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Distribution of values at 1 of symmetric power L-functions of Maass cusp forms

by

YINGNAN WANG (Shenzhen) and XUANXUAN XIAO (Macau)

1. Introduction. The distribution of values at $s = 1$ of symmetric power L-functions of Maass cusp forms was first investigated by Luo [12] who considered the case of symmetric square L-functions of Maass forms with motivation from spectral deformation theory. Luo's work was extended to symmetric power L-functions of holomorphic cusp forms with large squarefree levels in [13], [14], [15], [16], [3] and [1]. Inspired by [1] and [2], Lau and Wu [9, 10] investigated the distribution of values at $s = 1$ of symmetric power L-functions of holomorphic cusp forms in the weight aspect.

In this paper we consider the distribution of values at $s = 1$ of symmetric power L-functions of Maass cusp forms in the weight aspect. Similar results can be found in Xiao [20], Lau–Wu [9, 10], Liu–Royer–Wu [11] and Lamzouri [7] for holomorphic cusp forms. Although we use similar methods (the probability model proposed by Cogdell and Michel in [1]), we have to overcome the extra difficulties arising from the absence of the Generalized Ramanujan Conjecture (GRC). Our main idea is to use the result of Sarnak (see Lemma 3.1) to count the number of Maass forms which have "exceptional" Hecke eigenvalues (those not in the interval $[-2, 2]$) and find two dense subsets $H^{+,1}_{T,\mathrm{sym}^m}(\eta)$ and $H^{*}_{T,\mathrm{sym}^m}(\eta)$ (see (3.1) and (5.1) for the definitions) satisfying weak forms of GRC for small primes and the Generalized Riemann Hypothesis (GRH). Moreover, in order to find the moments of $L(1, \mathrm{sym}^m u_j)$ (Proposition 4.1), we use a truncated version of the Kuznetsov trace formula (see Lemma 2.2).

Let us begin with the setting of Maass cusp forms. Let $\Gamma = \mathrm{SL}_2(\mathbb{Z})$ and $\mathbb{H}$ be the open upper half-plane in $\mathbb{C}$. Denote by $\mathcal{C}(\Gamma\backslash\mathbb{H})$ the space spanned by the Maass cusp forms for Γ. Let Δ be the non-Euclidean Laplace operator

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and T_n be the nth Hecke operator, $n \in \mathbb{N}$. Then there exists a complete orthonormal basis $\{u_j : j \geq 0\}$ for $\mathcal{C}(\Gamma\backslash\mathbb{H})$ with

$$\Delta u_j = (1/4 + t_j^2)u_j \quad \text{and} \quad T_n u_j = \lambda_j(n)u_j,$$

where $0 < t_1 \leq t_2 \leq \cdots$. We know (Weyl's law) that

$$r(T) = \#\{j : 0 < t_j \leq T\} = \tfrac{1}{12}T^2 + O(T\log T). \tag{1.1}$$

Moreover, $\lambda_j(n) \in \mathbb{R}$, and GRC states that for any prime p and $j \geq 1$,

$$|\lambda_j(p)| \leq 2.$$

The above inequality is out of reach at present and the best result to date has been established by Kim and Sarnak [5], who proved that for any prime p and $j \geq 1$,

$$|\lambda_j(p)| \leq p^{\theta} + p^{-\theta} \tag{1.2}$$

where $\theta = 7/64$.

It is well-known that there exist two parameters $\alpha_{u_j}(p)$ and $\beta_{u_j}(p)$ such that

$$\alpha_{u_j}(p) + \beta_{u_j}(p) = \lambda_j(p) \quad \text{and} \quad \alpha_{u_j}(p)\beta_{u_j}(p) = 1.$$

For $m \in \mathbb{N}$, the *symmetric mth power L-function* of u_j is defined by

$$L(s, \operatorname{sym}^m u_j) = \prod_p \prod_{k=0}^{m} (1 - \alpha_{u_j}(p)^{m-2k}p^{-s})^{-1} = \sum_{n=1}^{\infty} \lambda_{\operatorname{sym}^m u_j}(n)n^{-s} \tag{1.3}$$

for $\operatorname{Re} s \gg 1$. For $m = 1, 2, 3, 4$, the above Dirichlet series converges absolutely for $\operatorname{Re} s > 1$, and $L(s, \operatorname{sym}^m u_j)$ is the L-function attached to a cuspidal automorphic form of $\mathrm{GL}_{m+1}(\mathbb{Q})$. Moreover, $L(s, \operatorname{sym}^m u_j)$ can be analytically continued to the entire complex plane and satisfies a functional equation for $m = 1, 2, 3, 4$. In this paper we are interested in the behavior of $\{L(1, \operatorname{sym}^m u_j) : t_j \leq T\}$ as $T \to \infty$.

In fact, $L(1, \operatorname{sym}^m u_j)$ is expected to behave very similarly to the values at $s = 1$ of symmetric mth power L-functions of holomorphic cusp forms under GRC. Similarly to [9, Theorem 3], we can see that under GRC and GRH, for $m = 1, 2, 3, 4$ and $t_j \leq T$,

$$\begin{aligned}\{1 + o(1)\}(2B_m^- \log_2 T)^{-A_m^-} \\ \leq L(1, \operatorname{sym}^m u_j) \leq \{1 + o(1)\}(2B_m^+ \log_2 T)^{A_m^+}\end{aligned} \tag{1.4}$$

where $\log_n$ is the n-fold iterated logarithm and the constants $A_m^{\pm}$ and $B_m^{\pm}$ are given by

$$
\begin{cases} A_m^+ = m+1, & B_m^+ = e^{\gamma} \qquad\qquad (m = 1,2,3,4),\\ A_m^- = m+1, & B_m^- = e^{\gamma}\zeta(2)^{-1} \quad (m = 1,3),\\ A_2^- = 1, & B_2^- = e^{\gamma}\zeta(2)^{-2},\\ A_4^- = 5/4, & B_4^- = e^{\gamma}B_{4,*}^-, \end{cases} \tag{1.5}
$$

and $B_{4,*}^-$ is an absolute constant given in [9, (1.16)]. Here γ is the Euler constant. On the other hand, Wang [19, Theorem 1.3] proved unconditionally that as $T \to \infty$, there exist u_{j_1}, u_{j_2} with $t_{j_1}, t_{j_2} \le T$ such that for $m = 1,2,3,4$,

$$
L(1, \mathrm{sym}^m u_{j_1}) \ge \{1 + o(1)\}(B_m^+ \log_2 T)^{A_m^+}, \tag{1.6}
$$

$$
L(1, \mathrm{sym}^m u_{j_2}) \le \{1 + o(1)\}(B_m^- \log_2 T)^{-A_m^-}. \tag{1.7}
$$

Thus the inequality (1.4) is sharp up to the constant 2 if it is unconditionally true. However, it is not known yet whether one can unconditionally obtain tight upper and lower bounds as in (1.4).

In this paper we prove the following unconditional result which indicates that as $T \to \infty$, for almost all $t_j \le T$, the magnitude of $L(1, \mathrm{sym}^m u_{j_2})$ lies between those of $(\log_2 T)^{-A_m^-}$ and $(\log_2 T)^{A_m^+}$.

