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Boundary asymptotics of the relative Bergman kernel metric for elliptic curves II: subleading terms

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Abstract. For a Legendre family of elliptic curves near the moduli space boundary, we study asymptotic behavior at a node of the relative Bergman kernel metric and show that its curvature form coincides with the Poincaré metric of $\mathbb{C} \setminus \{0, 1\}$. Four-term and three-term asymptotic expansion formulas near 0 are obtained for this metric and its Kähler potential, respectively.

1. Introduction. The Bergman kernel is a reproducing kernel of the space of L^2 holomorphic top degree forms on a complex manifold and is determined by the complex structure.

1.1. Background. The variations of the Bergman kernel on pseudoconvex domains were initially studied by Maitani and Yamaguchi [M-Y]; their results were later generalized to higher dimensional cases by Berndtsson [B1]. For the cases of Stein manifolds and complex projective algebraic manifolds of arbitrary dimension, see [B2], [T] and [B-P]. Moreover, recently these variations turned out to have a close relation to the optimal constant version of the Ohsawa–Takegoshi L^2 extension theorem (see [G-Z], [C], [Bl], [B-L], [O]). Roughly speaking, the plurisubharmonic variation results for the Bergman kernel state certain semipositivity properties of direct images of relative canonical bundles.

Being restricted to the one-dimensional case, namely a family of Riemann surfaces parametrized by one variable from the complex plane, the Bergman kernel on each fiber X_λ, denoted by B_λ, can be written as $B_\lambda = k_\lambda(z) dz \wedge d\bar{z}$ in some local coordinate z for some local function k_λ. Then, due to the

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plurisubharmonic variation results for the Bergman kernel, we have

$$L_{\lambda,z} := \sqrt{-1}\,\partial_\lambda\bar\partial_\lambda \log k_\lambda(z) \geq 0 \tag{1.1}$$

if the fiber X_λ is smooth. Note that $L_{\lambda,z}$ is independent of local coordinates, since the Jacobian determinants of transition functions are killed by the $\partial_\lambda\bar\partial_\lambda$-operator.

Now suppose some X_{λ_0} is a singular complex algebraic curve. Then a natural question is to characterize $L_{\lambda,z}$ as λ approaches λ_0. This limiting case is not fully understood, especially the asymptotic behavior. For bounded planar domains, an interesting formula for so-called generalized annuli was obtained in [W] using various elliptic functions. We consider the compact case. In the affine coordinate $(x, y) \in \mathbb{C}^2$, the so-called Legendre family of elliptic curves $X_\lambda := \{y^2 = x(x-1)(x-\lambda)\} \cup \{\infty\}$, $\lambda \in \mathbb{C} \setminus \{0, 1\}$, gives a general description of genus one compact Riemann surfaces (complex tori), since their moduli space is $\mathbb{C} \setminus \{0, 1\}$ (see [J, p. 75, p. 261]) and X_λ degenerates to a singular curve with a node as λ tends to the boundary of the moduli space, i.e., $\{0, 1, \infty\}$. Using the Weierstrass $\wp$-function's coordinate parametrization and the elliptic modular lambda function's Taylor expansion, the author [D1] showed that in this case $L_{\lambda,z}$ blows up and has hyperbolic growth as $\lambda \to 0$. Note that the Poincaré hyperbolic metric $\omega_{\mathbb{D}^*}$ on the punctured unit disk $\mathbb{D}^* := \mathbb{D} \setminus \{0\}$ has exactly the same asymptotic behavior near 0. In other words, they are asymptotically similar. Explicit asymptotic formulas of $L_{\lambda,z}$ near the moduli space boundary points 1 and ∞ are obtained in [D2].

1.2. Main results. In this paper, for the above Legendre family of elliptic curves X_λ, we calculate the following explicit four-term asymptotic expansion formula near 0, showing that in fact $L_{\lambda,z}$ and $\omega_{\mathbb{D}^*}$ are not the same. Another motivation of doing so is that we believe that each term and in particular its coefficient should contain certain geometrical interpretations which indicate the geometry of the base varieties and their singularities.

