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Weak completeness properties of the L^1-space of a spectral measure

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Contents

Abstract

Given a *Banach-space-valued* vector measure m, its associated space $L^1(m)$ is weakly complete iff it is finite-dimensional, weakly quasi-complete iff $L^1(m)$ is reflexive, and weakly sequentially complete iff $L^1(m)$ contains no copy of c_0. If m takes its values in a locally convex Hausdorff space Y, then the situation is more complicated. The first question is the *completeness* of $L^1(m)$ for its given L^1-topology.

It is shown that $L^1(m)$ is complete iff it is quasi-complete and, if Y is sequentially complete, that $L^1(m)$ is complete iff $\Sigma(m) := \{f \in L^1(m) : f \text{ is } \{0,1\}\text{-valued}\}$ is relatively weakly compact in $L^1(m)$. *Weak* completeness properties of $L^1(m)$ have no reasonable characterization, and little is known about the dual space $L^1(m)^*$. However, if $m := P$ is a *spectral measure* acting in a Banach space (with m strong operator σ-additive), then more can be said. Two remarkable features arise: $f \in L^1(P)$ iff $f \in L^\infty(P)$ and a concrete description of $L^1(P)^*$ is available. Some sample consequences are: $L^1(P)$ is complete iff it is quasi-complete, iff it is *weakly* quasi-complete, iff $\Sigma(P)$ is relatively weakly compact in $L^1(P)$. If $\Sigma(P)$ is not relatively weakly compact, then $L^1(P)$ may fail to be weakly sequentially complete, but never if P is *atomic*.

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1. Introduction and main results

For a positive measure μ defined on a measurable space (Ω, Σ), two of its fundamental properties are that $L^1(\mu)$ is *complete* for its usual norm $\|f\|_1 := \int_\Omega |f|\,d\mu$ (i.e., $L^1(\mu)$ is a Banach space) and that the Banach space $L^1(\mu)$ is *always weakly sequentially complete* (i.e., every sequence in $L^1(\mu)$ which is Cauchy for the weak topology $\sigma(L^1(\mu), L^1(\mu)^*)$ converges weakly to some element of $L^1(\mu)$) [9, Ch. IV, Theorem 8.6]. One may ask whether $L^1(\mu)$ has stronger completeness properties for $\sigma(L^1(\mu), L^1(\mu)^*)$, namely, is it weakly quasi-complete (i.e., bounded, weakly closed subsets are weakly complete), or perhaps even weakly complete? Except in trivial cases (i.e., when there is *no* infinite sequence of pairwise disjoint non-μ-null sets in Σ) the answer is negative in both cases. Indeed, we know that

$$\begin{aligned} L^1(\mu) \textit{ is weakly complete } &\Leftrightarrow\ L^1(\mu) \textit{ is weakly quasi-complete} \\ &\Leftrightarrow\ \dim(L^1(\mu)) < \infty. \end{aligned} \tag{1.1}$$

This is a routine consequence of the following general facts:

(i) $L^1(\mu)$ is reflexive $\Leftrightarrow \dim(L^1(\mu)) < \infty$ [30, Example 1.11.24].
(ii) A Banach space X is weakly complete $\Leftrightarrow \dim(X) < \infty$ [30, Proposition 2.5.15].
(iii) The closed unit ball $\mathbb{B}[X]$ of a Banach space X, being convex, is necessarily weakly closed [30, Theorem 2.5.16].
(iv) A bounded subset of a Banach space is weakly compact $\Leftrightarrow$ it is weakly complete [13, 0.6, p. 3].
(v) A Banach space X is reflexive $\Leftrightarrow \mathbb{B}[X]$ is weakly compact [30, Theorem 2.8.2].

Finally, whenever μ is a localizable measure (which includes all σ-finite measures), then the dual Banach space $L^1(\mu)^*$ of $L^1(\mu)$ is precisely $L^\infty(\mu)$, and so the weak topology of $L^1(\mu)$ is well understood; it is generated by the family of seminorms $f \mapsto |\int_\Omega fg\,d\mu|$, for $f \in L^1(\mu)$, as g varies through $L^\infty(\mu)$.

Suppose now that $m : \Sigma \to X$ is a *Banach-space-valued* vector measure (i.e., σ-additive). As for scalar measures, there is an associated Banach space $L^1(m)$ consisting of all the $\mathbb{C}$-valued, m-integrable functions defined on Ω, with norm again denoted by $\|\cdot\|_1$; see Remark 2.17 for the definitions. Moreover, $L^1(m)$ is always a (complex) Banach lattice with a weak order unit and its norm $\|\cdot\|_1$ is order continuous; actually *all* Banach lattices of this kind are isomorphic to $L^1(m)$ for a suitable Banach-space-valued vector measure m [2, Theorem 8], [37, Proposition 3.9]. Since the class of Banach lattices with order continuous norm and a weak order unit contains many reflexive spaces, their corresponding isomorphic spaces $L^1(m)$ will necessarily be weakly quasi-complete. Actually, $L^1(m)$ is reflexive if and only if it is weakly quasi-complete. This is true of any Banach space X. Indeed, if X is reflexive, then we can apply [30, Theorem 2.6.2 and Corollary 2.6.19] to the dual Banach space X^* and use $X^{**} = X$ to conclude that X is weakly quasi-complete. Conversely, if X is weakly quasi-complete, then by (iii) above, $\mathbb{B}[X]$ is weakly complete. So, (iv) shows that $\mathbb{B}[X]$ is weakly compact, and hence X is reflexive (via (v)). Of course, because of (ii) above, $L^1(m)$ will be weakly complete only in trivial cases.

Since $L^1(m)$ is a Banach lattice, its weak sequential completeness is also *characterized* by the fact that it does not contain an isomorphic copy of the Banach sequence space c_0; see [31, Theorem 2.5.6] for real spaces and [37, Proposition 3.38 together with the discussion after Remark 3.39] for complex spaces. Moreover, c_0 is a Banach lattice with order continuous norm and a weak order unit, and so it can be realized as $L^1(m)$ for some vector measure m. Accordingly, $L^1(m)$ spaces are *not always* weakly sequentially complete, in contrast to the spaces $L^1(\mu)$ when μ is a scalar-valued measure. Moreover, unlike for $L^1(\mu)$, there is no adequate description available for the dual space $L^1(m)^*$ when m is a general Banach-space-valued vector measure (although certain characterizations of individual members of $L^1(m)^*$ are known [33]). Accordingly, the weak topology of $L^1(m)$ is not so well understood in general. To compare the situation with (1.1) we have, for the spaces $L^1(m)$:

$$L^1(m) \textit{ is weakly complete } \Leftrightarrow \dim(L^1(m)) < \infty, \tag{1.2}$$

$$L^1(m) \textit{ is weakly quasi-complete } \Leftrightarrow L^1(m) \textit{ is reflexive}, \tag{1.3}$$

and

$$L^1(m) \textit{ is weakly sequentially complete } \Leftrightarrow L^1(m) \textit{ contains no isomorphic copy of } c_0. \tag{1.4}$$

The situation for a vector measure $m : \Sigma \to Y$, with Y a locally convex Hausdorff space (briefly, lcHs), is significantly different. Again there is an associated lcHs $L^1(m)$, equipped with the topology $\tau(m)$ of uniform convergence of indefinite integrals, which coincides with the norm topology $\|\cdot\|_1$ if Y is a Banach space; see Section 2 for the definitions. However, for spaces Y more general than Banach spaces, the lcHs $L^1(m)$ is no longer necessarily $\tau(m)$-complete (or even sequentially $\tau(m)$-complete). This new feature has an impact on the weak completeness properties of $L^1(m)$, especially when we recall that completeness and weak completeness in a general lcHs are closely connected [13, p. 4], [23, §18, 4.(4)]. So, in this more general setting of lcHs' we first need to address the question of the completeness of $L^1(m)$ for its *given topology* $\tau(m)$; this is the purpose of Section 2. A crucial role is played by the $\tau(m)$-closed subset $\Sigma(m) := \{\chi_E : E \in \Sigma\}$ of $L^1(m)$. A vector measure m is called *closed* if $\Sigma(m)$ is a $\tau(m)$-complete subset of $L^1(m)$.