THEOREM 1.1. *Let $m = 1,2,3,4$ and T be large enough. For any fixed $\eta \in (0, 10^{-5})$ and all $u_j \in H_{T,\mathrm{sym}^m}^*(\eta)$ (see (5.1) for the definition), we have*

$$
(\log_2 T)^{-A_m^-} \ll L(1, \mathrm{sym}^m u_j) \ll (\log_2 T)^{A_m^+}. \tag{1.8}
$$

REMARK 1.2. Here $H_{T,\mathrm{sym}^m}^*(\eta) \subset \{j : 0 < t_j \le T\}$ is defined by (5.1), and

$$
\lim_{T\to\infty} \frac{|H_{T,\mathrm{sym}^m}^*(\eta)|}{r(T)} = 1
$$

by (5.3). In other words, $H_{T,\mathrm{sym}^m}^*(\eta)$ is dense in $\{j : 0 < t_j \le T\}$ as $T \to \infty$. Moreover, for any $u_j \in H_{T,\mathrm{sym}^m}^*(\eta)$, $\lambda_j(p)$ has a good bound for small primes p, and $L(1, \mathrm{sym}^m u_j)$ has a large zero free region for $m = 1,2,3,4$ (by the definition of $H_{T,\mathrm{sym}^m}^+(\eta)$ in (2.1)). Therefore, $H_{T,\mathrm{sym}^m}^*(\eta)$ satisfies weak forms of GRC and GRH.

Next we estimate the size of the exceptional set for which (1.6) or (1.7) holds and consider the distribution functions

$$
F_T^+(t, \mathrm{sym}^m) = \frac{1}{r(T)} \sum_{\substack{t_j \le T \\ L(1,\mathrm{sym}^m u_j) > (B_m^+ t)^{A_m^+}}} 1,
$$

$$
F_T^-(t, \mathrm{sym}^m) = \frac{1}{r(T)} \sum_{\substack{t_j \le T \\ L(1,\mathrm{sym}^m u_j) < (B_m^- t)^{-A_m^-}}} 1.
$$

These functions are believed to satisfy the following analogue of Montgomery–Vaughan's first conjecture for symmetric power L-functions associated to Maass cusp forms: for each $m \in \mathbb{Z}$, there exist positive constants $T_0 = T_0(m)$ and $c_1 > c_2$ such that for $T > T_0$,

$$e^{-c_1(\log T)/\log_2 T} \le F_T^{\pm}(\log_2 T, \mathrm{sym}^m) \le e^{-c_2(\log T)/\log_2 T}. \tag{1.9}$$

Concerning the upper bound of (1.9), Wang [19, Theorem 1.4] proved that for $m = 1, 2, 3, 4$ and any $\varepsilon > 0$, there are positive constants $c = c(\varepsilon)$ and $T_0 = T_0(\varepsilon)$ such that

$$F_T^{\pm}(\log_2 T + r, \mathrm{sym}^m) \le \exp\left(-c(r+1)\frac{\log T}{(\log_2 T)(\log_3 T)^2(\log_4 T)^2}\right)$$

for $T \ge T_0$ and $\log \varepsilon \le r \le (9-\varepsilon)\log_2 T$.

In order to investigate the lower bound of (1.9), just as Liu, Royer and Wu [11], we consider the weighted distribution functions for Maass cusp forms, defined by

$$\mathscr{F}_T^{\pm}(t, \mathrm{sym}^m) := \Big(\sum_{t_j \le T} \omega(j)\Big)^{-1} \sum_{\substack{t_j \le T \\ L(1, \mathrm{sym}^m u_j) \gtreqless (B_m^{\pm} t)^{\pm A_m^{\pm}}}} \omega(j),$$

where

$$\omega(j) = \frac{\pi^2}{T^2} \cdot \frac{2}{L(1, \mathrm{sym}^2 u_j)}.$$

It is proved in [4] that

$$L(1, \mathrm{sym}^2 u_j) \gg (\log T)^{-1}, \tag{1.10}$$

and so

$$\omega(j) \ll \frac{\log T}{T^2}. \tag{1.11}$$

Another main result of this paper is as follows.

Theorem 1.3. *Let $m = 1, 2, 3, 4$. Then there is a positive constant c_3 such that*

$$\mathscr{F}_T^{\pm}(t, \mathrm{sym}^m) = \exp\left(-\frac{\mathrm{e}^{t-\mathscr{A}_m^{\pm}}}{t}\left\{1 + O\Big(\frac{1}{t}\Big)\right\}\right)$$

uniformly for $T \to \infty$ and

$$t \le \log_2 T - 2\log_3 T - \log_4 T - c_3,$$

where $\mathscr{A}_m^{\pm}$ are constants depending only on m defined in (6.1) *and* (6.2).

On noting (1.11) and

$$\sum_{t_j \le T} \omega(j) = 1 + O_\varepsilon(T^{-1+\varepsilon}) \tag{1.12}$$

by Lemma 2.2, Theorem 1.3 immediately implies the following corollary.

COROLLARY 1.4. *Let $m = 1, 2, 3, 4$. There are positive constants c_4 and T_1 such that*

$$F_T^{\pm}(\log_2 T - 2\log_3 T - \log_4 T, \operatorname{sym}^m) \geq \exp\left(-c_4 \frac{\log T}{(\log_2 T)^3(\log_3 T)}\right)$$

for $T \geq T_1$.

REMARK 1.5. Compared to the result on weighted distribution functions for holomorphic cusp forms, the domain of validity in Theorem 1.3 is slightly worse. It is closely related to the domain of validity of the result on moments in Proposition 4.1.

2. Preliminaries. One of our main tools is the following density theorem.

THEOREM 2.1 ([19, Theorem 1.1]). *Let $m = 1, 2, 3, 4$ and $r \geq 1$. Let $N(\alpha, H, \operatorname{sym}^m u_j)$ be the number of zeros $\rho = \beta + i\gamma$ of $L(s, \operatorname{sym}^m u_j)$ with $\beta \geq \alpha$ and $0 \leq \gamma \leq H$. Then for any $\varepsilon > 0$,*

$$\sum_{t_j \leq T} N(\alpha, H, \operatorname{sym}^m u_j) \ll_{\varepsilon, r} H^{1+1/r} T^{\frac{8((m+1)(m+r+1)+64)(1-\alpha)}{17-16\alpha}+\varepsilon}$$

uniformly for $15/16 + \varepsilon \leq \alpha \leq 1$ and $H \geq 1$. The implied constant depends on ε and r only.

Theorem 2.1 allows us to find a dense subset of $\{j : t_j \leq T\}$ as $T \to \infty$ such that a weak form of GRH is satisfied. Let $s = \sigma + i\tau$ and $m = 1, 2, 3, 4$. For any $\eta \in (0, 10^{-5})$, define

$$\begin{aligned} H^+_{T,\operatorname{sym}^m}(\eta) &= \{j : 0 < t_j \leq T \text{ and } L(s, \operatorname{sym}^m u_j) \neq 0,\ s \in \mathcal{S}\}, \\ H^-_{T,\operatorname{sym}^m}(\eta) &= \{j : 0 < t_j \leq T \text{ and } j \notin H^+_{T,\operatorname{sym}^m}(\eta)\}, \end{aligned} \tag{2.1}$$

where $\mathcal{S} = \{s : \sigma \geq 1 - \eta,\ |\tau| \leq 100T^{\eta}\} \cup \{s : \sigma \geq 1\}$. By Theorem 2.1 with $r = 1$, we have

$$\begin{aligned} |H^-_{T,\operatorname{sym}^m}(\eta)| &\leq \sum_{u_j \in H^-_{T,\operatorname{sym}^m}(\eta)} N(1-\eta, 100T^{\eta}, \operatorname{sym}^m u_j) \\ &\leq \sum_{0 < t_j \leq T} N(1-\eta, 100T^{\eta}, \operatorname{sym}^m u_j) \\ &\ll T^{8\eta((m+1)(m+2)+66)}. \end{aligned} \tag{2.2}$$

Hence, by Weyl's law (1.1),

$$|H^+_{T,\operatorname{sym}^m}(\eta)| = r(T) + O(T^{8\eta((m+1)(m+2)+66)}) \tag{2.3}$$

for $m = 1, 2, 3, 4$ and $\eta \in (0, 10^{-5})$.

