-}(\Omega)} + C\|u_{i-1}\|_{L^{p_*^-}(\Omega)} + C(\varepsilon) \le \varepsilon\|u\|_{L^{p_*^-}(\Omega)} + C(\varepsilon).
$$

By the imbedding inequality and Poincaré's inequality, for all $N \ge 1$,

$$
(3.9)\qquad I_1 \le C\varepsilon\|u\|_{W^{1,P}(\Omega)} + C(\varepsilon) \le \frac{1}{4}\int_\Omega P(|\nabla u|)\,dx + C.
$$

Next, we estimate $I_2 = \int_\Omega (|u|^{\alpha+1} + |u_{i-1}|^{\alpha+1})\,dx$ in two cases.

(i) $\alpha < p^- - 1$, hence $\alpha + 1 < p^- < p_*^-$. By Young's inequality and Poincaré's inequality, it is easy to show that

$$
\begin{aligned}
(3.10)\qquad I_2 &= C_1\int_\Omega (|u|^{\alpha+1} + |u_{i-1}|^{\alpha+1})\,dx \le \epsilon\int_\Omega |u|^{p^-}\,dx + C \\
&\le \frac{1}{4}\int_\Omega |\nabla u|^{p^-}\,dx + C \le \frac{1}{4}\int_\Omega P(|\nabla u|)\,dx + C.
\end{aligned}
$$

(ii) $\alpha = p^- - 1$ and $|\Omega|$ is sufficiently small. Using Poincaré's inequality and Young's inequality, we get

$$
\begin{aligned}
(3.11)\qquad I_2 &= C_1\int_\Omega (|u|^{p^-} + |u_{i-1}|^{p^-})\,dx \le C_1\int_\Omega |u|^{p^-}\,dx + C \\
&\le \frac{1}{4}\int_\Omega |\nabla u|^{p^-}\,dx + C \le \frac{1}{4}\int_\Omega P(|\nabla u|)\,dx + C.
\end{aligned}
$$

Combining the estimates (3.7)–(3.11), we get

$$
\psi^i(u) \ge \frac{1}{4}\int_\Omega P(|\nabla u|) - C(\Omega) \ge \frac{1}{4C}|u|_{W^{1,P}(\Omega)} - C(\Omega) \to \infty.
$$

Step 3. *$\psi^i(u)$ is weakly lower semicontinuous.*

By Lemma 3.1, we know that $\int_\Omega P(|\nabla u|)\,dx$ is convex and weakly lower semicontinuous. Now, consider the functional

$$I(u) = -\int_\Omega F^i(x,u)\,dx + \frac{1}{2h}\int_\Omega (u-u_{i-1})^2\,dx.$$

Since $v_l \to v$ in $W_0^{1,P}$, for any $0<\epsilon<p^-$ we have $v_l \rightharpoonup v$ in $W_0^{1,p^- -\epsilon}$. By the Sobolev compact imbedding theorem, we easily see that $v_l \to v$ in $L^{p_\epsilon^*}$, where

$$p_\epsilon^* = \begin{cases} \dfrac{N(p^- -\epsilon)}{N-(p^- -\epsilon)} & \text{if } N > p^- -\epsilon, \\ \dfrac{N+(p^- -\epsilon)}{N}(p^- -\epsilon) & \text{if } N \le p^- -\epsilon. \end{cases}$$

For ϵ small enough, we have $p_\varepsilon^* > \max\{r/r-1,\ 2\}$. Invoking (A) we may prove that I is weakly lower semicontinuous, so the functional $\psi^i(u)$ is weakly lower semicontinuous. By the above results and a standard argument (see [B]), we know that $\psi^i(u)$ achieves its minimum on S. ∎

LEMMA 3.3. *Assume* (A) *and* (B) *hold and* $u_{i-1} \in L^{p_*^-}(\Omega)$. *Then there exists a weak solution* u_i *of* (3.1) *such that* $u_i \in W_0^{1,P}(\Omega)$.

Proof. For $0<\epsilon<1$ and $\eta \in C_0^\infty$, we have $u_i \pm \epsilon\eta \in S$, and so

$$g(\epsilon) := \psi^i(u_i+\epsilon\eta) \ge \psi^i(u_i) = g(0), \quad g(-\epsilon) = \psi^i(u_i-\epsilon\eta) \ge \psi^i(u_i) = g(0).$$

Therefore

$$\lim_{\epsilon\to 0,\,\epsilon>0} \frac{g(-\epsilon)-g(0)}{-\epsilon} \le 0, \qquad \lim_{\epsilon\to 0,\,\epsilon>0} \frac{g(\epsilon)-g(0)}{\epsilon} \le 0.$$

Plugging in the definition of g, we get

$$\int_\Omega \frac{1}{h}(u_i-u_{i-1})\eta\,dx = -\int_\Omega a(|\nabla u_i|)\nabla u_i\cdot\nabla\eta\,dx + \int_\Omega \biggl(\frac{1}{h}\int_{ih}^{(i+1)h} f(x,\tau,u_i)\,d\tau\biggr)\eta\,dx.$$

Thus u_i is a weak solution of (3.1). ∎

4. Global existence of weak solutions. First, we assume that

$$lh \le T < (l+1)h,$$

where l is an integer. Define $u^h : \Omega\times[0,\infty) \to \mathbb{R}$ such that

$$u^h(\cdot,t) = u_i \quad \text{for } t\in[ih,(i+1)h),\ i=0,1,\ldots,l, \tag{4.1}$$

where u_i is a solution obtained in Lemma 3.3. We will prove that a subsequence of u^h converges and the limiting function is a solution of (1.1).