THEOREM 1.1. *For $\lambda \in \mathbb{C} \setminus \{0, 1\}$, let B_λ denote the Bergman kernel of X_λ and write $B_\lambda = k_\lambda(z)dz \wedge d\bar z$ in a local coordinate z. Then, as $\lambda \to 0$,*

$$L_{\lambda,z} = \frac{\sqrt{-1}\,d\lambda \wedge d\bar\lambda}{|\lambda|^2(-\log|\lambda|^2)^2} \times \left(1 + \frac{2\log 16}{\log|\lambda|} + 3\left(\frac{\log 16}{\log|\lambda|}\right)^2 + 4\left(\frac{\log 16}{\log|\lambda|}\right)^3 + \mathrm{O}\left(\frac{1}{(\log|\lambda|)^4}\right)\right).$$

As $\lambda \to 0$, X_λ degenerates to a singular curve $X_0 := \{y^2 = x^2(x-1)\} \cup \{\infty\}$. As we can see, even though the second and fourth terms tend to $-\infty$, $L_{\lambda,z}$ which is mainly affected by the leading term still tends to ∞. In general, the strict positivity of $L_{\lambda,z}$ is related to hyperellipticity and Weierstrass points (see [B3]). But in this special case, we find that $L_{\lambda,z}$ is strictly positive everywhere inside the moduli space and thus it defines a new Kähler metric, namely $L_{\lambda,z} > 0$ for all $\lambda \in \mathbb{C} \setminus \{0,1\}$.

QUESTION. What is the Gaussian curvature of $L_{\lambda,z}$?

After careful computations, the curvature is found to be identically -4. Moreover, the result is as follows.

THEOREM 1.2. *Under the assumptions of Theorem* 1.1*,* $L_{\lambda,z}$ *is the Poincaré metric of* $\mathbb{C} \setminus \{0,1\}$.

On the one hand, this result seems to suggest a connection between the Bergman kernel's variation and the moduli space's hyperbolic metric. On the other hand, a four-term expansion formula for the Poincaré metric of $\mathbb{C} \setminus \{0,1\}$ is obtained as a corollary.

COROLLARY 1.3. *Let* $\omega_{0,1}$ *denote the Poincaré metric of* $\mathbb{C}\setminus\{0,1\}$*. Then, as* $\lambda \to 0$*,*

$$\omega_{0,1} = \frac{\sqrt{-1}\, d\lambda \otimes d\bar{\lambda}}{|\lambda|^2(-\log|\lambda|^2)^2} \times \left(1 + \frac{2\log 16}{\log|\lambda|} + 3\left(\frac{\log 16}{\log|\lambda|}\right)^2 + 4\left(\frac{\log 16}{\log|\lambda|}\right)^3 + \mathrm{O}\left(\frac{1}{(\log|\lambda|)^4}\right)\right).$$

The leading term of the above expansion formula implies that near the origin, $\omega_{0,1}$ is asymptotically similar to $\omega_{\mathbb{D}^*}$, and the negative second term seems to indicate that the latter metric is bigger. Actually, we always have $\omega_{0,1} \le \omega_{\mathbb{D}^*}$ in $\mathbb{D}^*$ (see [S-V]). Finally, define $p(\lambda) := -\log(\operatorname{Im}\tau(\lambda))$, where $\tau(\cdot)$ is the inverse function of the elliptic modular lambda function and $\lambda \in \mathbb{C} \setminus \{0,1\}$. We have the following asymptotic expansion formula near 0.

THEOREM 1.4. *Under the assumptions of Corollary* 1.3*,* $p(\lambda)$ *is a Kähler potential of* $\omega_{0,1}$*. And as* $\lambda \to 0$*,*

$$p(\lambda) = -\log(-\log|\lambda|) + \log\pi + \frac{\log 16}{\log|\lambda|} + \mathrm{O}\left(\frac{1}{(\log|\lambda|)^2}\right).$$

We remark that the first three terms on the right hand side above (denoted by $\tilde{p}(\lambda)$) is a Kähler potential that exactly gives rise to the first two terms in the asymptotic expansion in Corollary 1.3.