Another main tool is the following truncated Kuznetsov trace formula (see [6, Theorem 6] or [8, Lemma 3.1]).

LEMMA 2.2 (Kuznetsov). *Let m, n be positive integers. Then for arbitrarily small $\varepsilon > 0$,*

$$\sum_{t_j \le T} \frac{2\lambda_j(m)\lambda_j(n)}{L(1, \mathrm{sym}^2 u_j)} = \frac{T^2}{\pi^2}\delta_{m,n} + O\big(T^{1+\varepsilon}(mn)^{7/64+\varepsilon} + (mn)^{1/4+\varepsilon}\big),$$

where $\delta_{m,n}$ is the Kronecker symbol.

REMARK 2.3. Here we have used the relation $\frac{|\rho_j(1)|^2}{\cosh \pi t_j} = \frac{2}{L(1,\mathrm{sym}^2 u_j)}$, where $\rho_j(1)$ is the first Fourier coefficient of u_j.

3. Treatment of "exceptional" Hecke eigenvalues. Although $H^+_{T,\mathrm{sym}^m}(\eta)$ satisfies a weak form of GRH, some difficulties arise from the absence of the generalized Ramanujan conjecture. In fact, the proof of Lemma 3.3 below indicates that these difficulties are caused by small primes. Thus we have to consider the number of $u_j \in H^+_{T,\mathrm{sym}^m}(\eta)$ such that $|\lambda_j(p)| > 2$ for small primes (see (3.2) below). In this direction, the first result was due to Sarnak [17].

LEMMA 3.1 (Sarnak). *Let p be a fixed prime. Then*

$$\#\{j : t_j \le T \text{ and } |\lambda_j(p)| \ge a \ge 2\} \ll T^{2-\frac{\log(a/2)}{\log p}},$$

where the implied constant is absolute.

Applying Lemma 3.1, we divide $H^+_{T,\mathrm{sym}^m}(\eta)$ into two parts. For any $\eta \in (0, 10^{-5}]$, define

$$\begin{aligned}(3.1)\qquad & H^{+,1}_{T,\mathrm{sym}^m}(\eta) = \\ & \{u_j \in H^+_{T,\mathrm{sym}^m}(\eta) : 2015^{-1} \le |\alpha_{u_j}(p)| \le 2015 \text{ for all } p \le (\log T)^{8/\eta}\}\end{aligned}$$

and

$$H^{+,2}_{T,\mathrm{sym}^m}(\eta) = H^+_{T,\mathrm{sym}^m}(\eta) \setminus H^{+,1}_{T,\mathrm{sym}^m}(\eta).$$

Then by Lemma 3.1,

$$|H^{+,2}_{T,\mathrm{sym}^m}(\eta)| \ll T^{2-\frac{\eta \log(2015/2)}{8\log_2 T}}(\log T)^{8/\eta}. \tag{3.2}$$

Combining (3.2) with (2.3), we get

$$|H^{+,1}_{T,\mathrm{sym}^m}(\eta)| = r(T) + O\big(T^{2-\frac{\eta \log(2015/2)}{8\log_2 T}}(\log T)^{8/\eta}\big). \tag{3.3}$$

Therefore, $H^{+,1}_{T,\mathrm{sym}^m}(\eta)$ is also dense in $\{j : t_j \le T\}$ as $T \to \infty$ and satisfies weak forms of GRH and GRC. Then we can get good estimates of $L(s, \mathrm{sym}^m u_j)$ in a certain region near the line $\mathrm{Re}\, s = 1$ for $u_j \in H^{+,1}_{T,\mathrm{sym}^m}(\eta)$ and $m = 1, 2, 3, 4$.

For $m = 1, 2, 3, 4$, define

$$\Lambda_{\mathrm{sym}^m u_j}(n) = \begin{cases} [\alpha_{u_j}(p)^{m\nu} + \alpha_{u_j}(p)^{(m-2)\nu} + \cdots + \alpha_{u_j}(p)^{-m\nu}]\log p & \text{if } n = p^\nu, \\ 0 & \text{otherwise.} \end{cases}$$

For $u_j \in H^+_{T,\mathrm{sym}^m}(\eta)$, following the arguments in [19, pp. 216–217], we can define the logarithm $\log L(s, \mathrm{sym}^m u_j)$ in $\mathcal{S}$ and have the absolutely convergent series

$$\log L(s, \mathrm{sym}^m u_j) = \sum_{n=2}^{\infty} \frac{\Lambda_{\mathrm{sym}^m u_j}(n)}{n^s \log n} \quad (\sigma > 1). \tag{3.4}$$

Moreover, for $s = \sigma + i\tau$, $\sigma > \sigma_0 = 1 - \eta$ and $|\tau| \le 100T^\eta$,

$$\log L(s, \mathrm{sym}^m u_j) \ll \frac{\log T}{\sigma - \sigma_0}, \tag{3.5}$$

where the implied constant is absolute.

LEMMA 3.2. *Let $\eta \in (0, 10^{-5}]$ be fixed, $\sigma_0 = 1 - \eta$, $m = 1, 2, 3, 4$ and $u_j \in H^+_{T,\mathrm{sym}^m}(\eta)$. Let $s = \sigma + i\tau$. Then*

$$\log L(s, \mathrm{sym}^m u_j) = \sum_{n=2}^{\infty} \frac{\Lambda_{\mathrm{sym}^m u_j}(n)}{n^s \log n} e^{-n/H} + R \tag{3.6}$$

uniformly for $3 \le H \le T^\eta$, $\sigma_0 < \sigma \le 3/2$ and $|\tau| \le H$, where

$$R \ll_\eta H^{-(\sigma-\sigma_0)/2} (\log T)/(\sigma - \sigma_0)^2. \tag{3.7}$$

Proof. It is well-known that for $c > 0$,

$$e^{-1/y} = \frac{1}{2\pi i} \int_{(c)} \Gamma(z) y^z \, dz. \tag{3.8}$$

Combining (3.8) with (3.4) yields

$$\sum_{n=2}^{\infty} \frac{\Lambda_{\mathrm{sym}^m u_j}(n)}{n^s \log n} e^{-n/H} = \frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} \Gamma(z - s) \log L(z, \mathrm{sym}^m u_j) H^{z-s} \, dz. \tag{3.9}$$

We shift the line of integration to the path $\mathcal{C}$ consisting of the straight lines joining

$$\kappa - i\infty, \quad \kappa - i2H, \quad \sigma_1 - i2H, \quad \sigma_1 + i2H, \quad \kappa + i2H, \quad \kappa + i\infty,$$

where $\kappa := 23/16 + 1/\log H$ and $\sigma_1 := (\sigma + \sigma_0)/2$. By the residue theorem,

$$\sum_{n=2}^{\infty}\frac{\Lambda_{\mathrm{sym}^m u_j}(n)}{n^s\log n}e^{-n/H} = \log L(s,\mathrm{sym}^m u_j) + \frac{1}{2\pi i}\int_{\mathcal{C}}\Gamma(z-s)\log L(z,\mathrm{sym}^m u_j)H^{z-s}\,dz.$$