If Y is metrizable, then every Y-valued vector measure is closed, but not in a general lcHs Y; see Example 2.13 below, for instance. The closedness of a vector measure $m : \Sigma \to Y$ is intimately related to the completeness of the lcHs $(L^1(m), \tau(m))$. As already noted, $L^1(m)$ need not be $\tau(m)$-complete or even sequentially $\tau(m)$-complete in general, even if the codomain space Y of m is itself complete; see, for example, [34, Examples 3.3 and 3.6]. This is a subtle point of the theory. On the other hand, even if Y is not sequentially complete, it can still happen that $L^1(m)$ is $\tau(m)$-complete; see Remark 2.6(iii) below. The following facts are known (references are given in Section 2). First,

$$L^1(m) \text{ is complete } \Leftrightarrow L^1(m) \text{ is quasi-complete.} \tag{1.5}$$

If, in addition, the lcHs Y is sequentially complete, then

$$L^1(m) \text{ is complete } \Leftrightarrow m \text{ is a closed measure.} \tag{1.6}$$

In Section 2 it is shown (still with Y sequentially complete) that (1.6) has a third *equivalence*, namely that

$$\Sigma(m) \text{ is a relatively weakly compact subset of } L^1(m); \tag{1.7}$$

see Remark 2.6(vi). This is a consequence of Proposition 2.4, one of the main results of Section 2, which states for *every* lcHs-valued vector measure m (with Y sequentially complete or not) that m is a closed measure whenever $\Sigma(m)$ is a relatively weakly compact subset of $L^1(m)$. The converse statement is false in general (cf. Example 2.5). However, another main result of Section 2 (i.e., Proposition 2.16) provides a class of vector measures m for which the converse does hold, namely whenever m is *atomic*.

The proofs of several of the results in Section 2 are somewhat technical. In order to maintain an ease of reading and to keep a clear overview of the section, these proofs have been placed in the Appendix at the end of the paper.

The weak completeness properties of $L^1(m)$ for a general lcHs-valued vector measure m cannot be characterized in any reasonable manner, not to mention the lack of concrete information concerning its dual space $L^1(m)^*$. Nevertheless, there is an important class of vector measures (within the family of operator-valued measures) for which interesting and non-trivial results can be achieved. This is the class of *spectral measures* acting in Banach spaces, which are analogues of the resolution of the identity for selfadjoint or more generally normal operators in a Hilbert space. The monographs [7], [10], [48] provide a detailed study of spectral measures and their corresponding spectral operators.

So, let X be a Banach space and $\mathcal{L}_s(X)$ be the lcHs of all continuous linear operators on X equipped with the strong operator topology. The first point is that $\mathcal{L}_s(X)$ is always *quasi-complete* and that its weak topology $\sigma(\mathcal{L}_s(X), \mathcal{L}_s(X)^*)$ is precisely the weak operator topology in $\mathcal{L}(X)$. Second, given any spectral measure $P : \Sigma \to \mathcal{L}_s(X)$ (see Section 3 for the definition) the space $L^1(P)$ of all P-integrable functions is precisely $L^\infty(P)$, as a vector space, that is, a Σ-measurable function is P-integrable if and only if it is P-essentially bounded! Of course, the lcH-topology $\tau(P)$ on $L^\infty(P) = L^1(P)$ is surely not the sup-norm topology (except in trivial cases). Third, the *integration operator* $I_P : (L^1(P), \tau(P)) \to \mathcal{L}_s(X)$, given by $f \mapsto \int_\Omega f\,dP$, is a bicontinuous isomorphism onto its range $I_P(L^1(P)) \subseteq \mathcal{L}_s(X)$; this is surely *not* the case for the integration operator $I_m : (L^1(m), \tau(m)) \to Y$ of a general lcHs-valued vector measure m. Fourth, the dual space $L^1(P)^*$ has a relatively simple description. Namely, $\xi \in L^1(P)^*$ if and only if there exist vectors $x \in X$ and $x^* \in X^*$ such that

$$\langle f, \xi \rangle = \int_\Omega f\, d\langle Px, x^* \rangle, \qquad f \in L^1(P), \tag{1.8}$$

where $\langle Px, x^* \rangle$ is the $\mathbb{C}$-valued measure $E \mapsto \langle P(E)x, x^* \rangle$ for $E \in \Sigma$. These four features (see Section 3 for precise details) play a special role in determining the completeness properties of $L^1(P)$, both for its given lcH-topology $\tau(P)$ and for its weak topology $\sigma(L^1(P), L^1(P)^*)$.

Let us now summarize the main results of Section 3 which address these completeness properties. Since the codomain space $\mathcal{L}_s(X)$ of a spectral measure P is quasi-complete, we are in the setting of (1.5)–(1.7) for $m := P$, that is:

$$\begin{aligned} &L^1(P) \text{ is } \tau(P)\text{-complete} \Leftrightarrow L^1(P) \text{ is } \tau(P)\text{-quasi-complete} \\ &\Leftrightarrow P \text{ is a closed measure} \Leftrightarrow \Sigma(P) \text{ is relatively weakly compact in } L^1(P). \end{aligned} \tag{1.9}$$

In Theorem 3.7 it is shown that a further condition equivalent to those in (1.9) is that

$$(L^1(P), \tau(P)) \text{ is weakly quasi-complete.} \tag{1.10}$$

The importance of the closedness of a spectral measure is clear from (1.9) and (1.10). Many additional criteria, equivalent to the closedness of P, are known (see those listed in Lemma 3.1(v) and the discussion after Lemma 3.1, for example). Some of these criteria are also formulated via certain properties of the *Boolean algebra of projections* $P(\Sigma) := \{P(E) : E \in \Sigma\} \subseteq \mathcal{L}(X)$; see Lemma 3.2.