In Section 2, we recall basic ingredients related to our problem and provide a new proof of a known fact that on any compact Riemann surface of genus $g \ge 1$, there exists a global holomorphic 1-form which does not vanish at a particular point. The proofs of the main results are given in Sections $3-6$. We use the symbol "$\sim$" to denote that the ratio of its both sides tends to 1 as $\lambda \to 0$.

2. Preliminaries

2.1. Modular lambda function. We will explain the relation between an elliptic curve and a complex torus by recalling the definition and basic properties of the elliptic modular lambda function. From [A, p. 264], one knows that for any $z \in T_\tau := \mathbb{C}/(\mathbb{Z} + \tau\mathbb{Z})$, the Weierstrass $\wp$-function with respect to the lattice $(1, \tau)$ ($\tau \in \mathbb{C}$, $\operatorname{Im}\tau > 0$) is defined to be

$$\wp(z) = \frac{1}{z^2} + \sum_{w \neq 0}\left(\frac{1}{(z-w)^2} - \frac{1}{w^2}\right),$$

where the sum ranges over all $w = n_1 + n_2\tau$ except 0, and $n_1, n_2 \in \mathbb{Z}$. Denote $e_1 := \wp(1/2)$, $e_2 := \wp(\tau/2)$ and $e_3 := \wp((1+\tau)/2)$. Then the *elliptic modular lambda function*

$$\lambda(\tau) := \frac{e_3 - e_2}{e_1 - e_2},$$

which is conformal, can be used to identify X_λ with a complex torus T_τ. Since the area of the parallelogram obtained from the lattice $(1, \tau)$ is $\operatorname{Im}\tau$, the normalized holomorphic 1-form is just $(1/\sqrt{\operatorname{Im}\tau})dz$. By definition, the Bergman kernel B_τ on T_τ is $(1/\operatorname{Im}\tau)dz \wedge d\bar{z}$, which means that $k_\lambda(z) = 1/\operatorname{Im}\tau$. Taking derivatives, one gets

$$\begin{aligned} l_{\lambda,z} &:= \frac{\partial^2(\log k_\lambda(z))}{\partial\lambda\partial\bar{\lambda}} = \frac{\partial^2(-\log \operatorname{Im}\tau)}{\partial\lambda\partial\bar{\lambda}} \\ &= -\frac{\partial^2\big(\log\big(\frac{\tau-\bar{\tau}}{2\sqrt{-1}}\big)\big)}{\partial\lambda\partial\bar{\lambda}} = -\frac{\partial\big(\frac{2\sqrt{-1}}{\tau-\bar{\tau}}\frac{\partial}{\partial\bar{\lambda}}\big(\frac{\tau-\bar{\tau}}{2\sqrt{-1}}\big)\big)}{\partial\lambda}. \end{aligned}$$

Since $\tau := \lambda^{-1}$ is holomorphic, implying that $\frac{\partial\tau}{\partial\bar{\lambda}} = 0$, we have

$$\begin{aligned} l_{\lambda,z} &= -\frac{\partial\big(\frac{2\sqrt{-1}}{\tau-\bar{\tau}}\frac{\partial}{\partial\bar{\lambda}}\big(\frac{-\bar{\tau}}{2\sqrt{-1}}\big)\big)}{\partial\lambda} = \frac{\partial\big(\frac{\overline{\tau'}}{\tau-\bar{\tau}}\big)}{\partial\lambda} = \frac{\frac{\partial\overline{\tau'}}{\partial\lambda}\cdot(\tau-\bar{\tau}) - \overline{\tau'}\frac{\partial(\tau-\bar{\tau})}{\partial\lambda}}{(\tau-\bar{\tau})^2} \\ &= \frac{0\cdot(\tau-\bar{\tau}) - \overline{\tau'}\frac{\partial(\tau)}{\partial\lambda}}{(\tau-\bar{\tau})^2} = \frac{-|\tau'|^2}{(\tau-\bar{\tau})^2} = \frac{|\tau'|^2}{4(\operatorname{Im}\tau)^2}. \end{aligned}$$