Denote

$$\partial^{(-h)}u^h(\cdot,t)=\frac{1}{-h}(u^h(\cdot,t-h)-u^h(\cdot,t)) = \begin{cases} \frac{1}{h}(u_i-u_{i-1})(\cdot) & \text{for } t\in[ih,(i+1)h),\ i=0,1,\ldots,l,\\ 0 & \text{for } t\in[0,h),\end{cases} \tag{4.2}$$

Define

$$f^{(h)}(x,t)=\frac{1}{h}\int_{ih}^{(i+1)h} f(x,\tau,u_i(x))\,d\tau,\quad t\in[ih,(i+1)h),\ i=0,1,\ldots,l, \tag{4.3}$$

$$\phi^{(h)}(x,t)=\frac{1}{h}\int_{ih}^{(i+1)h} \phi(x,\tau)\,d\tau,\qquad t\in[ih,(i+1)h),\ i=0,1,\ldots,l, \tag{4.4}$$

It can be proved easily by using Hölder's inequality and (A) that

$$|f^h(x,t,u)|\le C_0(\phi^{(h)}+|u^h|^\alpha). \tag{A$'$}$$

and

$$\iint_{Q_T}(\phi^{(h)})^r\,dx\,dt\le\iint_{Q_T}\phi^r\,dx\,dt, \tag{4.5}$$

when $\phi\in L^r(Q_T)$ with r given in (A).

In the following, we will estimate the maximum norm of the solution by adapting the method of [LGYC, Z1].

LEMMA 4.1. *Let* (A) *and* (B) *hold, and $u_0\in L^\infty(\Omega)\cap W_0^{1,P}(\Omega)$. Then for any integer $1\le q<\infty$, there is a constant $C(q)>0$ independent of h such that*

$$\|u^h\|_{L^{q+1}(Q_T)}\le C(q),\qquad \forall h>0.$$

Proof. Let $u_+=\max\{0,u\}$ and suppose $\|u_0\|_{L^\infty(\Omega)}\le k$. Multiplying (3.1) by $(u_i-k)_+^q$ and integrating over Ω we get

$$\begin{aligned}\frac{1}{h}\int_\Omega(u_i-k)_+^{(q+1)}\,dx+p^-q\int_\Omega P(|\nabla u_i|)(u_i-k)_+^{q-1}\,dx \\ \le\frac{1}{h}\int_\Omega(u_i-k)_+^{(q+1)}\,dx+q\int_\Omega a(|\nabla u_i|)(u_i-k)_+^{q-1}\nabla u_i\cdot\nabla u_i\,dx \\ =\frac{1}{h}\int_\Omega(u_i-k)_+^{q}(u_{i-1}-k)\,dx+\int_\Omega(u_i-k)_+^{q}f^{(h)}(x,ih)\,dx \\ \le\frac{1}{h}\int_\Omega(u_i-k)_+^{q}(u_{i-1}-k)_+\,dx+\int_\Omega(u_i-k)_+^{q}f^{(h)}(x,ih)\,dx.\end{aligned} \tag{4.6}$$

By Young's inequality,

$$(u_i-k)_+^q(u_{i-1}-k)_+ \le \frac{q}{q+1}(u_i-k)_+^{(q+1)} + \frac{1}{q+1}(u_{i-1}-k)_+^{(q+1)}.$$

Invoking (4.6), we deduce that

$$\begin{aligned}(4.7)\qquad &\int_\Omega \frac{1}{h}(u_i-k)_+^{(q+1)}\,dx + p^-q(q+1)\int_\Omega P(|\nabla u_i|)(u_i-k)_+^{q-1}\,dx \\ &\le \int_\Omega \frac{1}{h}(u_{i-1}-k)_+^{(q+1)}\,dx + (q+1)\int_\Omega (u_i-k)_+^q f^{(h)}(x,ih)\,dx, \quad i=1,\dots,l.\end{aligned}$$

Summing over i in (4.7) and considering the definition of u^h, we have

$$\begin{aligned}(4.8)\qquad &\int_\Omega (u^h-k)_+^{(q+1)}(\cdot,t)\,dx + p^-q(q+1)\int_h^{(l+1)h}\int_\Omega |\nabla u^h|^{p^-}(u^h-k)_+^{q-1}\,dx \\ &\le \int_\Omega (u^h-k)_+^{(q+1)}(\cdot,t)\,dx + p^-q(q+1)\int_h^{(l+1)h}\int_\Omega P(|\nabla u^h|)(u^h-k)_+^{q-1}\,dx \\ &\le \int_\Omega (u_0-k)_+^{(q+1)}\,dx + (q+1)\int_h^{(l+1)h}\int_\Omega (u^h-k)_+^q f^{(h)}\,dx\,dt \\ &= (q+1)\int_h^{(l+1)h}\int_\Omega (u^h-k)_+^q f^{(h)}\,dx\,dt =: (q+1)L_1,\end{aligned}$$

where $t\in[h,(l+1)h)$.

Denote

$$\mu(k) = |\{(x,t)\in\Omega\times(0,(l+1)h) : u^h \ge k\}|.$$

By (A′), we have

$$(4.9)\qquad L_1 \le C_0\int_h^{(l+1)h}\int_\Omega (u^h-k)_+^{q+\alpha}\,dx\,dt + C_0\int_h^{(l+1)h}\int_\Omega (u^h-k)_+^q\phi^{(h)}\,dx\,dt.$$

Since $\alpha < p^- - 1$, i.e., $q+\alpha < q+p^- - 1$, by Hölder's inequality, Poincaré's inequality and Young's inequality we get

$$\begin{aligned}(4.10)\qquad &\int_h^{(l+1)h}\int_\Omega (u^h-k)_+^{q+\alpha}\,dx\,dt \\ &\qquad\le C\int_h^{(l+1)h}\Big(\int_\Omega (u^h-k)_+^{q+p^--1}\,dx\Big)^{\frac{q+\alpha}{q+p^--1}}\,dt\end{aligned}$$

$$\le C(|\Omega|)\int_h^{(l+1)h}\Big(\int_\Omega |\nabla (u^h-k)_+^{\frac{q+p^--1}{p^-}}|^{p^-}\,dx\Big)^{\frac{q+\alpha}{q+p^--1}}\,dt$$
$$\le C\int_h^{(l+1)h}\int_\Omega |\nabla u^h|^{p^-}(u^h-k)_+^{q-1}\,dx + C(|Q_T|).$$