For *non-closed* spectral measures P, because of (1.9) and (1.10), the relevant question concerns the $\tau(P)$-*sequential completeness* of $L^1(P)$ and is more involved. Of course, the results of Section 2 enter here when applied to $m := P$ and $Y := \mathcal{L}_s(X)$. There exist spectral measures P, even in Hilbert space, for which $L^1(P)$ *fails* to be $\tau(P)$-sequentially complete; see Example 3.16. In particular, for this P the lcHs $L^1(P)$ is not weakly sequentially complete either. On the other hand, for *every* spectral measure P (closed or not) it is shown in Proposition 3.17 that the lcHs $(L^1(P), \tau(P))$ can never contain an isomorphic copy of c_0 (this should be compared with (1.4)). One of our main results (cf. Theorem 3.18) states that for any *atomic* spectral measure P (closed or not) the lcHs $L^1(P)$ is *always* weakly sequentially complete, and hence also $\tau(P)$-sequentially complete. Atomic spectral measures occur in abundance, for example, whenever the Banach space X has an unconditional basis or even an unconditional Schauder decomposition [32]. Spectral measures P for which $L^1(P)$ is weakly sequentially complete but not weakly quasi-complete are exhibited in Examples 3.22 and 3.23.

The dual space $L^1(P)^*$ is well understood in a certain sense (cf. (1.8)). For a large class of spectral measures P (namely, those admitting a separating vector) this can be significantly improved. It is shown for such P that there exists a finite, positive measure μ on Σ such that $L^1(P)$ equipped with its weak topology can be identified as the Banach space $L^\infty(\mu)$ equipped with its weak-$*$ topology $\sigma(L^\infty(\mu), L^1(\mu))$; see Proposition 3.12.

Finally, in Theorem 3.5 it is shown for the lcHs $L^1(P)$ that

$$(L^1(P), \tau(P)) \text{ is weakly complete} \Leftrightarrow \dim(L^1(P)) < \infty,$$

both conditions being also equivalent to the range $P(\Sigma)$ of P being a *finite subset* of $\mathcal{L}(X)$. Despite the fact that this is an essentially trivial situation, there exist infinite-dimensional Banach spaces X in which the *only* spectral measures are those having a finite range!

2. Preliminaries and general vector measures

Vector spaces to be considered are over $\mathbb{C}$ unless stated otherwise. Let Y be a lcHs with topological dual space Y^*. The duality between Y and Y^* is denoted by $\langle y, y^*\rangle := y^*(y)$ for $y \in Y$ and $y^* \in Y^*$. Let $\mathcal{P}(Y)$ denote the set of all continuous seminorms on Y. For each $q \in \mathcal{P}(Y)$, let $U_q := \{y \in Y : q(y) \leq 1\}$. Its corresponding *polar set* is denoted by $U_q^\circ := \{y^* \in Y^* : |\langle y, y^*\rangle| \leq q(y), \forall y \in U_q\}$. Unless stated otherwise, every linear subspace of Y is equipped with the topology induced by Y. Given a subset W of Y, we denote by $\overline{W}$ its closure in Y. Its linear span (resp. closed linear span) is denoted by $\operatorname{span} W$ (resp. $\overline{\operatorname{span}}\, W$). Furthermore, the closed convex hull (resp. closed, balanced, convex hull) of W is denoted by $\overline{\operatorname{co}}\, W$ (resp. $\overline{\operatorname{bco}}\, W$).

A subset W of Y is called *weakly complete* (resp. *weakly sequentially complete*) if it is complete (resp. sequentially complete) with respect to the weak topology $\sigma(Y, Y^*)$ of Y, that is, every $\sigma(Y, Y^*)$-Cauchy net (resp. $\sigma(Y, Y^*)$-Cauchy sequence) converges weakly in W. We say that Y is *weakly quasi-complete* if every bounded, weakly closed subset of Y is weakly complete. Of course, weak completeness implies weak quasi-completeness, which in turn implies weak sequential completeness. The $\sigma(Y, Y^*)$-closure of a subset $W \subseteq Y$ is called its *weak closure* and is denoted by $\overline{W}^\sigma$. A subset of Y is said to be *relatively weakly compact* if its weak closure is weakly compact (i.e., compact for $\sigma(Y, Y^*)$). When Y is equipped with its weak topology, we also write $Y_{\sigma(Y,Y^*)}$ or simply Y_σ if no confusion can occur.

Let Λ be a non-empty set. A (formal) series $\sum_{\lambda\in\Lambda} y_\lambda$ of elements $\{y_\lambda : \lambda \in \Lambda\}$ in Y is said to be *unconditionally convergent* if the net of partial sums taken over all finite subsets of Λ converges to an element y in Y. In this case we write $y = \sum_{\lambda\in\Lambda} y_\lambda$ and call y the sum of the series $\sum_{\lambda\in\Lambda} y_\lambda$; see, for example, [3, Ch. II, Definition 2.1].

Let $\mathcal{L}(Y, Z)$ denote the vector space of all continuous linear operators from Y into a lcHs Z. When $Y = Z$, we simply write $\mathcal{L}(Y) := \mathcal{L}(Y, Y)$.

Throughout this section, we denote by (Ω, Σ) a measurable space, that is, Σ is a σ-algebra of subsets of a non-empty set Ω. We set

$$\Sigma \cap F := \{E \cap F : E \in \Sigma\}, \qquad F \in \Sigma.$$

Clearly $\Sigma \cap F = \{G \in \Sigma : G \subseteq F\}$. The vector space of all $\mathbb{C}$-valued, Σ-measurable functions is denoted by $\mathcal{L}^0(\Sigma)$.

Let m be a *vector measure* on Σ, taking its values in Y, that is, m is σ-additive. For each $y^* \in Y^*$, the complex measure $\langle m, y^*\rangle : E \mapsto \langle m(E), y^*\rangle$ on Σ has a finite variation measure $|\langle m, y^*\rangle|$ [49, §6.1]. Consequently, the range $\mathcal{R}(m) := m(\Sigma)$ of m is bounded for the weak topology, and hence also bounded for the initial topology on Y.

LEMMA 2.1. *The range of a vector measure taking values in a quasi-complete lcHs is relatively weakly compact.*

The previous lemma occurs in [50, Theorem 3]. We refer to [35, §2] for further sufficient conditions for a vector measure to have relatively weakly compact range.

Let us return to a general lcHs-valued vector measure $m : \Sigma \to Y$. A function $f \in \mathcal{L}^0(\Sigma)$ is said to be *m-integrable* if it satisfies the following two conditions:

(I-1) f is $\langle m, y^*\rangle$-integrable for every $y^* \in Y^*$, and
(I-2) given $E \in \Sigma$, there exists a unique vector $\int_E f\,dm \in Y$ satisfying

$$\Big\langle \int_E f\,dm, y^* \Big\rangle = \int_E f\,d\langle m, y^*\rangle, \qquad y^* \in Y^*.$$

In this case, the Y-valued set function

$$m_f : E \mapsto \int_E f\,dm, \qquad E \in \Sigma,$$

is again σ-additive by the Orlicz–Pettis Theorem [29, Theorem 1], and will be called the *indefinite integral* of f with respect to m. Functions $f \in \mathcal{L}^0(\Sigma)$ which satisfy (I-1) are called *scalarly m-integrable*.