By (3.5), the integral over $\mathcal{C}$ is

$$\begin{aligned}
&\ll \frac{H^{\sigma_1-\sigma}\log T}{\sigma-\sigma_0}\int_{|y|\le 3H}|\Gamma(\sigma_1-\sigma+iy)|\,dy + H^{\kappa-\sigma}\int_{|y|\ge 2H}|\Gamma(\kappa-\sigma+iy)|\,dy\\
&\quad+\frac{\log T}{\sigma-\sigma_0}\int_{\sigma_1}^{\kappa}H^{x-\sigma}|\Gamma(x-\sigma+i(2H-\tau))|\,dx\\
&\ll \frac{H^{\sigma_1-\sigma}\log T}{(\sigma-\sigma_0)^2} + H^{\kappa-\sigma}\int_{|y|\ge 2H}|y-\tau|^{\kappa-\sigma-1/2}e^{-\pi|y-\tau|/2}\,dy\\
&\quad+\frac{\log T}{\sigma-\sigma_0}\int_{\sigma_1}^{\kappa}H^{2(x-\sigma)-1/2}e^{-\pi T/2}\,dx\\
&\ll \frac{H^{-(\sigma-\sigma_0)/2}\log T}{(\sigma-\sigma_0)^2}. \ \blacksquare
\end{aligned}$$

LEMMA 3.3. *Let $\eta\in(0,10^{-5}]$ be fixed, $m=1,2,3,4$ and $u_j\in H^{+,1}_{T,\mathrm{sym}^m}(\eta)$. Then*

$$\log L(s,\mathrm{sym}^m u_j)\ll_\eta \frac{H^{2\alpha}}{\alpha\log H}+\log_2 H \tag{3.10}$$

uniformly for $H=(\log T)^{4/\eta}$, $\sigma>1-\alpha>1-\frac12\eta$ and $|t|\le(\log T)^{4/\eta}$.

Proof. In fact, we have only to consider $1-\frac12\eta<\sigma\le 3/2$. Taking $H=(\log T)^{4/\eta}$ in Lemma 3.2, we have

$$\begin{aligned}
\log L(s,\mathrm{sym}^m u_j) &\ll_\eta \sum_p\frac{|\lambda_{\mathrm{sym}^m u_j}(p)|}{p^\sigma}e^{-p/H}+1 \qquad (3.11)\\
&\ll_\eta \sum_{p\le H^2}\frac{|\lambda_{\mathrm{sym}^m u_j}(p)|}{p^\sigma}e^{-p/H}+\sum_{p>H^2}\frac{|\lambda_{\mathrm{sym}^m u_j}(p)|}{p^\sigma}e^{-p/H}+1\\
&\ll_\eta \sum_{p\le H^2}\frac{1}{p^\sigma}+\sum_{p>H^2}\frac{1}{p^{\sigma-7/64}}e^{-p/H}+1\ll_\eta\sum_{p\le H^2}\frac{1}{p^\sigma}+\sum_{p>H^2}e^{-p/H}+1.
\end{aligned}$$

Hence we may assume $1-\frac12\eta<\sigma<1$ and by [18, Lemma 3.2] or [9, (3.20)], the first sum in (3.11) is

$$\sum_{p\le H^2}\frac{1}{p^\sigma}\ll\frac{H^{2(1-\sigma)}}{(1-\sigma)\log H}+\log_2 H.$$

By partial summation, the second sum in (3.11) is

$$\sum_{p>H^2} e^{-p/H} \ll \frac{1}{H}\int_{H^2}^{\infty} xe^{-x/H}\,dx \ll 1. \quad \blacksquare$$

4. Moments of $L(1,\mathrm{sym}^m u_j)$. For $\theta\in\mathbb{C}$, $m\in\mathbb{N}$ and $x\in\mathbb{C}$ satisfying $|x|\max_{0\le\ell\le m}|e^{i(m-2\ell)\theta}|<1$, we denote

$$\begin{aligned} g(\theta) &:= \mathrm{diag}[e^{i\theta}, e^{-i\theta}],\\ \mathrm{sym}^m[g(\theta)] &:= \mathrm{diag}[e^{im\theta}, e^{i(m-2)\theta}, \ldots, e^{-im\theta}],\\ D(x,\mathrm{sym}^m[g(\theta)]) &:= \det(I - x\cdot \mathrm{sym}^m[g(\theta)])^{-1}\\ &= \prod_{0\le\ell\le m}(1-e^{i(m-2\ell)\theta}x)^{-1}. \end{aligned} \tag{4.1}$$

And for $z\in\mathbb{C}$, $m\in\mathbb{N}$ and $\nu\ge 0$, define $\lambda_m^{z,\nu}[g(\theta)]$ by

$$D(x,\mathrm{sym}^m[g(\theta)])^z = \sum_{\nu\ge 0}\lambda_m^{z,\nu}[g(\theta)]x^\nu \qquad \big(|x|\max_{0\le\ell\le m}\{|e^{i(m-2\ell)\theta}|\}<1\big). \tag{4.2}$$

Then

$$\begin{aligned} \lambda_m^{1,1}[g(\theta)] &= \mathrm{tr}(\mathrm{sym}^m[g(\theta)]) = \frac{\sin[(m+1)\theta]}{\sin\theta},\\ \log D(x,\mathrm{sym}^m[g(\theta)]) &= \mathrm{tr}(\mathrm{sym}^m[g(\theta)])x + O\Big(x^2\max_{0\le\ell\le m}|e^{i(m-2\ell)\theta}|^2\Big) \end{aligned} \tag{4.3}$$

for $|x|\max_{0\le\ell\le m}|e^{i(m-2\ell)\theta}|<1$. We denote $\alpha_{u_j}(p)=e^{i\theta_j(p)}$ where $\theta_j(p)\in\mathbb{C}$. Then

$$\lambda_j(p^m) = \frac{\sin[(m+1)\theta_j(p)]}{\sin\theta_j(p)} = \mathrm{tr}(\mathrm{sym}^m[g(\theta_j(p))]) = \lambda_m^{1,1}[g(\theta_j(p))]. \tag{4.4}$$

By (4.1) and (1.3), we have

$$L(s,\mathrm{sym}^m u_j)^z = \prod_p D(p^{-s},\mathrm{sym}^m[g(\theta_j(p))])^z,$$

and it admits a Dirichlet series

$$L(s,\mathrm{sym}^m u_j)^z = \sum_{n\ge 1}\lambda^z_{\mathrm{sym}^m u_j}(n)n^{-s} \qquad (\sigma>1).$$

Define

$$M^z_{\mathrm{sym}^m} = \prod_p \frac{2}{\pi}\int_0^{\pi} D(p^{-1},\mathrm{sym}^m[g(\theta)])^z\sin^2\theta\,d\theta. \tag{4.5}$$

Another main tool is the following analogue of [9, Proposition 6.1].

PROPOSITION 4.1. *Let $\eta \in (0, 10^{-5})$ be fixed. Then there exist positive constant $\delta = \delta(\eta)$ and $c = c(\eta)$ such that*

$$\sum_{u_j \in H^{+,1}_{T,\mathrm{sym}^m}(\eta)} \omega(j) L(1, \mathrm{sym}^m u_j)^z = M^z_{\mathrm{sym}^m} + O_\eta\big(e^{-\delta \log T/(\log_2 T)^2}\big)$$

uniformly for

$$|z| \le \frac{c \log T}{(\log_2 T)^2 \log_3 T}.$$

REMARK 4.2. To prove Proposition 4.1, we need six lemmas which are analogous to [9, Lemmas 6.1–6.6] with similar proofs. The main difference is that we use a truncated version of the Kuznetsov trace formula. Here we give detailed proofs of these six lemmas for the sake of completeness. In our proposition above, the moment is defined in the dense subset $H^{+,1}_{T,\mathrm{sym}^m}(\eta)$ where the proper upper bound for $\log L(s, \mathrm{sym}^m u_j)$ in Lemma 3.3 is needed.