Similarly, by Hölder's inequality and (A′), we have

$$(4.11)\qquad \int_h^{(l+1)h}\int_\Omega (u^h-k)_+^q\phi^{(h)}\,dx\,dt$$
$$\le \Big(\int_h^{(l+1)h}\int_\Omega (u^h-k)_+^{qq_1}\,dx\,dt\Big)^{1/q_1}\Big(\int_h^{(l+1)h}\int_\Omega (\phi^{(h)})^{q_2}\,dx\,dt\Big)^{1/q_2}$$
$$\le C(|\Omega|)\Big(\int_h^{(l+1)h}\int_\Omega (u^h-k)_+^{q+p^--1+\frac{p^-}{N}(q+1)}\,dx\,dt\Big)^{1/q_1}$$

where

$$q_1=\frac{q+p^--1+\frac{p^-}{N}(q+1)}{q}>1,$$
$$q_2=\frac{q+p^--1+\frac{p^-}{N}(q+1)}{p^--1+\frac{p^-}{N}(q+1)}<\frac{p^-+N}{p^-}<r.$$

Using the imbedding theorem (see [LSU, Z1]) and Young's inequality, we see that

$$(4.12)\qquad \int_h^{(l+1)h}\int_\Omega (u^h-k)_+^q\phi^{(h)}\,dx\,dt$$
$$\le C\Big(\sup_{t\in(h,(l+1)h)}\int_\Omega (u^h-k)_+^{q+1}\,dx+\int_h^{(l+1)h}\int_\Omega \Big|\nabla (u^h-k)_+^{\frac{q+p^--1}{p^--}}\Big|^{p^-}\,dx\,dt\Big)^{1/q_1}$$
$$\le C\sup_{t\in(h,(l+1)h)}\int_\Omega (u^h-k)_+^{(q+1)}(\cdot,t)\,dx$$
$$+\,C\int_h^{(l+1)h}\int_\Omega \Big|\nabla (u^h-k)_+^{\frac{q+p^--1}{p^--}}\Big|^{p^-}\,dx\,dt + C(|Q_T|)$$
$$= C\sup_{t\in(h,(l+1)h)}\int_\Omega (u^h-k)_+^{(q+1)}(\cdot,t)\,dx$$
$$+\,C\int_h^{(l+1)h}\int_\Omega |\nabla u^h|^{p^-}(u^h-k)_+^{q-1}\,dx + C(|Q_T|).$$

Choose the coefficients small enough in Young's inequalities so that (4.9)–(4.12) can be absorbed in (4.8). Then we get

$$\sup_{t\in(0,(l+1)h)} \int_\Omega (u^h-k)_+^{(q+1)}(\cdot,t)\,dx \le C(|Q_T|). \tag{4.13}$$

If $\alpha = p^- - 1$ and $|\Omega|$ is sufficiently small, then by the Poincaré inequality, in (4.10), $C|\Omega| \to 0$ as $|\Omega| \to 0$. We can also derive (4.13). Similarly, we may prove

$$\sup_{t\in(0,(l+1)h)} \int_\Omega (-u^h-k)_+^{(q+1)}(\cdot,t)\,dx \le C(|Q_T|).$$

Thus $\|u^h\|_{L^{q+1}(Q_T)} \le C(q)$, where $C(q)$ is independent of h. ∎

Corollary 4.1.

$$\int_h^{(l+1)h} \int_\Omega P(|\nabla u^h|)\,dx\,dt \le C. \tag{4.14}$$

Proof. Multiplying (3.1) by $(u_i)^+$ and integrating over Ω, taking the same procedure as (4.6)–(4.8), we get

$$\begin{aligned}
&\sup_{t\in(h,(l+1)h)} \int_\Omega (u^h)_+^2(\cdot,t)\,dx + 2p^- \int_h^{(l+1)h} \int_\Omega P(|\nabla u^h|)\,dx \\
&\le \sup_{t\in(h,(l+1)h)} \int_\Omega (u^h)_+^2(\cdot,t)\,dx + 2\int_h^{(l+1)h} \int_\Omega p(|\nabla u^h|)|\nabla u^h|\,dx \\
&\le \int_\Omega (u_0)_+^2(\cdot,t)\,dx + \int_h^{(l+1)h} \int_\Omega (u^h)_+ f^{(h)}\,dx\,dt \\
&\le C + C_0 \int_h^{(l+1)h} \int_\Omega (\phi(x,\tau) + |u^h|^\alpha)(u^h)_+\,dx\,dt \\
&\le C + C_0 \Big(\int_h^{(l+1)h} \int_\Omega |\phi(x,\tau)|^r\,dx\,dt\Big)^{1/r} \Big(\int_h^{(l+1)h} \int_\Omega (u^h)_+^{\frac{r}{r-1}}\,dx\,dt\Big)^{\frac{r-1}{r}} \\
&\quad + C_0 \int_h^{(l+1)h} \int_\Omega |u^h|^{\alpha+1}\,dx\,dt \le C
\end{aligned} \tag{4.15}$$

where the last inequality holds by Lemma 4.1. Similarly, the corollary holds for $-u$. ∎

In order to obtain a uniform estimate of the maximum norm of the solution, we need the following propositions.

Proposition 4.1 (see [D]). *Let $\{Y_n\}$, $n = 0, 1, 2, \ldots$, be a sequence of positive numbers satisfying*

$$Y_{n+1} \le Bb^n Y_n^{1+\beta}$$

where $B, b > 1$ and $\beta > 0$ are given numbers. If $Y_0 \le B^{-1/\beta} b^{-1/\beta^2}$, then Y_n converges to zero as $n \to \infty$.

Proposition 4.2 (see [D]). *Let $k, p \ge 1$ and consider the Banach spaces*

$$V_0^{k,p}(\Omega_T) \equiv L^\infty(0, T; L^k(\Omega)) \cap L^p(0, T; W_0^{1,p}(\Omega)),$$

equipped with the norm

$$\|u\|_{V^{k,p}(\Omega_T)} \equiv \operatorname*{ess\,sup}_{0<t<T} \|v(\cdot, t)\|_{k,\Omega} + \|Dv\|_{p,\Omega_T}.$$

Then there exists a constant γ depending only upon N, p, k such that for every $v \in V_0^{k,p}(\Omega_T)$,

$$\iint_{\Omega_T} |v(x,t)|^q \, dx\, dt \le \gamma^q \iint_{\Omega_T} |Dv(x,t)|^p \, dx\, dt \cdot \Big(\operatorname*{ess\,sup}_{0<t<T} \int_\Omega |v(x,t)|^k \, dx\Big)^{p/N},$$

where $q = p(N + k)/N$.