Let $\mathcal{L}^1(m)$ denote the linear subspace of $\mathcal{L}^0(\Sigma)$ consisting of all m-integrable functions on Ω. Clearly the linear subspace $\operatorname{sim}\Sigma \subseteq \mathcal{L}^0(\Sigma)$ of all $\mathbb{C}$-valued, Σ-simple functions is contained in $\mathcal{L}^1(m)$ because the characteristic function χ_E of each set $E \in \Sigma$ is m-integrable with $\int_F \chi_E\,dm = m(E \cap F)$ for $F \in \Sigma$ and because $\operatorname{sim}\Sigma = \operatorname{span}\{\chi_E : E \in \Sigma\}$. Moreover, given $f \in \mathcal{L}^1(m)$ and $E \in \Sigma$, we have

$$f\chi_E \in \mathcal{L}^1(m) \quad \text{and} \quad \int_F f\chi_E\,dm = \int_{E\cap F} f\,dm \quad \text{for } F \in \Sigma. \tag{2.1}$$

Given $q \in \mathcal{P}(Y)$, define a function $q(m)$ on $\mathcal{L}^1(m)$ by

$$q(m)(f) := \sup_{y^* \in U_q^\circ} \int_\Omega |f|\,d|\langle m, y^*\rangle|, \qquad f \in \mathcal{L}^1(m), \tag{2.2}$$

in which case

$$\sup_{E\in\Sigma} q\Big(\int_E f\,dm\Big) \le q(m)(f) \le 4 \sup_{E\in\Sigma} q\Big(\int_E f\,dm\Big), \qquad f \in \mathcal{L}^1(m). \tag{2.3}$$

Equivalently, given $f \in \mathcal{L}^1(m)$, its indefinite integral m_f satisfies

$$\sup_{E\in\Sigma} q(m_f(E)) \le \sup_{y^*\in U_q^\circ} |\langle m_f, y^*\rangle|(\Omega) \le 4 \sup_{E\in\Sigma} q(m_f(E))$$

[25, p. 158]. Indeed, this inequality and the identity $\sup_{y^*\in U_q^\circ} |\langle m_f, y^*\rangle| = q(m)(f)$ (see [25, Theorem 2.2(1)]) yield (2.3).

Next, since the vector measure $m_f : \Sigma \to Y$ has bounded range, it follows from (2.3) that $q(m)$ is $[0,\infty)$-valued. In particular, $q(m)$ is a *seminorm* on $\mathcal{L}^1(m)$.

The *mean convergence topology* on $\mathcal{L}^1(m)$ is defined as the locally convex topology generated by the class of seminorms $q(m)$ for all $q \in \mathcal{P}(Y)$. Via (2.3), it is the topology of uniform convergence of indefinite integrals. The linear subspace $\operatorname{sim}\Sigma$ is sequentially dense in $\mathcal{L}^1(m)$ with respect to this topology; see [34, Proposition 1.2] (which is obtained as an application of [25, Theorem 2.4]). The mean convergence topology may not be Hausdorff. The associated lcHs is the quotient space

$$L^1(m) := \mathcal{L}^1(m)/\mathcal{N}(m)$$

with respect to the closed subspace

$$\mathcal{N}(m) := \bigcap_{q\in\mathcal{P}(Y)} q(m)^{-1}(\{0\}).$$

The corresponding quotient topology is denoted by $\tau(m)$. The lcHs $L^1(m)$ is equipped with this (initial) topology $\tau(m)$ unless stated otherwise. To emphasize this, we may speak of the lcHs $(L^1(m), \tau(m))$. Another topology on $L^1(m)$ which we shall also discuss is the weak topology $\sigma(L^1(m), L^1(m)^*)$.

Functions belonging to $\mathcal{N}(m)$ are said to be *m-null*. These are exactly those m-integrable functions whose indefinite integral is the zero vector measure. A set $E \in \Sigma$ is called *m-null* if $\chi_E \in \mathcal{N}(m)$. The family of all m-null sets is denoted by $\mathcal{N}_0(m)$. Clearly a set $E \in \Sigma$ is m-null if and only if $m(F) = 0$ for all $F \in \Sigma$ with $F \subseteq E$. A property holding outside an m-null set is said to hold m-almost everywhere, briefly m-a.e. Observe that a function $f \in \mathcal{L}^0(\Sigma)$ is equal to 0 pointwise m-a.e. if and only if f is both m-integrable and m-null.

The integration operator $I_m : \mathcal{L}^1(m) \to Y$ is defined by

$$I_m(f) := \int_\Omega f\,dm = m_f(\Omega), \qquad f \in \mathcal{L}^1(m), \tag{2.4}$$

which is clearly linear. Moreover, I_m is continuous via (2.3). The range of I_m is known to lie within the closed linear span of the range of m, that is,

$$I_m(\mathcal{L}^1(m)) \subseteq \overline{\operatorname{span}}\,\mathcal{R}(m) \tag{2.5}$$

[34, p. 347]. Moreover, the equality $I_m(\mathcal{N}(m)) = \{0\}$ implies that I_m induces a unique Y-valued, continuous linear operator on $L^1(m)$, namely (in standard coset notation)

$$f + \mathcal{N}(m) \mapsto I_m(f), \qquad f \in \mathcal{L}^1(m). \tag{2.6}$$

Let $E \in \Sigma$. Because of (2.1) we can define a linear multiplication operator $M_E : \mathcal{L}^1(m) \to \mathcal{L}^1(m)$ via

$$M_E : f \mapsto f\chi_E, \qquad f \in \mathcal{L}^1(m). \tag{2.7}$$

Since $q(m)(M_E(f)) \le q(m)(f)$ for $q \in \mathcal{P}(Y)$ and $f \in \mathcal{L}^1(m)$, the operator M_E is continuous. As $M_E(\mathcal{N}(m)) \subseteq \mathcal{N}(m)$, the operator M_E induces a unique continuous linear operator from $L^1(m)$ into $L^1(m)$ given by

$$f + \mathcal{N}(m) \mapsto M_E(f) + \mathcal{N}(m), \qquad f \in \mathcal{L}^1(m). \tag{2.8}$$

The subset

$$\Sigma(m) := \{\chi_E + \mathcal{N}(m) : E \in \Sigma\} \subseteq L^1(m)$$

is always $\tau(m)$-closed in $L^1(m)$. This was first stated in [22, Ch. IV, proof of Theorem 4.1] without proof; its proof can be found in [35, p. 8], [38, Lemma 2.10(i)]. A vector measure m is called *closed* if $\Sigma(m)$ is a $\tau(m)$-complete subset of $L^1(m)$ [22, p. 71].

Define an order on $\Sigma(m)$ by

$$\chi_E + \mathcal{N}(m) \le \chi_F + \mathcal{N}(m) \tag{2.9}$$

whenever $E, F \in \Sigma$ satisfy $\chi_E \le \chi_F$ pointwise m-a.e. Then $\Sigma(m)$ is a lattice for the operations $\wedge$ and $\vee$ given by

$$(\chi_E + \mathcal{N}(m)) \wedge (\chi_F + \mathcal{N}(m)) := \chi_{E\cap F} + \mathcal{N}(m),$$
$$(\chi_E + \mathcal{N}(m)) \vee (\chi_F + \mathcal{N}(m)) := \chi_{E\cup F} + \mathcal{N}(m)$$

for $E, F \in \Sigma$. $\Sigma(m)$ is a lattice with unit element $\chi_\Omega + \mathcal{N}(m)$ and zero element $\mathcal{N}(m)$. Furthermore, $\Sigma(m)$ is distributive and complemented. Therefore, $\Sigma(m)$ is a *Boolean algebra*, briefly B.a. [11, p. 5]. Actually $\Sigma(m)$ is isomorphic to the quotient B.a. $\Sigma/\mathcal{N}_0(m)$ [14, §41], [18, p. 53].