Next, by the inverse function theorem, $\tau'(b) = (\lambda^{-1})'(b) = 1/\lambda'(a)$ for any $b = \lambda(a)$ (here λ' being the derivative of λ with respect to τ). Therefore,

$$l_{\lambda,z} = \frac{|\tau'|^2}{4(\operatorname{Im}\tau)^2} = \frac{1}{4(\operatorname{Im}\tau\cdot|\lambda'(\tau)|)^2} > 0. \tag{2.1}$$

The last inequality is due to the fact that the derivative of the elliptic modular lambda function is nowhere vanishing in the domain of definition. Thus, $L_{\lambda,z} = \sqrt{-1}\, l_{\lambda,z}\, d\lambda \wedge d\bar{\lambda}$ is a true metric on $\mathbb{C} \setminus \{0, 1\}$.

2.2. Base point freeness. Here we recall a basic known fact that on a compact connected Riemann surface of genus $g \geq 1$ the Bergman kernel never vanishes, which is true because the canonical bundle is base point free (see the following proposition). We sketch a proof of this fact without using the Riemann–Roch theorem, and an even simpler proof by using the Riemann–Roch theorem can be found in [Bo]. The non-compact version of this classical result is sometimes named "Virtanen theorem". Therefore, it makes sense to take the logarithm of the Bergman kernel (since it is positive) and further consider its variations.

PROPOSITION 2.1. *Let X be a connected compact Riemann surface of genus $g \geq 1$. Then, for each $p \in X$, there exists a holomorphic 1-form s on X such that $s(p) \neq 0$.*

Proof. Assume that $g > 1$. For any $p \in X$, consider the short exact sequence

$$0 \to K \otimes m_p \xrightarrow{\iota} K \otimes \mathcal{O} \xrightarrow{r} K \otimes \mathbb{C}_p \to 0,$$

where K is the canonical bundle, m_p the sheaf of germs of holomorphic functions vanishing at p, $\mathcal{O}$ the sheaf of germs of holomorphic functions, $\mathbb{C}_p$ the skyscraper sheaf, ι the inclusion map and r the restriction map. Notice that by Serre duality, $H^0(X, K \otimes \mathbb{C}_p) \cong \mathbb{C}$, $H^1(X, K \otimes \mathcal{O}) \cong \mathbb{C}$ and $H^1(X, K \otimes \mathbb{C}_p) = 0$. Denote by $\mathcal{O}(p)$ the sheaf associated to the divisor p (taking value 1 at p and 0 otherwise). Then $H^1(X, K \otimes m_p) \cong H^0(X, \mathcal{O}(p))^\star \cong \mathbb{C}$, since a non-constant holomorphic map from X to $\mathbb{P}^1$ must have degree > 1 (1). Let ς be the operator between $H^0(X, K \otimes \mathbb{C}_p)$ and $H^1(X, K \otimes m_p)$. We get the induced long exact sequence

$$0 \to H^0(X, K \otimes m_p) \xrightarrow{\iota_1} H^0(X, K) \xrightarrow{r_1} \mathbb{C} \xrightarrow{\varsigma} \mathbb{C} \xrightarrow{\iota_2} \mathbb{C} \xrightarrow{r_2} 0.$$

The exactness implies the surjectivity of r_1 and completes the proof. ∎

3. Proof of the second term in Theorem 1.1. By [D1, Theorem 1.3], we know that

$$l_{\lambda,z} \sim \frac{1}{4|\lambda|^2(\log|\lambda|)^2}$$

as $\lambda \to 0$. In order to get the second term of $l_{\lambda,z}$, we define

$$J_\lambda := l_{\lambda,z} - \frac{1}{4|\lambda|^2(\log|\lambda|)^2}, \tag{3.1}$$

and analyze its asymptotic behavior as $\lambda \to 0$. Using $q := \exp(\pi\sqrt{-1}\,\tau)$, we rewrite the elliptic modular lambda function as

$$\lambda(\tau) = 16q - 128q^2 + 704q^3 - 3072q^4 + \cdots = 16q - 128q^2 + \mathrm{O}(q^3). \tag{3.2}$$

Thus, $|\lambda| = |16q - 128q^2 + \mathrm{O}(q^3)| = |q| \cdot |16 - 128q + \mathrm{O}(q^2)|$, yielding

$$\log|\lambda| = \log|q| + \log|16 - 128q + \mathrm{O}(q^2)|.$$

(1) The author thanks Prof. R. Kobayashi for clarifying this fact.