Lemma 4.2. *Let the assumptions of Lemma 4.1 hold. Then there is a constant $M_1 > 0$ depending only on $T, |\Omega|, N, p^-, r, \|u_0\|_{L^\infty(Q_T)}$ such that*

$$\|u^h\|_{L^\infty(Q_T)} \le M_1, \quad \forall h > 0.$$

Proof. Let $k \ge 1$ be chosen so that $\|u_0\|_{L^\infty(\Omega)} \le k$, and set

$$J_k = \sup_{t\in(h,(l+1)h)} \int_\Omega (u^h - k)_+^2(\cdot, t)\, dx + \int_0^{(l+1)h} \int_\Omega |\nabla(u^h - k)_+|^{p^-} \, dx\, dt.$$

Take $q = 1$ in (4.8). Then by (A′),

$$J_k \le C_1\Big(\int_0^{(l+1)h} \int_\Omega (\phi^{(h)} + |u^h|^\alpha)(u^h - k)_+ \, dx\, dt + \mu(k)\Big). \tag{4.16}$$

By Lemma 4.1 and Hölder's inequality,

$$\begin{aligned}\int_0^{(l+1)h} \int_\Omega \phi^{(h)}(u^h - k)_+ \, dx\, dt &\le C_2\Big(\int_0^{(l+1)h} \int_\Omega (u^h - k)_+^{\frac{r}{r-1}} \, dx\, dt\Big)^{\frac{r-1}{r}} \\ &\le C_2\Big[\Big(\int_0^{(l+1)h} \int_\Omega (u^h - k)_+^{\frac{r}{r-1}\cdot\frac{(r-1)(p^-N+2p^-)}{Nr}} \, dx\, dt\Big)^{\frac{Nr}{(r-1)(p^-N+2p^-)}} \\ &\qquad \cdot \mu(k)^{1-\frac{Nr}{(r-1)(p^-N+2p^-)}}\Big]^{\frac{r-1}{r}} \\ &= C_2\Big(\int_0^{(l+1)h} \int_\Omega (u^h - k)_+^{\frac{p^-N+2p^-}{N}} \, dx\, dt\Big)^{\frac{N}{p^-N+2p^-}} \mu(k)^{\frac{r-1}{r}-\frac{N}{p^-N+2p^-}}.\end{aligned} \tag{4.17}$$

By Proposition 4.2 and Young's inequality, we can derive from (4.17) that

$$(4.18)\qquad \Big(\int_0^{(l+1)h}\int_\Omega (u^h-k)_+^{\frac{p^-N+2p^-}{N}}\,dx\,dt\Big)^{\frac{N}{p^-N+2p^-}}$$

$$\le C_3\gamma^q\Big[\Big(\int_0^{(l+1)h}\int_\Omega |\nabla(u^h-k)_+|^{p^-}\,dx\,dt\Big)\Big(\operatorname*{ess\,sup}_{0<t<(l+1)h}\int_\Omega (u^h-k)_+^2\,dx\Big)^{\frac{p^-}{N}}\Big]^{\frac{N}{p^-N+2p^-}}$$

$$= C_4\Big[\Big(\int_0^{(l+1)h}\int_\Omega |\nabla(u^h-k)_+|^{p^-}\,dx\,dt\Big)^{\frac{N}{N+p^-}}$$

$$\cdot\Big(\operatorname*{ess\,sup}_{0<t<(l+1)h}\int_\Omega (u^h-k)_+^2\,dx\Big)^{\frac{p^-}{N+p^-}}\Big]^{\frac{N+p^-}{p^-N+2p^-}}$$

$$\le C_4\Big[\Big(\int_0^{(l+1)h}\int_\Omega |\nabla(u^h-k)_+|^{p^-}\,dx\,dt\Big)+\Big(\operatorname*{ess\,sup}_{0<t<(l+1)h}\int_\Omega (u^h-k)_+^2\,dx\Big)\Big]^{\frac{N+p^-}{p^-N+2p^-}}$$

where γ is as in Proposition 4.2 depending on N, p^-, and $q=\frac{Np^-+2p^-}{N}$.

Combining (4.17) and (4.18), we get

$$(4.19)\qquad \int_0^{(l+1)h}\int_\Omega \phi^{(h)}(u^h-k)_+\,dx\,dt \le C_4J_k^{\frac{N+p^-}{p^-N+2p^-}}\cdot\mu(k)^{\frac{r-1}{r}-\frac{N}{p^-N+2p^-}}.$$

Also, by Lemma 4.1 and (4.17), we have

$$(4.20)\qquad \Big(\int_0^{(l+1)h}\int_\Omega (u^h)^\alpha(u^h-k)_+\,dx\,dt\Big)$$

$$\le\Big(\int_0^{(l+1)h}\int_\Omega (u^h-k)_+^{\frac{r}{r-1}}\,dx\,dt\Big)^{\frac{r-1}{r}}\cdot\Big(\int_0^{(l+1)h}\int_\Omega (u^h)^{\alpha r}\,dx\,dt\Big)^{1/r}$$

$$\le C_5\Big(\int_0^{(l+1)h}\int_\Omega (u^h-k)_+^{\frac{r}{r-1}}\,dx\,dt\Big)^{\frac{r-1}{r}}$$

$$\le C_6J_k^{\frac{N+p^-}{p^-N+2p^-}}\cdot\mu(k)^{\frac{r-1}{r}-\frac{N}{p^-N+2p^-}}.$$

Substituting (4.17)–(4.20) into (4.16), we obtain

$$J_k\le C_7J_k^{\frac{N+p^-}{p^-N+2p^-}}\cdot\mu(k)^{\frac{r-1}{r}-\frac{N}{p^-N+2p^-}}+C_7\mu(k).$$