Henceforth, we will identify $\mathcal{L}^1(m)$ with $L^1(m) = \mathcal{L}^1(m)/\mathcal{N}(m)$, i.e., its associated quotient space, except when we need to distinguish these two spaces (this may be indicated by speaking of individual functions in $\mathcal{L}^1(m)$). Accordingly, the integration operator I_m defined by (2.4) is identified with the Y-valued linear operator (2.6) defined on $L^1(m)$. Similarly, the multiplication operator M_E defined by (2.7) is identified with the corresponding operator (2.8), for each $E \in \Sigma$. In the same spirit, the subsets $\{\chi_E : E \in \Sigma\} \subseteq \mathcal{L}^1(m)$ and $\Sigma(m) \subseteq L^1(m)$ are identified. With this identification, when $\chi_E \le \chi_F$ pointwise m-a.e. on Ω (i.e., (2.9) holds in $\Sigma(m)$), we say that $\chi_E \le \chi_F$ in the order of $\Sigma(m)$.

The following result was originally presented in [6, Proposition 1.1] with some extra assumptions; for the current general form see [35, Lemma 1.4].

LEMMA 2.2. *A lcHs-valued vector measure $m : \Sigma \to Y$ is closed if and only if $\Sigma(m)$ is complete as an abstract B.a. and has the property that whenever $\{\chi_{E(\gamma)}\}_{\gamma\in\Gamma}$ is a net in $\Sigma(m)$ which is downwards filtering to 0, then the net $\{m(E(\gamma))\}_{\gamma\in\Gamma}$ is convergent to 0 in Y.*

Let us now provide some criteria for closedness of vector measures, which will be needed in what follows. Parts (i) and (ii) of the following result are from [22, Theorems 7.1 and 7.3, Ch. IV], respectively.

LEMMA 2.3. *Let $m : \Sigma \to Y$ be a lcHs-valued vector measure.*

(i) *If Y is metrizable, then m is closed.*
(ii) *If there exists a localizable measure $\mu : \Sigma \to [0, \infty]$ such that $\langle m, y^*\rangle \ll \mu$ for every $y^* \in Y^*$, then m is closed.*

Every Banach-space-valued vector measure is closed by (i) above. For the definition of a *localizable measure*, see [14, 64A]. Every σ-finite measure is localizable [14, 64H]. We refer to [36, §1] for further criteria ensuring the closedness of general vector measures.

The next result is new and plays an important role in what follows.

PROPOSITION 2.4. *Let m be a lcHs-valued vector measure defined on Σ. If the subset $\Sigma(m) \subseteq L^1(m)$ is relatively weakly compact, then it is $\tau(m)$-complete in $L^1(m)$ (i.e., m is a closed measure).*

Proof. By assumption, the weak closure $\overline{\Sigma(m)}^{\sigma}$ of $\Sigma(m)$ in $L^1(m)$ is weakly compact, and hence is a weakly complete subset of $L^1(m)$ [13, 0.6, p. 3]. Thus, $\overline{\Sigma(m)}^{\sigma}$ is also $\tau(m)$-complete [13, p. 4], [23, §18, 4.(4)]. The $\tau(m)$-closed subset $\Sigma(m)$ of $L^1(m)$ is also a closed subset of the $\tau(m)$-complete set $\overline{\Sigma(m)}^{\sigma} \subseteq L^1(m)$. Accordingly, $\Sigma(m)$ is $\tau(m)$-complete. ∎

We point out that the set $\overline{\Sigma(m)}^{\sigma} \subseteq L^1(m)$ that occurs in the proof of Proposition 2.4 can be genuinely larger than the $\tau(m)$-closed set $\Sigma(m) \subseteq L^1(m)$; see Example 2.11 below.

The converse statement to Proposition 2.4 fails to hold, in general.

EXAMPLE 2.5. Let $\Omega := [0, 1]$ and Σ be the Borel σ-algebra of Ω. Equip the linear subspace $Y := \operatorname{sim}\Sigma \subseteq L^\infty(\mu)$ with the topology induced by the weak-$*$ topology $\sigma(L^\infty(\mu), L^1(\mu))$. Remark 1.8(iii) in [35] shows that the Y-valued set function $m : E \mapsto \chi_E$ on Σ is a closed vector measure (by using Lemma 2.3(ii) for the Lebesgue measure μ), and that its range

$\mathcal{R}(m) \subseteq Y$ is not relatively weakly compact. The latter fact is a consequence of Y failing to be sequentially complete. Now, since $\mathcal{R}(m)$ equals the image of $\Sigma(m)$ under I_m and $I_m \in \mathcal{L}(L^1(m), Y)$, and because I_m maps relatively weakly compact sets to relatively weakly compact sets, the subset $\Sigma(m) \subseteq L^1(m)$ cannot be relatively weakly compact. □

In Example 2.5 above, it is crucial that Y is not sequentially complete. Indeed, the converse statement of Proposition 2.4 *does hold* for a vector measure with values in a sequentially complete lcHs (see Remark 2.6(vi) below).

Remark 2.6. Let $m : \Sigma \to Y$ be a lcHs-valued vector measure.

(i) The lcHs $L^1(m)$ is $\tau(m)$-complete if and only if it is $\tau(m)$-quasi-complete. This fact occurs in this form in [38, Lemma 2.10(iii)] but it was essentially known earlier (see [22, Ch. IV, Theorem 4.1], [44, Theorem 1]). For the case when Y is a real lcHs, $L^1(m)$ is a real lcH-Riesz space [38, Proposition 3.7(i)], and so one can also apply [52, Proposition 1.4] to deduce the equivalence of the completeness and quasi-completeness of $L^1(m)$. This approach also applies when Y is a lcHs over $\mathbb{C}$ *provided that* $L^1(m)$ is a complex Riesz space, which is not always the case; see [38, Example 3.9(iv), (v)].

(ii) If $L^1(m)$ is $\tau(m)$-complete, then its $\tau(m)$-closed subset $\Sigma(m)$ is necessarily $\tau(m)$-complete, i.e., m is closed.

(iii) If m is closed and its codomain space Y is sequentially complete, then $L^1(m)$ is $\tau(m)$-complete [44, Theorem 2]. There exists a vector measure m, with values in an incomplete normed space, such that $L^1(m)$ is $\tau(m)$-complete [44, Example 1].