LEMMA 4.3. *For $m \in \mathbb{N}$, $z \in \mathbb{C}$ and integer $\nu > 0$, we have*

$$\lambda^z_{\mathrm{sym}^m u_j}(p^\nu) = \sum_{0 \le \nu' \le m\nu} \mu^{z,\nu}_{m,\nu'} \lambda_j(p^{\nu'}), \tag{4.6}$$

where

$$\mu^{z,\nu}_{m,\nu'} = \frac{2}{\pi} \int_0^\pi \lambda^{z,\nu}_m[g(\theta)] \sin[(\nu'+1)\theta] \sin\theta \, d\theta. \tag{4.7}$$

Furthermore,

$$\begin{aligned} \mu^{z,1}_{m,\nu'} &= z\delta(m,\nu') \qquad (0 \le \nu' \le m), \\ |\mu^{z,\nu}_{m,\nu'}| &\le \binom{(m+1)|z| + \nu - 1}{\nu} \qquad (0 \le \nu' \le m\nu), \end{aligned} \tag{4.8}$$

where $\delta(a,b)$ is 1 for $a = b$ and 0 otherwise.

Proof. The proof is very similar to the proof of [9, Lemma 6.1] with obvious modifications and so we omit it. ∎

LEMMA 4.4. *Let $m, n \in \mathbb{N}$ and $z \in \mathbb{C}$. Then*

$$\sum_{t_j \le T} \omega(j) \lambda^z_{\mathrm{sym}^m u_j}(n) = \lambda^z_{\mathrm{sym}^m}(n) + O_{m,\varepsilon}\big(T^{-1+\varepsilon} n^{m/4+\varepsilon} r^z_m(n)\big), \tag{4.9}$$

where $\lambda^z_{\mathrm{sym}^m}(n)$ and $r^z_m(n)$ are the multiplicative functions defined by

$$\lambda^z_{\mathrm{sym}^m}(p^\nu) = \mu^{z,\nu}_{m,0}, \qquad r^z_m(p^\nu) = (m\nu + 1)\binom{(m+1)|z| + \nu - 1}{\nu}. \tag{4.10}$$

Furthermore there is a constant $c = c(m)$ such that

$$\sum_{n \le t} r^z_m(n) \ll_m t[\log(wt)]^{z_m - 1} e^{c|z| \log_2(|z|+3)} \tag{4.11}$$

uniformly for $t \geq 1$ *and* $z \in \mathbb{C}$, *where* $z_m = (m+1)^2 z^*$ *and* z^* *is the smallest integer* n *such that* $n \geq |z|$.

Proof. The proof follows the method of [9, Lemma 6.2] by using the Kuznetsov trace formula of Lemma 2.2 instead of the Petersson trace formula. ∎

For notational convenience, we set

$$\omega^z_{\mathrm{sym}^m u_j}(x) = \sum_{n=1}^{\infty} \frac{\lambda^z_{\mathrm{sym}^m u_j}(n)}{n} e^{-n/x}. \tag{4.12}$$

LEMMA 4.5. *Let* $m \in \mathbb{N}$, $x \geq 3$ *and* $z \in \mathbb{C}$. *Then for any* $\varepsilon > 0$,

$$\sum_{t_j \leq T} \omega(j)\omega^z_{\mathrm{sym}^m u_j}(x) = \sum_{n=1}^{\infty} \frac{\lambda^z_{\mathrm{sym}^m}(n)}{n} e^{-n/x} + O_m\big(T^{-1+\varepsilon} x^{m/4+\varepsilon}[(z_m+1)\log x]^{z_m}\big).$$

Proof. By the definition of $\omega^z_{\mathrm{sym}^m f}(x)$ and (4.9), we have

$$\begin{aligned}\sum_{t_j \leq T} \omega(j)\omega^z_{\mathrm{sym}^m u_j}(x) &= \sum_{t_j \leq T} \omega(j) \sum_{n=1}^{\infty} \frac{\lambda^z_{\mathrm{sym}^m f}(n)}{n} e^{-n/x} \\ &= \sum_{n=1}^{\infty} \frac{e^{-n/x}}{n} \sum_{t_j \leq T} \omega(j)\lambda^z_{\mathrm{sym}^m f}(n) \\ &= \sum_{n=1}^{\infty} \frac{\lambda^z_{\mathrm{sym}^m}(n)}{n} e^{-n/x} \\ &\quad + O\Big(T^{-1+\varepsilon} \sum_{n=1}^{\infty} n^{-1+m/4+\varepsilon} e^{-n/x} r^z_m(n)\Big).\end{aligned}$$

Integrating by parts, with (4.11) we obtain

$$\begin{aligned}\sum_{n=1}^{\infty} n^{-1+m/4+\varepsilon} e^{-n/x} r^z_m(n) &= \int_{1-}^{\infty} t^{-1+m/4+\varepsilon} e^{-t/x}\, d\sum_{n \leq t} r^z_m(n) \\ &\ll_m e^{c|z|\log_2(|z|+3)} \int_1^{\infty} \frac{[\log(3t)]^{z_m}}{t^{1-m/4-\varepsilon}} e^{-t/x}\Big(1+\frac{t}{x}\Big)\, dt.\end{aligned}$$

On the other hand,

$$\int_1^{x} \frac{[\log(3t)]^{z_m}}{t^{1-m/4-\varepsilon}} e^{-t/x}\Big(1+\frac{t}{x}\Big)\, dt \ll x^{m/4+\varepsilon}(\log x)^{z_m},$$

$$\int_x^{\infty} \frac{[\log(3t)]^{z_m}}{t^{1-m/4-\varepsilon}} e^{-t/x}\Big(1+\frac{t}{x}\Big)\, dt \ll x^{m/4+\varepsilon} \int_1^{\infty} u^{m/4+\varepsilon} e^{-u}[\log(3ux)]^{z_m}\, du$$

$$\ll x^{m/4+\varepsilon}(\log x)^{z_m}\sum_{\nu=0}^{z_m}\binom{z_m}{\nu}\int_1^\infty u^{m+\nu}e^{-u}\,du$$
$$\ll x^{m/4+\varepsilon}[(z_m+1)\log x]^{z_m}. \ \blacksquare$$

LEMMA 4.6 ([9, Lemma 6.4]). *Let $m\in\mathbb{N}$, $z\in\mathbb{C}$ and define $z'_m := (m+1)|z|+3$. Then there exists a constant $c=c(m)>0$ such that*

$$\sum_n \frac{|\lambda^z_{\mathrm{sym}^m}(n)|}{n^\sigma} \le \exp\biggl(cz'_m\biggl(\log_2 z'_m + \frac{{z'_m}^{(1-\sigma)/\sigma}-1}{(1-\sigma)\log z'_m}\biggr)\biggr) \tag{4.13}$$

for any $\sigma\in(1/2,1]$. Further

$$\sum_n \frac{\lambda^z_{\mathrm{sym}^m}(n)}{n} = \prod_p \frac{2}{\pi}\int_0^\pi D(p^{-1},\mathrm{sym}^m[g(\theta)])^z\sin^2\theta\,d\theta. \tag{4.14}$$

LEMMA 4.7 ([9, Lemma 6.5]). *Let $m\in\mathbb{N}$, $\sigma\in[0,1/3)$, $x\ge 3$ and $z\in\mathbb{C}$. There exists a constant $c=c(m)$ such that*

$$\sum_{n=1}^\infty \frac{\lambda^z_{\mathrm{sym}^m}(n)}{n}e^{-n/x}$$
$$= M^z_{\mathrm{sym}^m} + O\biggl(x^{-\sigma}\exp\biggl\{cz'_m\biggl(\log_2 z'_m + \frac{{z'_m}^{\sigma/(1-\sigma)}-1}{\sigma\log z'_m}\biggr)\biggr\}\biggr).$$

The implied constant depends on m only.