(3.2) also implies that $\lambda'(\tau) = \frac{\partial\lambda}{\partial q}\cdot\frac{\partial q}{\partial\tau} = (16 - 256q + \mathrm{O}(q^2))\cdot q\cdot\sqrt{-1}\,\pi$. Therefore,

$$|\lambda'(\tau)| = |16 - 256q + \mathrm{O}(q^2)|\cdot|q|\cdot\pi \quad\text{and}\quad |q| = \exp(-\pi\cdot\mathrm{Im}\,\tau),$$

i.e.,

$$\mathrm{Im}\,\tau = \frac{\log|q|}{-\pi}. \tag{3.3}$$

Substituting these into (2.1), we get

$$l_{\lambda,z} = \frac{1}{4(\log|q|)^2\cdot|q|^2\cdot|16-256q+\mathrm{O}(q^2)|^2}.$$

Therefore,

$$\begin{aligned} 4|q|^2\cdot J_\lambda &= \frac{1}{(\log|q|)^2|16-256q+\mathrm{O}(q^2)|^2} \\ &\quad - \frac{1}{|16-128q+\mathrm{O}(q^2)|^2(\log|q|+\log|16-128q+\mathrm{O}(q^2)|)^2} \\ &= \frac{|16+\mathrm{O}(q)|^2\big(2(\log|q|)\cdot\log|16+\mathrm{O}(q)| + (\log|16+\mathrm{O}(q)|)^2\big)}{(\log|q|)^2\cdot|16+\mathrm{O}(q)|^2\cdot|16+\mathrm{O}(q)|^2\cdot\big(\log|q|+\log|16+\mathrm{O}(q)|\big)^2} \\ &\sim \frac{2\cdot|16+\mathrm{O}(q)|^2\cdot\log 16}{(\log|q|)\cdot|16+\mathrm{O}(q)|^2\cdot|16+\mathrm{O}(q)|^2\cdot(\log|q|)^2} \sim \frac{2\cdot\log 16}{16^2\cdot(\log|q|)^3}. \end{aligned}$$

As $q\to 0$ (implying $\lambda\to 0$), it follows that

$$J_\lambda \sim \frac{\log 16}{2|\lambda|^2(\log|\lambda|)^3}.$$

Finally, applying (3.1) one obtains the second term in Theorem 1.1.

An alternative proof of the first two terms in Theorem 1.1, without using special properties of elliptic functions, is given in [D3]. Let us generalize Theorem 1.3(i) of [D1] by proving the following lemma ([2]), which will be used in the next section.

Lemma 3.1. *Under the assumptions of Theorem* 1.1, *as* $\lambda\to 0$,

$$k_\lambda(z) = \frac{\pi}{-\log|\lambda| + \log 16 - \mathrm{Re}\,\lambda/2 + \mathrm{O}(\lambda^2)}. \tag{3.4}$$

Proof. The preliminary section says that

$$\frac{1}{k_\lambda(z)} = \mathrm{Im}\,\tau = \frac{-\log|q|}{\pi}.$$

As $q\to 0$, we have

$$\frac{1}{k_\lambda(z)} \sim \frac{-\log|\lambda|}{\pi}.$$

([2]) If one drops the lower terms, then one gets $\log k_\lambda(z) \sim -\log(-\log|\lambda| + \log 16)$.