(iv) A useful way to analyze $\Sigma(m)$ is to realize it as the range of the $L^1(m)$-valued vector measure

$$[m] : E \mapsto \chi_E, \qquad E \in \Sigma.$$

That $[m]$ is indeed a vector measure with range $\mathcal{R}([m]) = \Sigma(m)$ and satisfying the equality $(L^1([m]), \tau([m])) = (L^1(m), \tau(m))$ as lcHs' is a special case of [34, Proposition 3.1]. In particular, $\Sigma([m])$ and $\Sigma(m)$ are the same subset of $L^1([m]) = L^1(m)$. Accordingly, $[m]$ is closed if and only if m is. From this fact and Lemma 2.3(i), with $Y := L^1(m)$ and $[m]$ in place of m, it follows that m is closed whenever $L^1(m)$ is $\tau(m)$-metrizable.

(v) If $L^1(m)$ is $\tau(m)$-complete, then its subset $\Sigma(m)$ is relatively weakly compact. This is a consequence of Lemma 2.1 applied to the $L^1(m)$-valued vector measure $[m]$ (see part (iv) above).

(vi) Assume, in addition now, that Y is sequentially complete. Then the following assertions are equivalent:

(a) m is closed.
(b) $L^1(m)$ is $\tau(m)$-complete.
(c) $\Sigma(m)$ is relatively weakly compact in $L^1(m)$.

Indeed, part (iii) above yields (a)⇒(b). For (b)⇒(c), see (v) above. The implication (c)⇒(a) is clear from Proposition 2.4.

In particular, the converse of Proposition 2.4 holds whenever Y happens to be sequentially complete. We point out that there exists a non-closed vector measure taking its values in a sequentially complete lcHs (see Examples 2.13 and 2.14 below). □

The converse statement of Proposition 2.4 also holds whenever m is atomic. This result, which is included in Proposition 2.16 below, is part of the special features of atomic vector measures to which we devote the remainder of this section.

The concept of an atom for a vector measure is analogous to that for a scalar measure [20], [22]. To be precise, let $m : \Sigma \to Y$ be a lcHs-valued vector measure. A set $E \in \Sigma$ is an *m-atom* if $m(E) \neq 0$ and if, for every $F \in \Sigma$, either $m(E \cap F) = 0$ or $m(E \setminus F) = 0$ [22, p. 32]. In this case

$$m(\Sigma \cap E) = \{0, m(E)\}. \tag{2.10}$$

A vector measure m is called *atomic* if every non-m-null set contains an m-atom. If m has no m-atoms, then it will be called *atomless*; in the literature the terminology non-atomic is also common.

Let us recall the well known concept of atoms in a B.a. $\mathcal{B}$ [18, p. 69]. A non-zero element $a \in \mathcal{B}$ is called an *atom* if the only elements $b \in \mathcal{B}$ satisfying $b \leq a$ (i.e., a dominates b) are $b = 0$ and $b = a$. The B.a. $\mathcal{B}$ is called *atomic* if every non-zero element dominates an atom. We say that $\mathcal{B}$ is *atomless* if it has no atoms.

The following result (whose proof is presented in the Appendix) shows that the concept of an m-atom and that of an atom in the B.a. $\Sigma(m)$ are essentially the same.

LEMMA 2.7. *Let $m : \Sigma \to Y$ be a lcHs-valued vector measure.*

(i) *Let $E \in \Sigma$ be an m-atom. Then a set $G \in \Sigma \cap E$ is m-null if and only if $m(G) = 0$.*

(ii) *The following conditions on a set $E \in \Sigma$ are equivalent:*

   (a) *E is an m-atom.*
   (b) *$E \notin \mathcal{N}_0(m)$ and, for each $F \in \Sigma$, either $E \cap F \in \mathcal{N}_0(m)$ or $(E \setminus F) \in \mathcal{N}_0(m)$.*
   (c) *E is an $[m]$-atom for the vector measure $[m] : \Sigma \to L^1(m)$.*
   (d) *χ_E is an atom of the B.a. $\Sigma(m)$.*

(iii) *The vector measure m is atomic (resp. atomless) if and only if the vector measure $[m]$ is atomic (resp. atomless) if and only if the B.a. $\Sigma(m)$ is atomic (resp. atomless).*

Note that a set $E \in \Sigma$ satisfying condition (b) in Lemma 2.7(ii) is a proper m-atom in the terminology of [20, (7), p. 7]. The equivalence (a)$\Leftrightarrow$(d) in Lemma 2.7(ii) is stated without proof in [22, p. 32].

Let m be a lcHs-valued vector measure. The family of all the atoms in the B.a. $\Sigma(m)$ is denoted by $\{\chi_{F(\alpha)}\}_{\alpha \in \mathcal{A}(m)}$. We assume, for distinct labels $\alpha, \beta \in \mathcal{A}(m)$, that their corresponding atoms $\chi_{F(\alpha)}$ and $\chi_{F(\beta)}$ are disjoint, that is, $\chi_{F(\alpha)} \wedge \chi_{F(\beta)} = 0$ in $\Sigma(m)$ (equivalently $F(\alpha) \cap F(\beta) \in \mathcal{N}_0(m)$). So, $\{\chi_{F(\alpha)}\}_{\alpha \in \mathcal{A}(m)}$ is an "enumeration" of all the atoms in $\Sigma(m)$.

Note that $\mathcal{A}(m) = \emptyset$ if an only if m is atomless. We say that m is *σ-atomic* if m is atomic and if the index set $\mathcal{A}(m)$ is countable. Every σ-atomic vector measure is necessarily closed [35, Proposition 1.9]. The atomic measures treated in [20, (9), p. 7] are restricted to the class of σ-atomic measures.

The proofs of the following two lemmas are postponed to the Appendix.

LEMMA 2.8. *Suppose that a lcHs-valued vector measure $m : \Sigma \to Y$ satisfies $\mathcal{A}(m) \neq \emptyset$. Let $f \in L^1(m)$. For each $\alpha \in \mathcal{A}(m)$, with corresponding atom $\chi_{F(\alpha)} \in \Sigma(m)$, there exists* *a unique complex number* $a(\alpha, f)$ *satisfying both*

$$\int_{F(\alpha)} f\,dm = a(\alpha, f)m(F(\alpha)) \tag{2.11}$$

and

$$f\chi_{F(\alpha)} = a(\alpha, f)\chi_{F(\alpha)} \quad \textit{pointwise } m\textit{-a.e. on } \Omega. \tag{2.12}$$

LEMMA 2.9. *Let* $m : \Sigma \to Y$ *be a lcHs-valued, atomic vector measure. A set* $E \in \Sigma$ *is* m*-null if and only if* $E \cap F(\alpha)$ *is* m*-null for every* m*-atom* $F(\alpha)$, $\alpha \in \mathcal{A}(m)$.

Lemmas 2.8 and 2.9 are crucial for establishing Proposition 2.10 below (for its proof we refer to the Appendix) as well as for further results in Section 3.

PROPOSITION 2.10. *Let* m *be an atomic, lcHs-valued vector measure. Then the subset* $\Sigma(m) \subseteq L^1(m)$ *is necessarily weakly closed.*

The weak closedness of $\Sigma(m)$ may not be enjoyed by a general vector measure m, as demonstrated by the following example. See also Remark 3.9(ii) below and the discussion after Proposition 3.21.