By Young's inequality,

$$(4.21)\qquad J_k\le C_8\Big(\mu(k)^{1+\frac{p^-(r-N-2)}{rN(p^--1)+rp^-}}+\mu(k)\Big).$$

Notice that, for all $1 \le k_1 \le k_2$,

$$(k_2 - k_1)\mu(k_2) = \int_0^{(l+1)h} \int_{D(k_2)} (k_2 - k_1)\, dx\, dt \le \int_0^{(l+1)h} \int_{D(k_2)} (u^h - k_1)_+\, dx\, dt \le \Big(\int_0^{(l+1)h} \int_{D(k_2)} (u^h - k_1)_+^{\frac{p^- N + 2N}{N}}\, dx\, dt \Big)^{\frac{N}{p^- N + 2N}} \cdot \mu(k_2)^{1 - \frac{N}{p^- N + 2N}}.$$

That is,

(4.22)
$$(k_2 - k_1)\mu(k_2)^{\frac{N}{p^- N + 2N}} \le \Big(\int_0^{(l+1)h} \int_{D(k_2)} (u^h - k_1)_+^{\frac{p^- N + 2N}{N}}\, dx\, dt \Big)^{\frac{N}{p^- N + 2N}} \le C_9 J_{k_1}^{\frac{N+p^-}{p^- N + 2p^-}} \le C_{10}\Big(\mu(k_1)^{1 + \frac{p^-(r-N-2)}{rN(p^- - 1) + rp^-}} + \mu(k_1) \Big)^{\frac{N+p^-}{p^- N + 2p^-}},$$

where C_{10} is a constant depending only on N, p^-, $|\Omega|$ and T.

If we take $k_2 = \|u_0\|_{L^\infty(\Omega)} + j$ $(j > 1)$ and $k_1 = \|u_0\|_{L^\infty(\Omega)} + 1$, then

$$\mu(k_2)^{\frac{N}{p^- N + 2N}} \le \frac{C_{10}}{j-1}\Big(((T+1)\Omega)^{1 + \frac{p^-(r-N-2)}{rN(p^- - 1) + rp^-}} + (T+1)|\Omega| \Big)^{\frac{N+p^-}{p^- N + 2p^-}}.$$

Hence, there exists a constant $j_0 > 1$ depending on $N, p^-, |\Omega|, T, r$ such that

$$\mu(k_2) \le 1 \quad \text{for } j \ge j_0.$$

We take $k_n = \tilde{M}(2 - 2^{-n})$, $n = 0, 1, 2, \ldots$, where $\tilde{M} \ge \|u_0\|_{L^\infty(\Omega)} + j_0$ is a constant. Using Proposition 4.1 and following a similar procedure to [LGYC, Z1], we may prove that

$$\mu(k_n) \to 0 \quad \text{as } n \to \infty.$$

Thus, $\|u^h\|_{L^\infty(Q_T)} \le 2\tilde{M} =: M_1$. ∎

Lemma 4.3. *Let the assumptions of Lemma 4.2 hold. Then for any integer $1 \le \tilde{l} \le l$, we have*

$$\frac{1}{2} \int_0^{(\tilde{l}+1)h} \int_\Omega |\partial^{(-h)} u^h|^2\, dx\, dt + \int_\Omega P(|\nabla u^h(x, \tilde{l}h)|)\, dx \le \int_\Omega P(|\nabla u_0|)\, dx. \tag{4.23}$$

Proof. Since $u_i, u_{i-1} \in S$, and u_i is the minimum point of $\psi(u)$, we have $\psi(u_i) \le \psi(u_{i-1})$ and so

$$\int_\Omega \frac{1}{2h}|u_i - u_{i-1}|^2\,dx + \int_\Omega P(|\nabla u_i|)\,dx \le \int_\Omega P(|\nabla u_{i-1}|)\,dx + \int_\Omega \int_{u_{i-1}}^{u_i} \left(\frac{1}{h} \int_{ih}^{(i+1)h} f(x,\tau,s)\,d\tau \right) ds\,dx, \qquad i = 1,\dots,\tilde{l}.$$

Summing over i, we obtain

$$\sum_{i=1}^{\tilde{l}} \int_\Omega \frac{1}{2h}|u_i - u_{i-1}|^2\,dx + \int_\Omega P(|\nabla u_{\tilde{l}}|)\,dx \le \int_\Omega P(|\nabla u_0|)\,dx + \sum_{i=1}^{\tilde{l}} \int_\Omega \int_{u_{i-1}}^{u_i} \left(\frac{1}{h} \int_{ih}^{(i+1)h} f(x,\tau,s)\,d\tau \right) ds\,dx.$$

By (A) and Young's inequality, we have

$$\int_\Omega \int_{u_{i-1}}^{u_i} \left(\frac{1}{h} \int_{ih}^{(i+1)h} f(x,\tau,s)\,d\tau \right) ds\,dx \le C \int_\Omega |u_i - u_{i-1}|\,dx \le \frac{1}{4h} \int_\Omega |u_i - u_{i-1}|^2\,dx + 4Ch.$$

Thus

$$\sum_{i=1}^{\tilde{l}} \frac{1}{4h} \int_\Omega |u_i - u_{i-1}|^2\,dx\,dt + \int_\Omega P(|\nabla u_{\tilde{l}}(x,\tilde{l}h)|)\,dx \le \int_\Omega P(|\nabla u_0|)\,dx + 4CT. \tag{4.24}$$

The conclusion follows by the definition of u^h. ∎

Define a new auxiliary function:

$$w^h(\cdot,t) = \begin{cases} (t/h - i)u_i + [1 - (t/h - i)]u_{i-1}, & t \in [ih,(i+1)h),\ i = 1,\dots,l, \\ u_0, & t \in [0,h). \end{cases}$$