LEMMA 4.8. *Let $\eta\in(0,10^{-5}]$ be fixed, $m=1,2,3,4$ and $u_j\in H^{+,1}_{T,\mathrm{sym}^m}(\eta)$. Then*

$$L(1,\mathrm{sym}^m u_j)^z = \omega^z_{\mathrm{sym}^m u_j}(x) + O_\eta\bigl((x^{-1/\log_2 T} + x^{c|z|}e^{-(\log T)^4})e^{c|z|\log_3 T}\bigr).$$

Proof. Note that

$$\omega^z_{\mathrm{sym}^m u_j}(x) = \frac{1}{2\pi i}\int_{(1)} L(s+1,\mathrm{sym}^m u_j)^z\Gamma(s)x^s\,ds.$$

Shift the line of integration to the path $\mathcal{C}$ consisting of the straight lines joining

$$\kappa_1 - i\infty,\quad \kappa_1 - iH,\quad -\kappa_2 - iH,\quad -\kappa_2 + iH,\quad \kappa_1 + iH,\quad \kappa_1 + i\infty,$$

where $\kappa_1 := 1/\log x$, $\kappa_2 := 1/\log_2 T$ and $H := (\log T)^4$. Then

$$\omega^z_{\mathrm{sym}^m u_j}(x) = L(1,\mathrm{sym}^m u_j)^z + \frac{1}{2\pi i}\int_{\mathcal{C}} L(s+1,\mathrm{sym}^m u_j)^z\Gamma(s)x^s\,ds.$$

By using Lemma 3.3 and (3.5), we get

$$\frac{1}{2\pi i}\int_{\mathcal{C}} L(s+1,\mathrm{sym}^m u_j)^z\Gamma(s)x^s ds \ll_\eta x^{-\kappa_2}e^{c|z|\log_3 T}\int_{|y|\le H}|\Gamma(1-\kappa_1+iy)|\,dy$$
$$+ e^{c|z|\log_3 T}\int_{-\kappa_2}^{\kappa_1}|\Gamma(1+\alpha+iH)|d\alpha + e^{c|z|\log T}\int_{|y|>H}|\Gamma(1+\kappa_1+iy)|\,dy,$$

which implies the result by the Stirling formula. $\blacksquare$

Now we are ready to prove Proposition 4.1. We have

$$\sum_{u_j\in H^{+,1}_{T,\mathrm{sym}^m}(\eta)} \omega(j)L(1,\mathrm{sym}^m u_j)^z = \sum_{u_j\in H^{+,1}_{T,\mathrm{sym}^m}(\eta)} \omega(j)\omega^z_{\mathrm{sym}^m u_j}(x) + O_\eta(R_1), \tag{4.15}$$

where

$$R_1 = \big(x^{-1/\log_2 T} + x^{c|z|}e^{-(\log T)^4}\big)e^{c|z|\log_3 T}.$$

Here we have used

$$\sum_{u_j\in H^{+,1}_{T,\mathrm{sym}^m}(\eta)} \omega(j) \le \sum_{t_j\le T}\omega(j) = 1 + O_\varepsilon(T^{-1+\varepsilon})$$

by (1.12).

On the other hand, for $x \ge 3$ and $z\in\mathbb{C}$, we have

$$\begin{aligned}\omega^z_{\mathrm{sym}^m u_j}(x) = \frac{1}{2\pi i}\int_{(1)} L(s+1,\mathrm{sym}^m u_j)^z\Gamma(s)x^s\,ds &\ll 1000^{|\mathrm{Re}\, z|}x\int_{(1)}|\Gamma(s)|\,ds \\ &\ll 1000^{|\mathrm{Re}\, z|}x.\end{aligned}$$

Thus, by (1.11), (2.2) and (3.2),

$$\Big|\sum_{u_j\in H^{+,2}_{\mathrm{sym}^m}(1;\eta)} \omega(j)\omega^z_{\mathrm{sym}^m u_j}(x)\Big| \ll x1000^{|\mathrm{Re}\, z|}e^{-\frac{\log T}{\log_2 T}}$$

$$\Big|\sum_{u_j\in H^{-}_{T,\mathrm{sym}^m}(\eta)} \omega(j)\omega^z_{\mathrm{sym}^m u_j}(x)\Big| \ll x1000^{|\mathrm{Re}\, z|}T^{100\eta-2}\log T.$$

Therefore,

$$\sum_{u_j\in H^{+,1}_{T,\mathrm{sym}^m}(\eta)} \omega(j)L(1,\mathrm{sym}^m u_j)^z = \sum_{t_j\le T}\omega(j)\omega^z_{\mathrm{sym}^m u_j}(x) + O_\eta(R_2),$$

where

$$R_2 = R_1 + x1000^{|\mathrm{Re}\, z|}e^{-\frac{\log T}{\log_2 T}}.$$

Hence by Lemmas 4.5 and 4.7,

$$\sum_{u_j\in H^{+,1}_{T,\mathrm{sym}^m}(\eta)} \omega(j)L(1,\mathrm{sym}^m u_j)^z = M^z_{\mathrm{sym}^m} + O_\eta(R_3),$$

where

$$\begin{aligned}R_3 = R_2 &+ T^{-1+\varepsilon}x^{m/4+\varepsilon}[(z_m+1)\log x]^{z_m} \\ &+ x^{-\sigma}\exp\Big\{c_3 z'_m\Big(\log_2 z'_m + \frac{(z'_m)^{\sigma/(1-\sigma)}-1}{\sigma\log z'_m}\Big)\Big\}.\end{aligned}$$

Take $\varepsilon = 10^{-5}$, $\sigma = 1/\log(|z|+8)$ and $x = e^{\frac{\log T}{10\log_2 T}}$. It is easy to verify that there are positive constants c and δ depending at most on η such that the error term is $\ll e^{-\delta \log T/(\log_2 T)^2}$ uniformly for $|z| \le \frac{c\log T}{(\log_2 T)^2 \log_3 T}$.

5. Proof of Theorem 1.1. The key point here is to establish an analogue of [9, (8.3)]. However, we may have $|\lambda_j(p)| > 2$, i.e. $\theta_j(p) \notin [0,\pi]$ for some j and prime p due to the absence of GRC. Therefore, we have to control the contribution of these possible "exceptional" Hecke eigenvalues. To this end, we introduce a new dense subset of $\{j : t_j \le T\}$: for any $\eta \in (0, 10^{-5}]$, define

$$(5.1)\qquad H^*_{T,\mathrm{sym}^m}(\eta) = \\ \Big\{u_j \in H^+_{T,\mathrm{sym}^m}(\eta) : \Big(1+\frac{1}{\log_3 T}\Big)^{-1} \le |\alpha_{u_j}(p)| \le 1+\frac{1}{\log_3 T}, \\ \forall p \le (\log T)^{8/\eta}\Big\}$$

with $H^+_{T,\mathrm{sym}^m}(\eta)$ defined by (2.1). Then by Lemma 3.1,

$$(5.2)\qquad |H^+_{T,\mathrm{sym}^m}(\eta) \setminus H^*_{T,\mathrm{sym}^m}(\eta)| \ll T^{2-\frac{\eta}{16(\log_2 T)(\log_3 T)^2}} (\log T)^{8/\eta}.$$

Thus by (2.3),

$$(5.3)\qquad |H^*_{T,\mathrm{sym}^m}(\eta)| = r(T) + O\big(T^{2-\frac{\eta}{16(\log_2 T)(\log_3 T)^2}} (\log T)^{8/\eta}\big).$$

By Lemma 3.2 with $\eta \in (0, 10^{-5}]$ being fixed and $H = (\log T)^{16/\eta}$, we see that for $u_j \in H^*_{T,\mathrm{sym}^m}(\eta)$,

$$\log L(1, \mathrm{sym}^m u_j) = \sum_{n=2}^{\infty} \frac{\Lambda_{\mathrm{sym}^m u_j}(n)}{n^s \log n} e^{-n/H} + o(1).$$