EXAMPLE 2.11. Let $m : \Sigma \to Y$ be a lcHs-valued, *atomless* vector measure. Recall from Remark 2.6(iv) that the associated vector measure $[m] : \Sigma \to L^1(m)$ satisfies $\mathcal{R}([m]) = \Sigma(m)$ and, via Lemma 2.7(iii), that $[m]$ is also atomless. Assume further that

$$\overline{\mathcal{R}([m])}^{\sigma} = \overline{\mathrm{co}}\,\mathcal{R}([m]) \quad \text{equivalently} \quad \overline{\Sigma(m)}^{\sigma} = \overline{\mathrm{co}}\,\mathcal{R}([m]) \tag{2.13}$$

in $L^1(m)$. Then $\Sigma(m)$ is *not* weakly closed in $L^1(m)$ because $\frac{1}{2}\chi_\Omega + \frac{1}{2}\chi_\emptyset = \frac{1}{2}\chi_\Omega$ and (2.13) imply that

$$\tfrac{1}{2}\chi_\Omega \in (\overline{\mathrm{co}}\,\Sigma(m)) \setminus \Sigma(m) = \overline{\Sigma(m)}^{\sigma} \setminus \Sigma(m).$$

We note that either of the conditions that

(a) the domain Σ of $[m]$ is a countably generated σ-algebra, or
(b) the codomain space $L^1(m)$ of $[m]$ is $\tau(m)$-metrizable

is sufficient for (2.13) to be satisfied; see Theorem 1 in [22, Ch. V, Section 6], applied to the vector measure $[m]$, for example, whereas [22, p. 111] provides earlier references.

Condition (a) is satisfied if Σ is the Borel σ-algebra of a topological space with a countable open base. For example, every separable metric space, such as the closed unit interval $[0, 1]$, admits a countable open base.

If the codomain space Y of m is metrizable, then we find both that m is closed (see Lemma 2.3(i)) and that $L^1(m)$ is $\tau(m)$-metrizable (in view of the definition of $\tau(m)$). In short, every atomless vector measure m with values in a metrizable lcHs has the property that $\Sigma(m)$ is $\tau(m)$-complete but *not* weakly closed in $L^1(m)$. □

PROPOSITION 2.12. *Let* $m : \Sigma \to Y$ *be a lcHs-valued, vector measure which is both closed and atomic. Then:*

(i) *The range* $\mathcal{R}(m) \subseteq Y$ *is compact.*
(ii) *Given any* $E \in \Sigma$, *the series* $\sum_{\alpha \in \mathcal{A}(m)} m(E \cap F(\alpha))$ *is unconditionally convergent in* Y *and its sum is precisely* $m(E)$.

According to [20, Theorem 10], every σ-atomic vector measure has compact range. This is a special case of Proposition 2.12(i) above because every σ-atomic vector measure is necessarily closed; see the discussion prior to Lemma 2.8. Our proof of Proposition 2.12 (provided in the Appendix) is an extension of that of [20, Theorem 10].

Let us now present some examples of atomic vector measures. Further examples, of a different nature, will be presented in Section 3.

Example 2.13. Let $\Omega := [0,1]$ and $\Sigma \subseteq 2^\Omega$ be any σ-algebra such that $\{\omega\} \in \Sigma$ for all $\omega \in \Omega$. Let $Y := \mathcal{L}^0(\Sigma) \subseteq \mathbb{C}^\Omega$ be equipped with the topology of pointwise convergence on Ω. Then Y is a sequentially complete lcHs whose initial topology and weak topology coincide.

Note that $m : \Sigma \to Y$ defined by $m(E) := \chi_E$ for $E \in \Sigma$ is an atomic vector measure whose m-atoms are all the singleton sets $\{\omega\}$ for $\omega \in \Omega$. It is clear that

$$\mathcal{N}(m) = \{0\} \quad \text{and} \quad \Sigma(m) = \mathcal{R}(m) = \{\chi_E : E \in \Sigma\}$$

as sets, that $L^1(m) = Y$ as lcHs', and that the integration operator $I_m : L^1(m) \to Y$ is the identity map. Consequently, the initial topology $\tau(m)$ on $L^1(m)$ is exactly the pointwise convergence topology.

The claim is that m is closed if and only if $\Sigma = 2^\Omega$. To see this, assume first that $\Sigma \subsetneq 2^\Omega$. Fix any $E \in 2^\Omega \setminus \Sigma$ and let $\mathcal{F}(E)$ denote the family of all finite subsets of E, directed by inclusion. Then the net $\{\chi_F\}_{F \in \mathcal{F}(E)}$ is $\tau(m)$-Cauchy in $\Sigma(m)$ but has no $\tau(m)$-limit. Hence, $\Sigma(m)$ is not τ-complete, that is, m is not closed. So, the 'only if' portion holds as we have established its contrapositive statement. On the other hand, if $\Sigma = 2^\Omega$, then $\Sigma(m) = \{\chi_E : E \in 2^\Omega\} = \{0,1\}^\Omega$ is $\tau(m)$-compact (as $\tau(m)$ is the product topology on $\{0,1\}^\Omega$), and hence is τ-complete. This verifies the 'if' portion.

Returning to the case when $\Sigma \subsetneq 2^\Omega$, it is routine to verify that the non-closed, atomic vector measure m satisfies

$$m(E) = \sum_{\omega \in \Omega} m(E \cap \{\omega\}), \qquad E \in \Sigma,$$

with the right-hand side being unconditionally convergent in Y. In other words, part (ii) of Proposition 2.12 still holds for this m even though it is non-closed. This is due to the fact that $\mathcal{A}(m)$ (in our standing notation) can be taken as Ω. □

Example 2.14. Let Ω and Y be as in Example 2.13 with Σ now being the Borel σ-algebra. Denoting the Lebesgue measure on Σ by μ, define a Y-valued vector measure by

$$m(E) := \chi_E + \mu(E)\chi_{\{1\}}, \qquad E \in \Sigma.$$

Then m is atomic and its m-atoms are all the singletons $\{\omega\}$ with $\omega \in \Omega$. It is clear that $\mathcal{N}(m) = \{0\}$ and that $L^1(m)$ equals the vector space $\mathcal{L}^1(\mu)$ of all *individual* μ-integrable functions. Moreover, we have

$$I_m(f) = f + \Big(\int_\Omega f\,d\mu\Big)\chi_{\{1\}}, \qquad f \in L^1(m).$$

Since m restricted to $\Sigma \cap [0,1)$ is not closed (by applying the argument in Example 2.13 to $[0,1)$ in place of $[0,1]$), the vector measure m itself is not closed.

The claim is that now part (ii) of Proposition 2.12 *fails* to hold (with the index set $\mathcal{A}(m) := \Omega$). Indeed, for any set $E \in \Sigma$ with $\mu(E) > 0$,

$$m(E) = \chi_E + \mu(E)\chi_{\{1\}} \neq \chi_E = \sum_{\omega \in \Omega} m(E \cap \{\omega\})$$

with the right-hand side being unconditionally convergent in Y. □

REMARK 2.15. (i) Example 2.5 above provides a closed, atomless vector measure, with values in a lcHs equipped with its weak topology, whose range is not relatively compact (= not relatively weakly compact). Therefore, we cannot omit the assumption in Proposition 2.12(i) that the vector measure is atomic.