By Lemma 4.3, we may prove that (see [LGYC])

$$\int_0^T \int_\Omega |w^h - u^h|^2\,dx\,dt \to 0, \qquad h \to 0. \tag{4.25}$$

Lemma 4.4. *Let the assumptions of Lemma* 4.3 *hold. Then there exists a subsequence of* $\{u^h\}$*, still denoted by* $\{u^h\}$*, and a function* u *such that,* *as* $h \to 0$,

$$u^h \to u \quad \text{in } L^2(Q_T), \tag{4.26}$$

$$\nabla u^h \overset{w}{\rightharpoonup} \nabla u \quad \text{in } L^P(Q_T), \tag{4.27}$$

$$\partial^{(-h)} u^h \overset{w}{\rightharpoonup} \partial^{(-h)} u \quad \text{in } L^2(Q_T), \tag{4.28}$$

$$u^h \to u \quad \text{a.e. in } Q_T. \tag{4.29}$$

Proof. By Lemma 4.3 and Young's inequality, for any $t \in (0, (l+1)h)$,

$$\Big(\int_\Omega |\nabla u^h(\cdot,t)|^2\,dx\Big)^{p^+/2} \le C\int_\Omega |\nabla u^h(\cdot,t)|^{p^+} \le \int_\Omega P(|\nabla u^h(\cdot,t)|) \le C \tag{4.30}$$

for $|\nabla u^h(\cdot,t)| < 1$, and

$$\Big(\int_\Omega |\nabla u^h(\cdot,t)|^2\,dx\Big)^{p^-/2} \le C\int_\Omega |\nabla u^h(\cdot,t)|^{p^-} \le \int_\Omega P(|\nabla u^h(\cdot,t)|) \le C. \tag{4.31}$$

for $|\nabla u^h(\cdot,t)| > 1$. By Poincaré's inequality, we have

$$\int_0^T\int_\Omega |u^h(\cdot,t)|^2\,dx\,dt \le CT. \tag{4.32}$$

Therefore, by Lemma 3.3 there exists a subsequence of u^h (not relabeled) and a function u such that

$$u^h \overset{w}{\rightharpoonup} u \quad \text{in } L^2(Q_T), \tag{4.33}$$

$$\nabla u^h \overset{w}{\rightharpoonup} \nabla u \quad \text{in } L^P(Q_T). \tag{4.34}$$

Since

$$\nabla w^h = (\nabla u_i - \nabla u_{i-1})(t/h - i) + \nabla u_{i-1}, \quad t \in [ih, (i+1)h),\ i = 1, \ldots, l,$$

by Corollary 4.1, (4.25) and the above, we know that w^h and ∇w^h are uniformly bounded in $L^2(Q_T)$. Since

$$(w^h)_t = \partial^{(-h)} u^h = \begin{cases} h^{-1}(u_i - u_{i-1}), & t \in [ih, (i+1)h),\ i = 1, \ldots, l, \\ 0, & t \in [0, h), \end{cases}$$

by Lemma 4.3 we have $(w^h)_t \in L^2(Q_T)$. Combining the above estimates, we find that there exists a subsequence of w^h (not relabeled) and a function u_*

such that

$$w^h \to u_* \quad \text{in } L^2(Q_T),$$
$$\nabla w^h \overset{w}{\rightharpoonup} \nabla u_* \quad \text{in } L^2(Q_T),$$
$$\partial^{(-h)} w^h \overset{w}{\rightharpoonup} (u_*)_t \quad \text{in } L^2(Q_T).$$

By Lemma 3.3 and (4.25), we get $u_* = u$, thus $u^h \to u$ in $L^2(Q_T)$, and $u^h \to u$ a.e. in Q_T. ■

REMARK 4.1. We know from Lemma 4.3 that $u \in L^\infty(0, T; W_0^{1,P}(\Omega))$.

LEMMA 4.5. *Let the assumptions of Lemma 4.2 hold. Then*

$$f^{(h)} \to f(\cdot, \cdot, u) \quad \text{in } L^1(Q_T) \text{ as } h \to 0. \tag{4.35}$$

Proof. This can be proved much as in [LGYC]. ■

Proof of Theorem 2.1. STEP 1. We will prove that there exists a subsequence such that

$$a(|\nabla u^h|)(u^h)_{x_i} \overset{w}{\rightharpoonup} a(|\nabla u|)u_{x_i} \quad \text{in } L^{\tilde{P}}(Q_T). \tag{4.36}$$

By Lemma 4.4, we have $u^h \in L^P(Q_T)$ and $\nabla u^h \overset{w}{\rightharpoonup} \nabla u$ in $L^P(Q_T)$. Hence for any $\phi \in C_0^\infty(Q_T)$,

$$\begin{aligned} \int_0^T \int_\Omega \phi a(|\nabla u^h|)\nabla u^h \cdot (\nabla u^h - \nabla u)\,dx\,dt &\le cp^+ \int_0^T \int_\Omega \phi \frac{P(|\nabla u^h|)}{|\nabla u^h|} |\nabla u^h - \nabla u|\,dx\,dt \\ &\le cp^+ \left| \frac{P(|\nabla u^h|)}{|\nabla u^h|} \right|_{\tilde{P}} |\nabla u^h - \nabla u|_P \\ &\le cp^+ |\nabla u^h|_P |\nabla u^h - \nabla u|_P \to 0, \end{aligned} \tag{4.37}$$

and similarly

$$\int_0^T \int_\Omega \phi a(|\nabla u|)\nabla u \cdot (\nabla u^h - \nabla u)\,dx\,dt \to 0. \tag{4.38}$$

Notice that

$$\begin{aligned} &\big(a(|\nabla u^h|)\nabla u^h - a(|\nabla u|)\nabla u\big) \cdot \nabla(u^h - u) \\ &\quad = \int_0^1 \frac{d}{ds}\big[a(|s\nabla u^h + (1-s)\nabla u|)(s\nabla u^h + (1-s)\nabla u)\big]\nabla(u^h - u) \end{aligned} \tag{4.39}$$

$$= \int_0^1 \big[a'(|s\nabla u^h + (1-s)\nabla u|)|s\nabla u^h + (1-s)\nabla u| + a(|s\nabla u^h + (1-s)\nabla u|)\big]|\nabla(u^h - u)|^2\,ds$$

$$\geq a^- \int_0^1 a(|s\nabla u^h + (1-s)\nabla u|)|\nabla(u^h - u)|^2\,ds$$

Combining (4.37)–(4.39), we get

$$\lim_{h\to 0}\int_0^T\int_\Omega \phi\int_0^1 a(|s\nabla u^h + (1-s)\nabla u|)\,ds\,|\nabla(u^h - u)|^2\,dx\,dt = 0. \tag{4.40}$$