Considering their difference, from (3.2) one gets

$$\frac{1}{k_\lambda(z)} - \frac{-\log|\lambda|}{\pi} = \frac{1}{\pi}\log\left|\frac{\lambda}{q}\right| = \frac{1}{\pi}\log|16-128q+\mathrm{O}(q^2)|.$$

Furthermore, $|16-128q+\mathrm{O}(q^2)|^2 = 16^2 - 32\cdot 128\,\mathrm{Re}\,q + \mathrm{O}(q^2)$, which implies that

$$\frac{1}{k_\lambda(z)} - \frac{-\log|\lambda|}{\pi} = \frac{1}{2\pi}\log\bigl(16^2 - 32\cdot 128\,\mathrm{Re}\,q + \mathrm{O}(q^2)\bigr).$$

The Taylor expansion of $\log t$ at $t = 16^2$ says that

$$\log\bigl(16^2 - 32\cdot 128\,\mathrm{Re}\,q + \mathrm{O}(q^2)\bigr) = \log(16^2) - 16\,\mathrm{Re}\,q + \mathrm{O}(q^2),$$

which yields

$$\frac{1}{k_\lambda(z)} = \frac{-\log|\lambda| + \log 16 - 8\,\mathrm{Re}\,q + \mathrm{O}(q^2)}{\pi}$$

as $q \to 0$. Since $\mathrm{Re}\,q \sim \mathrm{Re}\,\lambda/16$ and $\mathrm{O}(q^2) = \mathrm{O}(\lambda^2)$, the proof is complete. ∎

4. Proof of the third and fourth terms in Theorem 1.1. From Lemma 3.1, as $\lambda \to 0$, we know that

$$\log k_\lambda(z) \sim -\log\left(\frac{-\log|\lambda| + \log 16 - \mathrm{Re}\,\lambda/2}{\pi}\right) =: \mathrm{RHS}.$$

After some elementary calculations one gets

$$\frac{\partial^2(\mathrm{RHS})}{\partial\lambda\partial\bar\lambda} = \frac{1 + \mathrm{Re}\,\lambda + \frac{1}{4}|\lambda|^2}{4|\lambda|^2(-\log|\lambda| + \log 16 - \mathrm{Re}\,\lambda/2)^2}.$$

STEP 1: *estimating the third term.* We have

$$\begin{aligned}
&\frac{\partial^2(\mathrm{RHS})}{\partial\lambda\partial\bar\lambda} - \frac{1}{4|\lambda|^2(\log|\lambda|)^2} - \frac{\log 16}{2|\lambda|^2(\log|\lambda|)^3}\\
&= \frac{\bigl(\mathrm{Re}\,\lambda + \frac{1}{4}|\lambda|^2\bigr)\cdot(\log|\lambda|)^2 + 2\log|\lambda|(\log 16 - \mathrm{Re}\,\lambda/2) - (\log 16 - \mathrm{Re}\,\lambda/2)^2}{4|\lambda|^2(\log|\lambda|)^2(-\log|\lambda| + \log 16 - \mathrm{Re}\,\lambda/2)^2}\\
&\quad - \frac{\log 16}{2|\lambda|^2(\log|\lambda|)^3}\\
&\sim \frac{-(\log 16 - \mathrm{Re}\,\lambda/2)^2\log|\lambda| + 4(\log 16)^2\log|\lambda| - 2(\log 16)^3}{4|\lambda|^2(\log|\lambda|)^3(-\log|\lambda| + \log 16 - \mathrm{Re}\,\lambda/2)^2}\\
&\sim \frac{3(\log 16)^2}{4|\lambda|^2(\log|\lambda|)^4},
\end{aligned}$$

which means that

$$\frac{\partial^2(\log k_\lambda(z))}{\partial\lambda\partial\bar\lambda} = \frac{1}{4|\lambda|^2(\log|\lambda|)^2}\left(1+\frac{2\log 16}{\log|\lambda|}+3\left(\frac{\log 16}{\log|\lambda|}\right)^2+\mathrm{O}\left(\frac{1}{(\log|\lambda|)^3}\right)\right).$$