By the definition of $H^*_{T,\mathrm{sym}^m}(\eta)$ and Lebesgue's dominated convergence theorem, we have

$$\sum_p \sum_{\nu \ge 2} \frac{\Lambda_{\mathrm{sym}^m u_j}(p^\nu)}{p^\nu \log p^\nu} (e^{-p^\nu/H} - e^{-\nu p/H}) \to 0 \qquad (T \to \infty).$$

Following the arguments in [9, p. 468], we get

$$(5.4)\qquad \log L(1, \mathrm{sym}^m u_j) = \sum_{p \le H} \log D(p^{-1}, \mathrm{sym}^m[g(\theta_j(p))]) + o(1).$$

Let $\theta^{\pm}_{m,p} \in [0,\pi]$ be determined by

$$(5.5)\qquad \begin{cases} D(p^{-1}, \mathrm{sym}^m[g(\theta^+_{m,p})]) = \max_{\theta\in[0,\pi]} D(p^{-1}, \mathrm{sym}^m[g(\theta)]), \\ D(p^{-1}, \mathrm{sym}^m[g(\theta^-_{m,p})]) = \min_{\theta\in[0,\pi]} D(p^{-1}, \mathrm{sym}^m[g(\theta)]). \end{cases}$$

Then (see [9, (7.3)])

$$A_m^{\pm} = \max_{\theta\in[0,\pi]} \pm\mathrm{tr}(\mathrm{sym}^m[g(\theta)]) = \pm\mathrm{tr}(\mathrm{sym}^m[g(\theta_m^{\pm})]), \tag{5.6}$$

$$B_m^{\pm} = \exp\biggl\{\gamma_0 + \frac{1}{A_m^{\pm}}\sum_p \biggl(\pm\log D(p^{-1}, \mathrm{sym}^m[g(\theta_{m,p}^{\pm})]) - \frac{A_m^{\pm}}{p}\biggr)\biggr\}, \tag{5.7}$$

where γ_0 is a constant satisfying

$$\sum_{p\le t}\frac{1}{p} = \log_2 t + \gamma_0 + O\biggl(\frac{1}{\log t}\biggr).$$

Next we will divide the summation on the right-hand side of (5.4) into two parts,

$$\biggl(\sum_{\substack{p\le H\\ \theta_j(p)\in[0,\pi]}} + \sum_{\substack{p\le H\\ \theta_j(p)\notin[0,\pi]}}\biggr)\log D(p^{-1}, \mathrm{sym}^m[g(\theta_j(p))]). \tag{5.8}$$

If $u_j \in H^*_{T,\mathrm{sym}^m}(\eta)$ and $\theta_j(p) \notin [0,\pi]$, then $\theta_j(p) \in i(0, 1/\log_3 T] \cup \pi + i(0, 1/\log_3 T]$ and $D(x, \mathrm{sym}^m[g(\theta_j(p))])$ is real by its definition in (4.1). Moreover, it is easy to deduce that

$$\sum_{\substack{p\le H\\ \theta_j(p)\notin[0,\pi]}} \log D(p^{-1}, \mathrm{sym}^m[g(\theta_j(p))]) \le \sum_{\substack{p\le H\\ \theta_j(p)\notin[0,\pi]}} \log D(p^{-1}, \mathrm{sym}^m[g(i/\log_3 T)]).$$

Combining the above formula with (4.3), we get

$$\sum_{\substack{p\le H\\ \theta_j(p)\notin[0,\pi]}} \log D(p^{-1}, \mathrm{sym}^m[g(\theta_j(p))]) \le \sum_{\substack{p\le H\\ \theta_j(p)\notin[0,\pi]}} \biggl(\frac{\mathrm{tr}(\mathrm{sym}^m[g(i/\log_3 T)])}{p} + O\biggl(\frac{1}{p^2}\biggr)\biggr).$$

Noting that (see (1.5) and the definition of $\mathrm{sym}^m[g(\theta)]$ in (4.1))

$$\mathrm{tr}(\mathrm{sym}^m[g(i/\log_3 T)]) = \mathrm{tr}(\mathrm{sym}^m[g(0)]) + O\biggl(\frac{1}{\log_3 T}\biggr) = A_m^+ + O\biggl(\frac{1}{\log_3 T}\biggr),$$

we deduce that

$$\begin{aligned}\sum_{\substack{p\le H\\ \theta_j(p)\notin[0,\pi]}} \log D(p^{-1}, \mathrm{sym}^m[g(\theta_j(p))]) &\le \sum_{\substack{p\le H\\ \theta_j(p)\notin[0,\pi]}} \biggl(\frac{A_m^+}{p} + O\biggl(\frac{1}{p\log_3 T} + \frac{1}{p^2}\biggr)\biggr)\\
&= \sum_{\substack{p\le H\\ \theta_j(p)\notin[0,\pi]}} \biggl(\frac{\mathrm{tr}(\mathrm{sym}^m[g(\theta_{m,p}^+)])}{p} + \frac{A_m^+ - \mathrm{tr}(\mathrm{sym}^m[g(\theta_{m,p}^+)])}{p}\\
&\qquad + O\biggl(\frac{1}{p\log_3 T} + \frac{1}{p^2}\biggr)\biggr).\end{aligned}$$

By using (4.3) and [9, (8.4)], we obtain

$$\begin{aligned}
(5.9)\qquad &\sum_{\substack{p\le H\\ \theta_j(p)\notin[0,\pi]}} \log D(p^{-1},\mathrm{sym}^m[g(\theta_j(p))]) \\
&\le \sum_{\substack{p\le H\\ \theta_j(p)\notin[0,\pi]}} \bigg(\log D(p^{-1},\mathrm{sym}^m[g(\theta^+_{m,p})]) + \frac{A^+_m - \mathrm{tr}(\mathrm{sym}^m[g(\theta^+_{m,p})])}{p} \\
&\qquad\qquad + O\Big(\frac{1}{p\log_3 T} + \frac{1}{p^2}\Big)\bigg) \\
&= \sum_{\substack{p\le H\\ \theta_j(p)\notin[0,\pi]}} \bigg(\log D(p^{-1},\mathrm{sym}^m[g(\theta^+_{m,p})]) + O\Big(\frac{1}{p\log_3 T} + \frac{1}{p^2}\Big)\bigg).
\end{aligned}$$

For the other direction, if $m=2,4$ and $\theta_j(p)\in i(0,1/\log_3 T]\cup \pi + i(0,1/\log_3 T]$, then $D(p^{-1},\mathrm{sym}^m[g(\theta_j(p))])>1$. On the other hand, it is easy to see that

$$D(p^{-1},\mathrm{sym}^m[g(\theta^-_{m,p})])\le 1.$$

Therefore, for $m=2,4$,

$$\begin{aligned}
(5.10)\qquad &\sum_{\substack{p\le H\\ \theta_j(p)\notin[0,\pi]}} \log D(p^{-1},\mathrm{sym}^m[g(\theta_j(p))]) \\
&\qquad > \sum_{\substack{p\le H\\ \theta_j(p)\notin[0,\pi]}} \log D(p^{-1},\mathrm{sym}^m[g(\theta^-_{m,p})]).
\end{aligned}$$

If $m=1,3$, then by the definition of $D(x,\mathrm{sym}^m[g(\theta)])$ in (4.1), it is easy to see that

$$\begin{aligned}
&\sum_{\substack{p\le H\\ \theta_j(p)\notin[0,\pi]}} \log D(p^{-1},\mathrm{sym}^m[g(\theta_j(p))]) \\
&\qquad \ge \sum_{\substack{p\le H\\ \theta_j(p)\notin[0,\pi]}} \log D(p^{-1},\mathrm{sym}^m[g(\pi + i/\log_3 T)]).
\end{aligned}$$