Since $\lim_{h\to 0}\int_0^T\int_\Omega\int_0^1 a(|s\nabla u^h + (1-s)\nabla u|)\,ds\,dx\,dt \leq C$ and

$$\begin{aligned}|a(|\nabla u^h|)(u^h)_{x_i} - a(|\nabla u|)u_{x_i}| &= \Big|\int_0^1 \frac{d}{ds}a(|s\nabla u^h + (1-s)\nabla u|)[s(u^h)_{x_i} + (1-s)u_{x_i}]\,ds\Big| \\ &= \Big|\int_0^1 \big[a'(|s\nabla u^h + (1-s)\nabla u|)|s\nabla u^h + (1-s)\nabla u| \\ &\qquad + a(|s\nabla u^h + (1-s)\nabla u|)\big]\nabla(u^h - u)\,ds\Big| \\ &\leq a^+\int_0^1 a(|s\nabla u^h + (1-s)\nabla u|)\cdot|\nabla(u^h - u)|\,ds,\end{aligned} \tag{4.41}$$

we obtain

$$\begin{aligned}&\Big|\int_0^T\int_\Omega \phi[a(|\nabla u^h|)(u^h)_{x_i} - a(|\nabla u|)u_{x_i}]\,dx\,dt\Big| \\ &\leq C\Big(\int_0^T\int_\Omega \phi\int_0^1 |a(|s\nabla u^h + (1-s)\nabla u|)\,ds\cdot|\nabla(u^h - u)|^2\Big)^{1/2} \\ &\quad\times\Big(\int_0^T\int_\Omega \phi\int_0^1 a(|s\nabla u^h + (1-s)\nabla u|)\,ds\,dx\,dt\Big)^{1/2} \to 0.\end{aligned} \tag{4.42}$$

Thus (4.36) is derived.

STEP 2. For each $\phi \in C_0^\infty(Q_T)$ and any constant $\tilde\tau \in [0,T]$, we have $\phi(\cdot,\tilde\tau) \in C_0^\infty(\Omega)$. Hence, by Lemma 3.2,

$$\int_\Omega \partial^{(-h)}u^h\phi(x,\tilde\tau)\,dx = \int_\Omega a(|\nabla u_i|)\nabla u_i\nabla\phi(x,\tilde\tau)\,dx + \int_\Omega f^{(h)}\phi(x,\tilde\tau)\,dx.$$

Integrating over $\tilde\tau$ and invoking Lemmas 4.4 and 4.5 and Corollary 4.1 we may prove that u is a weak solution of (1.1).

Now we prove that u satisfies the initial condition (2.5).

For problem (3.1), taking a test function $\tilde{\phi} \in C_0^\infty(\Omega)$, we get

$$(4.43)\qquad \int_\Omega (u_i - u_{i-1})\tilde{\phi}\,dx + \int_{ih}^{(i+1)h}\int_\Omega a(|\nabla u_i|)\nabla u_i \nabla\tilde{\phi}\,dx = \int_\Omega \Big(\int_{ih}^{(i+1)h} f(x,\tau,u_i)\,d\tau\Big)\tilde{\phi}\,dx, \qquad i = 1,2,\ldots.$$

Summing over i, we obtain

$$(4.44)\qquad \int_\Omega (u_{\tilde{l}} - u_0)\tilde{\phi}\,dx = -\int_h^{(\tilde{l}+1)h}\int_\Omega a(|\nabla u^h|)\nabla u^h \nabla\tilde{\phi}\,dx\,dt + \int_h^{(\tilde{l}+1)h}\int_\Omega (f(x,\tau,u^h)\tilde{\phi}\,dx\,dt,$$

where $\tilde{l}$ is an integer. Then by Hölder's inequality and the boundedness of $|\nabla u^u|_P$, we have

$$(4.45)\qquad \Big|\int_h^{(\tilde{l}+1)h}\int_\Omega a(|\nabla u^h|)\nabla u^h \nabla\tilde{\phi}\,dx\,dt\Big| \le \sup|\nabla\tilde{\phi}| \int_\Omega a(|\nabla u^h|)|\nabla u^h|\,dx\,dt$$
$$= \sup|\nabla\tilde{\phi}| \int_\Omega p(|\nabla u^h|)\frac{|\nabla u^h|}{|\nabla u^h|}\,dx\,dt \le p^+ \sup|\nabla\tilde{\phi}| \int_\Omega \frac{P(|\nabla u^h|)}{|\nabla u^h|}\cdot 1\,dx\,dt$$
$$\le 2p^+ \sup|\nabla\tilde{\phi}| \Big|\frac{P(|\nabla u^h|)}{|\nabla u^h|}\Big|_{\tilde{P}} \cdot |1|_P \le C|\nabla u^h|_P |1|_P \le C(\tilde{l}h)^\delta,$$

where $\delta > 0$ is a constant depending only on p^- and p^+. By the differentiability of f, we have

$$(4.46)\qquad \Big|\int_h^{(\tilde{l}+1)h}\int_\Omega f(x,\tau,u^h)\tilde{\phi}\,dx\,dt\Big| \le C(\tilde{l}h).$$

If $\tilde{l}h < 1$, there exists a constant $\delta_2 > 0$ depending on p^- and p^+ such that

$$\int_\Omega (u_i - u_{i-1})\tilde{\phi}\,dx \le C(\tilde{l}h)^{\delta_2}.$$

Thus

$$(4.47)\qquad \sup_{t\in[h,(\tilde{l}+1)h)} \Big|\int_\Omega (u^h(x,t) - u_0)\tilde{\phi}\,dx\Big| \le Ct^{\delta_2}.$$

For $t \in [0,h)$,

$$\int_\Omega (u^h(x,t) - u_0)\tilde{\phi}\,dx = 0.$$

Letting $h \to 0$, we get (2.5).