STEP 2: *estimating the fourth term.* Similarly,

$$\begin{aligned}
&\frac{\partial^2(\mathrm{RHS})}{\partial\lambda\partial\bar{\lambda}}-\frac{1}{4|\lambda|^2(\log|\lambda|)^2}-\frac{\log 16}{2|\lambda|^2(\log|\lambda|)^3}-\frac{3(\log 16)^2}{4|\lambda|^2(\log|\lambda|)^4}\\
&\quad\sim\frac{3\log|\lambda|(\log 16)^2-2(\log 16)^3}{4|\lambda|^2(\log|\lambda|)^3(-\log|\lambda|+\log 16-\operatorname{Re}\lambda/2)^2}-\frac{3(\log 16)^2}{4|\lambda|^2(\log|\lambda|)^4}\\
&\quad\sim\frac{-2(\log|\lambda|)(\log 16)^3-3(\log 16)^2\big(-2(\log|\lambda|)(\log 16)+(\log 16)^2\big)}{4|\lambda|^2(\log|\lambda|)^4(-\log|\lambda|+\log 16-\operatorname{Re}\lambda/2)^2}\\
&\quad\sim\frac{4\log|\lambda|(\log 16)^3-3(\log 16)^4}{4|\lambda|^2(\log|\lambda|)^4(-\log|\lambda|+\log 16-\operatorname{Re}\lambda/2)^2}\sim\frac{(\log 16)^3}{|\lambda|^2(\log|\lambda|)^5},
\end{aligned}$$

as $\lambda\to 0$, which finishes the proof of Theorem 1.1.

REMARK. As $\lambda\to 0$, we do not know why our results on the asymptotic behavior of Bergman kernels depend only on $|\lambda|$. Moreover, we will see in the next section that the positivity of the above third term contributes to the completeness argument in the proof of Theorem 1.2.

5. Proof of Theorem 1.2. We first compute the Gaussian curvature of the Kähler metric $L_{\lambda,z}$ on $\mathbb{C}\setminus\{0,1\}$. From the preliminary section, it is known that

$$L_{\lambda,z}=\frac{\sqrt{-1}\cdot|\tau'|^2}{4(\operatorname{Im}\tau)^2}\,d\lambda\wedge d\bar{\lambda}=:\sqrt{-1}\,(J_\lambda)^2\,d\lambda\wedge d\bar{\lambda}.$$

Therefore,

$$\frac{-4\partial^2\log(J_\lambda)}{\partial\lambda\partial\bar{\lambda}}=\frac{-4\partial^2\log\big(\frac{|\tau'|}{2\cdot\operatorname{Im}\tau}\big)}{\partial\lambda\partial\bar{\lambda}}=\frac{-4\partial^2\log(|\tau'|)}{\partial\lambda\partial\bar{\lambda}}+\frac{4\partial^2\log(2\cdot\operatorname{Im}\tau)}{\partial\lambda\partial\bar{\lambda}}.$$

Since $\tau(\cdot)$, the inverse function of the elliptic modular function, is also conformal, $\log(|\tau'|)$ is harmonic with respect to λ. So,

$$\frac{-4\partial^2\log(J_\lambda)}{\partial\lambda\partial\bar{\lambda}}=\frac{4\partial^2\log(2\cdot\operatorname{Im}\tau)}{\partial\lambda\partial\bar{\lambda}}=-\frac{|\tau'|^2}{(\operatorname{Im}\tau)^2}.$$

Furthermore,

$$\operatorname{Curv}(L_{\lambda,z})=\frac{\frac{-4\partial^2\log(J_\lambda)}{\partial\lambda\partial\bar{\lambda}}}{(J_\lambda)^2}=\frac{-\frac{|\tau'|^2}{(\operatorname{Im}\tau)^2}}{\big(\frac{|\tau'|}{2\cdot\operatorname{Im}\tau}\big)^2}\equiv-4.$$

To prove that $L_{\lambda,z}$ is complete at 0, we use our asymptotic result of Theorem 1.1. Since the subleading terms are all incomplete near 0, the sum of the first and second terms becomes a complete metric (with a non-constant curvature) on $\mathbb{D}^*$, denoted by $\omega'_{\mathbb{D}^*}$. Then, due to the positivity of the third term we get $L_{\lambda,z}>\omega'_{\mathbb{D}^*}$, which guarantees the completeness of $L_{\lambda,z}$ at 0. For the completeness at other boundary points, 1 and ∞, we make use of the behavior of the elliptic modular lambda function under the composition with inverse or translation mappings (cf. [K-R, D2]).