Combining the above formula with (4.3), we get

$$\begin{aligned}
&\sum_{\substack{p\le H\\ \theta_j(p)\notin[0,\pi]}} \log D(p^{-1},\mathrm{sym}^m[g(\theta_j(p))]) \\
&\qquad \ge \sum_{\substack{p\le H\\ \theta_j(p)\notin[0,\pi]}} \bigg(\frac{\mathrm{tr}(\mathrm{sym}^m[g(\pi + i/\log_3 T)])}{p} + O\Big(\frac{1}{p^2}\Big)\bigg).
\end{aligned}$$

Noting that for $m = 1, 3$ (see (1.5) and the definition of $\mathrm{sym}^m[g(\theta)]$ in (4.1)),

$$\begin{aligned}\mathrm{tr}(\mathrm{sym}^m[g(\pi + i/\log_3 T)]) &= -\mathrm{tr}(\mathrm{sym}^m[g(0)]) + O\biggl(\frac{1}{\log_3 T}\biggr)\\ &= A_m^- + O\biggl(\frac{1}{\log_3 T}\biggr),\end{aligned}$$

we infer that

$$\begin{aligned}&\sum_{\substack{p\le H\\ \theta_j(p)\notin[0,\pi]}} \log D(p^{-1}, \mathrm{sym}^m[g(\theta_j(p))]) \qquad (5.11)\\
&\ge \sum_{\substack{p\le H\\ \theta_j(p)\notin[0,\pi]}} \biggl(\frac{A_m^-}{p} + O\biggl(\frac{1}{p\log_3 T} + \frac{1}{p^2}\biggr)\biggr)\\
&= \sum_{\substack{p\le H\\ \theta_j(p)\notin[0,\pi]}} \biggl(\frac{\mathrm{tr}(\mathrm{sym}^m[g(\theta_{m,p}^-)])}{p} + \frac{A_m^- - \mathrm{tr}(\mathrm{sym}^m[g(\theta_{m,p}^-)])}{p}\\
&\qquad\qquad + O\biggl(\frac{1}{p\log_3 T} + \frac{1}{p^2}\biggr)\biggr)\\
&= \sum_{\substack{p\le H\\ \theta_j(p)\notin[0,\pi]}} \biggl(\log D(p^{-1}, \mathrm{sym}^m[g(\theta_{m,p}^-)]) + \frac{A_m^- - \mathrm{tr}(\mathrm{sym}^m[g(\theta_{m,p}^-)])}{p}\\
&\qquad\qquad + O\biggl(\frac{1}{p\log_3 T} + \frac{1}{p^2}\biggr)\biggr)\\
&= \sum_{\substack{p\le H\\ \theta_j(p)\notin[0,\pi]}} \biggl(\log D(p^{-1}, \mathrm{sym}^m[g(\theta_{m,p}^-)]) + O\biggl(\frac{1}{p\log_3 T} + \frac{1}{p^2}\biggr)\biggr).\end{aligned}$$

Here we have applied (4.3) in the third step and applied [9, (8.4)] in the last step.

Combining (5.9), (5.10), (5.11), (5.4) and (5.5), we obtain an analogue of [9, (8.3)]:

$$\begin{aligned}\sum_{p\le H} \log D(p^{-1}, \mathrm{sym}^m[g(\theta_{m,p}^+)]) + O(1) &\ge \log L(1, \mathrm{sym}^m u_j) \qquad (5.12)\\ &\ge \sum_{p\le H} \log D(p^{-1}, \mathrm{sym}^m[g(\theta_{m,p}^-)]) + O(1).\end{aligned}$$

The arguments in [9, p. 469] yield

$$\sum_{p\le H} \log D(p^{-1}, \mathrm{sym}^m[g(\theta_{m,p}^\pm)]) = \pm A_m^\pm \log(B_m^\pm \log H) + O(1).$$

Then Theorem 1.1 follows plainly.

6. Proof of Theorem 1.3. To prove Theorem 1.3, we follow Xiao's method [20] which is a variant of the idea of Lamzouri [7] . In order to get a better error term, the method needs a little more careful calculations to give a good estimation of $\log M^{\pm r}_{\mathrm{sym}^m}$ (see [20, Lemma 7.2]). Since the case of Maass cusp forms is similar to that of holomorphic cusp forms, we just sketch the proof here.

Define

$$h_m^+(t) = \log\bigg(\frac{2}{\pi}\int_0^\pi \exp\bigg(\frac{t}{m+1}\sum_{j=0}^m \cos(\theta(m-2j))\bigg)\sin^2\theta\,d\theta\bigg),$$

$$h_m^-(x) = \begin{cases} \log\bigg(\dfrac{2}{\pi}\displaystyle\int_0^\pi \exp\bigg(-\dfrac{\mathrm{tr}(\mathrm{sym}^m[g(\theta)])}{m+1}x\bigg)\sin^2\theta\,d\theta\bigg) & \text{if } x<1,\\ \log\bigg(\dfrac{2}{\pi}\displaystyle\int_0^\pi \exp\bigg(-\dfrac{\mathrm{tr}(\mathrm{sym}^m[g(\theta)])}{m+1}x\bigg)\sin^2\theta\,d\theta\bigg) - \dfrac{A_m^-}{m+1}x & \text{if } x\ge 1. \end{cases}$$

Define also

$$\mathscr{A}_m^+ = 1 + \int_0^1 (h_m^+(t)/t^2)\,dt + \int_1^\infty ((h_m^+(t)-t)/t^2)\,dt, \tag{6.1}$$

$$\mathscr{A}_m^- = 1 + \int_0^\infty \frac{h_m^-(u)}{u^2}\,du + \log\frac{m+1}{A_m^-}. \tag{6.2}$$

Set

$$\mathfrak{F}_T^\pm(t) := \sum_{\substack{t_j\le T\\ L(1,\mathrm{sym}^m u_j)\gtreqless (B_m^\pm t)^{\pm A_m^\pm}}} \omega(j), \quad \mathfrak{F}_T^{\pm,*}(t) := \sum_{\substack{u_j\in H^{+,1}_{T,\mathrm{sym}^m}(\eta)\\ L(1,\mathrm{sym}^m u_j)\gtreqless (B_m^\pm t)^{\pm A_m^\pm}}} \omega(j).$$

Then for some constant c_η depending on η we have

$$\mathfrak{F}_T^\pm(t) = \mathfrak{F}_T^{\pm,*}(t) + O(\exp(-c_\eta \log T/\log_2 T)). \tag{6.3}$$

Following the method of Xiao [20, Section 7], for any $t \le \log_2 T - 2\log_3 T - \log_4 T - c_5$, we can get

$$\mathfrak{F}_T^{+,*}(t) = \exp\bigg(-\frac{e^{t-\mathscr{A}_m^+}}{t}\bigg\{1 + O\bigg(\frac{1}{t}\bigg)\bigg\}\bigg).$$

Together with (6.3) and (1.12) the estimate for $\mathscr{F}_T^+(t,\mathrm{sym}^m)$ follows. A similar estimate holds for $\mathscr{F}_T^-(t,\mathrm{sym}^m)$.

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Yingnan Wang
College of Mathematics and Statistics
Shenzhen University
Shenzhen, Guangdong 518060, P.R. China
E-mail: ynwang@szu.edu.cn

Xuanxuan Xiao
Faculty of Information Technology
Macau University of Science and Technology
Avenida Wai Long, Taipa, Macau
E-mail: xiaoxuan.uhp@gmail.com