STEP 3. Finally, we prove the uniqueness of solution. Let u, v be two solutions of (1.1). Taking $u - v$ as a test function, we obtain

$$\frac{1}{2}\int_{\Omega}(u-v)^2\,dx + \iint_{Q_t}[a(|\nabla u|)\nabla u - a(|\nabla v|)\nabla v]\nabla(u-v)\,dx\,d\tau = \iint_{Q_t}(f(x,\tau,u) - f(f,\tau,v))(u-v)\,dx\,d\tau.$$

Since $[a(|\nabla u|)\nabla u - a(|\nabla v|)\nabla v]\nabla(u-v) \geq 0$ and u, v are bounded, and since $f \in C^1$, we have

$$\int_{\Omega}(u-v)^2\,dx \leq C\iint_{Q_t}(u-v)^2\,dx.$$

Obviously, Gronwall's inequality implies that $u = v$.

The proof is complete. ∎

References

[AF] R. Adams and J. F. Fournier, *Sobolev Spaces*, 2nd ed., Academic Press, 2003.

[AS] S. Antontsev and S. Shmarev, *Anisotropic parabolic equations with variable nonlinearity*, Publ. Mat. 53 (2009), 355–399.

[B] M. Berger, *Nonlinearity and Functional Analysis. Lectures on Nonlinear Problems in Mathematical Analysis*, Pure Appl. Math., Academic Press, New York, 1977.

[BAH] R. Bird, R. Armstrong and O. Hassager, *Dynamics of Polymeric Liquids,* Vol. 1, *Fluid Mechanics*, Wiley, 1987.

[BMR] G. Bonanno, G. Molica Bisci and V. Rădulescu, *Quasilinear elliptic non-homogeneous Dirichlet problems through Orlicz–Sobolev spaces*, Nonlinear Anal. 75 (2012), 4441–4456.

[CM] A. Cianchi and V. G. Maz'ya, *Global Lipschitz regularity for a class of quasilinear elliptic equations*, Comm. Partial Differential Equations 36 (2011), 100–133.

[CGMS] Ph. Clément, M. García-Huidobro, R. Manásevich and K. Schmitt, *Mountain pass type solutions for quasilinear elliptic equations*, Calc. Var. Partial Differential Equations 11 (2000), 33–62.

[D] E. DiBenedetto, *Degenerate Parabolic Equations*, Springer, New York, 1993.

[FZo] X. L. Fan and D. Zhao, *On the spaces $L^{p(x)}(\Omega)$ and $W^{m,p(x)}(\Omega)$*, J. Math. Anal. Appl. 263 (2001), 424–446.

[FT] F. Fang and Z. Tan, *Existence and multiplicity of solutions for a class of quasilinear elliptic equations: an Orlicz–Sobolev space setting*, J. Math. Anal. Appl. 389 (2012), 420–428.

[FZu] F. Fang and Z. Zhou, *Relationship between solutions to a quasilinear elliptic equation in Orlicz spaces*, Electron. J. Differential Equations 2014, no. 265, 10 pp.

[FL] M. Fuchs and M. Li, *Variational inequalities for energy functionals with nonstandard growth conditions*, Abstr. Appl. Anal. 3 (1998), 41–64.

[FO] M. Fuchs and V. Osmolovski, *Variational integrals on Orlicz–Sobolev spaces*, Z. Anal. Anwendungen 17 (1998), 393–415.

[FIN] N. Fukagai, M. Ito and K. Narukawa, *Positive solutions of quasilinear elliptic equations with critical Orlicz–Sobolev nonlinearity on* $\boldsymbol{R}^N$, Funkcial. Ekvac. 49 (2006), 235–267.

[FN] N. Fukagai and K. Narukawa, *Nonlinear eigenvalue problem for a model equation of an elastic surface*, Hiroshima Math. J. 25 (1995), 19–41.

[GLMS] M. García-Huidobro, V. Le, R. Manásevich and K. Schmitt, *On principal eigenvalues for quasilinear elliptic differential operators: an Orlicz–Sobolev space setting*, Nonlinear Differential Equations Appl. 6 (1999), 207–225.

[KR] O. Kováčik and J. Rákosník, *On spaces* $L^{p(x)}$ *and* $W^{k,p(x)}$, Czechoslovak Math. J. 41 (116) (1991), 592–618.

[LSU] O. A. Ladyženskaja, V. A. Solonnikov, and N. N. Ural'ceva, *Linear and Quasilinear Equations of Parabolic Type*, Amer. Math. Soc., Transl. Math. Monogr., Amer. Math. Soc., 1968.

[LGYC] S. Lian, W. J. Gao, H. J. Yuan and C. L. Cao, *Existence of solutions to an initial Dirichlet problem of evolutional* $p(x)$*-Laplace equations*, Ann. Inst. H. Poincaré Anal. Linéaire 29 (2012), 377–399.

[RR1] M. M. Rao and Z. D. Ren, *Theory of Orlicz Spaces*, Monogr. Textbooks Pure Appl. Math. 146, Dekker, New York, 1991.

[RR2] M. M. Rao and Z. D. Ren, *Applications of Orlicz Spaces*, Monogr. Textbooks Pure Appl. Math. 250, Dekker, New York, 2002.

[TF] Z. Tan and F. Fang, *Orlicz–Sobolev versus Hölder local minimizer and multiplicity results for quasilinear elliptic equations*, J. Math. Anal. Appl. 402 (2013), 348-370.

[Z1] J. N. Zhao, *Existence and nonexistence of solutions for* $u_t = \mathrm{div}(|\nabla u|^{p-2}\nabla u) + f(\nabla u, u, x, t)$, J. Math. Anal. Appl. 172 (1993), 130–146.

[Z2] J. N. Zhao, *On the Cauchy problem and initial traces for the evolution p-Laplacian equations with strongly nonlinear sources*, J. Differential Equations 121 (1995), 329–383.

Zheng Zhou
School of Applied Mathematical Sciences
Xiamen University of Technology
Xiamen 361024, China
E-mail: zhouzhengslx@163.com

Fei Fang (corresponding author)
Department of Mathematics
Beijing Technology and Business University
Beijing 100048, China
E-mail: fangfei68@163.com