Corollary 1.3 follows from Theorems 1.2 and 1.1.

6. Proof of Theorem 1.4. The computations in the preliminary section show that

$$0 < \frac{1}{4(\operatorname{Im}\tau \cdot |\lambda'(\tau)|)^2} = \frac{\partial^2(\log(k_\lambda(z)))}{\partial\lambda\partial\bar{\lambda}} = \frac{\partial^2(p(\lambda))}{\partial\lambda\partial\bar{\lambda}}.$$

From Theorem 1.2, it follows that $p(\lambda)$ is a Kähler potential of $\omega_{0,1}$. First, let us consider the leading term of the asymptotic expansion. By (3.2) and (3.3), as $\lambda \to 0$, it can be seen that

$$p(\lambda) \sim -\log(-\log|\lambda|) =: p_1(\lambda).$$

Actually, $p_1(\lambda)$ is the potential's leading term near $\lambda = 0$ and satisfies

$$\frac{\partial^2(p_1(\lambda))}{\partial\lambda\partial\bar{\lambda}} = \frac{1}{4|\lambda|^2(\log|\lambda|)^2}.$$

By (3.4), in order to get the second term, we use $p(\lambda)$ to subtract $p_1(\lambda)$ and analyze their difference:

$$\begin{aligned} p(\lambda) - p_1(\lambda) &\sim -\log(-\log|\lambda| + \log 16) + \log\pi + \log(-\log|\lambda|) \\ &= \log\pi - \log\left(1 - \frac{\log 16}{\log|\lambda|}\right) \sim \log\pi + \frac{\log 16}{\log|\lambda|} \sim \log\pi =: p_2(\lambda). \end{aligned}$$

The second to last similarity relation holds due to the Taylor expansion of $\log(1+t)$ at 0. Similarly, we see that the third term is just $\frac{\log 16}{\log|\lambda|} =: p_3(\lambda)$.

Remark. We now verify our claim in Section 1 that $\tilde{p}(\lambda)$ is a Kähler potential that exactly gives rise to the first two terms in the asymptotic expansion in Corollary 1.3. To check this, we make the following computations:

$$\partial\left(\frac{1}{\log|\lambda|}\right) = \frac{-d\lambda}{2\lambda(\log|\lambda|)^2}, \quad \bar{\partial}\left(\frac{1}{\log|\lambda|}\right) = \frac{-d\bar{\lambda}}{2\bar{\lambda}(\log|\lambda|)^2},$$

$$\begin{aligned} \partial\bar{\partial}\left(\frac{1}{\log|\lambda|}\right) &= \partial\Big(\frac{-d\bar{\lambda}}{2\bar{\lambda}(\log|\lambda|)^2}\Big) = \frac{2\partial\big(\bar{\lambda}(\log|\lambda|)^2\big) \wedge d\bar{\lambda}}{4\bar{\lambda}^2(\log|\lambda|)^4} \\ &= \frac{2\bar{\lambda}\partial\Big((\log|\lambda|)^2\Big) \wedge d\bar{\lambda}}{4\bar{\lambda}^2(\log|\lambda|)^4} = \frac{2\bar{\lambda}2(\log|\lambda|)\frac{d\lambda}{2\lambda} \wedge d\bar{\lambda}}{4\bar{\lambda}^2(\log|\lambda|)^4} = \frac{d\lambda \wedge d\bar{\lambda}}{2|\lambda|^2(\log|\lambda|)^3}. \end{aligned}$$

Thus, since $p_2(\lambda)$ is a constant,

$$\frac{\partial^2(p_2(\lambda) + p_3(\lambda))}{\partial\lambda\partial\bar{\lambda}} = \frac{\log 16}{2|\lambda|^3(\log|\lambda|)^2}.$$

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