(ii) There also exist closed, atomless vector measures which do have compact range. Indeed, take any atomless vector measure with values in a finite-dimensional Banach space. Then it is closed by Lemma 2.3(i), and it has compact range by Lyapunov's Theorem [4, Ch. IX, Corollary 1.5]. For the infinite-dimensional case, see [4, Ch. IX, Example 1.9 and the subsequent comment] which provides a c_0-valued, atomless vector measure with compact range; such a vector measure is closed (again by Lemma 2.3(i)). □

We conclude this section with a result which determines exactly when an atomic vector measure m is closed in terms of compactness properties of $\Sigma(m)$. Its proof will be provided in the Appendix.

PROPOSITION 2.16. *The following assertions for a lcHs-valued, atomic vector measure m defined on Σ are equivalent:*

(a) *m is closed.*
(b) *$\Sigma(m)$ is a $\tau(m)$-compact subset of $L^1(m)$.*
(c) *$\Sigma(m)$ is a weakly compact subset of $L^1(m)$.*
(d) *$\Sigma(m)$ is a relatively weakly compact subset of $L^1(m)$.*

REMARK 2.17. Let $m : \Sigma \to X$ be a *Banach-space-valued* vector measure. Then the corresponding seminorm (2.2), with q replaced by the norm of X, is the norm

$$f \mapsto \|f\|_1 := \sup_{\|x^*\|_{X^*} \le 1} \int_\Omega |f| \, d|\langle m, x^* \rangle|, \qquad f \in L^1(m).$$

Here $\|\cdot\|_{X^*}$ is the norm of the dual Banach space X^* of X. Then $(L^1(m), \|\cdot\|_1)$ turns out to be a Banach space; it is a natural analogue of $(L^1(\mu), \|\cdot\|_1)$ whenever μ is a positive scalar measure. Actually, $L^1(m)$ is a (complex) Banach lattice and $\|\cdot\|_1$ is a lattice norm. Some of the important features of $L^1(m)$ have been alluded to in Section 1; for further properties see [37, Ch. 3] and the references therein, for instance. □

3. Weak completeness of L^1 for spectral measures

Let X be a (complex) Banach space with norm $\|\cdot\|_X$ and with closed unit ball $\mathbb{B}[X] := \{x \in X : \|x\|_X \le 1\}$. Recall that the norm of its dual Banach space X^* is denoted by $\|\cdot\|_{X^*}$. The vector space $\mathcal{L}(X)$ is also an algebra for the multiplication defined by composition; its unit is the identity operator $\mathbf{I}$.

The *uniform operator topology* on $\mathcal{L}(X)$ is defined by the operator norm

$$\|T\|_{\mathrm{op}} := \sup_{x\in\mathbb{B}[X]} \|Tx\|_X, \qquad T \in \mathcal{L}(X).$$

In this case we write $\mathcal{L}_u(X)$ and note that $\mathcal{L}_u(X)$ is a unital Banach algebra with respect to $\|\cdot\|_{\mathrm{op}}$.

By $\mathcal{L}_s(X)$ we denote $\mathcal{L}(X)$ equipped with the *strong operator topology* τ_s (i.e., the topology of pointwise convergence on X). It is a lcHs whose topology is generated by the family of seminorms

$$\rho_x : T \mapsto \|Tx\|_X, \qquad T \in \mathcal{L}(X), \tag{3.1}$$

with $x \in X$ varying. It turns out that $\mathcal{L}_s(X)$ is always quasi-complete because of the completeness of X and the Banach–Steinhaus Theorem. The multiplication $(S, T) \mapsto ST$ is clearly separately continuous from $\mathcal{L}_s(X) \times \mathcal{L}_s(X)$ into $\mathcal{L}_s(X)$. So, $\mathcal{L}_s(X)$ is a locally convex algebra. To describe the dual space $\mathcal{L}_s(X)^*$ of $\mathcal{L}_s(X)$, fix any $x \in X$ and $x^* \in X^*$. The linear functional

$$x \otimes x^* : T \mapsto \langle Tx, x^*\rangle, \qquad T \in \mathcal{L}_s(X), \tag{3.2}$$

is clearly continuous, i.e., $x \otimes x^* \in \mathcal{L}_s(X)^*$. Conversely, every continuous linear functional on $\mathcal{L}_s(X)$ is necessarily the sum of finitely many linear functionals of the form (3.2). In other words,

$$\mathcal{L}_s(X)^* = \Big\{\sum_{j=1}^{n} x_j \otimes x_j^* : x_j \in X,\, x_j^* \in X^*,\, j = 1, \ldots, n,\, n \in \mathbb{N}\Big\}; \tag{3.3}$$

see [9, Ch. VI, proof of Theorem 1.4]. Thus, the weak topology $\sigma(\mathcal{L}_s(X), \mathcal{L}_s(X)^*)$ in the lcHs $\mathcal{L}_s(X)$ coincides with the *weak operator topology* τ_w on $\mathcal{L}(X)$ [9, Ch. VI, Definition 1.3], in which case we write $\mathcal{L}_w(X)$.

We point out that X is weakly sequentially complete if and only if $\mathcal{L}_s(X)$ is weakly sequentially complete. The 'only if' portion appears in [19, Theorem 2.15.2], and the 'if' portion follows from the fact that X is topologically isomorphic to a closed linear subspace of $\mathcal{L}_s(X)$ [24, §39, 1.(2′)].

Throughout this section, (Ω, Σ) denotes an arbitrary measurable space except when we consider specific measure spaces. A vector measure with values in the lcHs $\mathcal{L}_s(X)$ is called an *operator-valued measure*. Since $\mathcal{L}_s(X)$ is quasi-complete, every operator-valued measure has relatively weakly compact range (see Lemma 2.1), and so its range is uniformly bounded (i.e., bounded in $\mathcal{L}_u(X)$). An operator-valued measure $P : \Sigma \to \mathcal{L}_s(X)$ is called a *spectral measure* if $P(\Omega) = \mathbf{I}$ and

$$P(E \cap F) = P(E)P(F), \qquad E, F \in \Sigma. \tag{3.4}$$

The σ-additivity of P means exactly that $\lim_{n\to\infty} \|P(E_n)x\|_X = 0$ for each $x \in X$ whenever $E_n \downarrow \emptyset$ in Σ, that is,

$$Px : E \mapsto P(E)x, \qquad E \in \Sigma, \tag{3.5}$$

is an X-valued vector measure for each $x \in X$.

Given a spectral measure $P : \Sigma \to \mathcal{L}_s(X)$ and vectors $x \in X$, $x^* \in X^*$, it follows that $\langle P(E), x \otimes x^*\rangle = \langle P(E)x, x^*\rangle$ for $E \in \Sigma$, a formula in which the left-hand side is the duality between $\mathcal{L}_s(X)$ and $\mathcal{L}_s(X)^*$, whereas the right-hand side is the duality between X and X^*. In short, we have equality between two scalar measures on Σ:

$$\langle P, x \otimes x^* \rangle = \langle Px, x^* \rangle, \quad x \in X,\, x^* \in X^*. \tag{3.6}$$

Therefore, (3.3) shows that a function $f \in \mathcal{L}^0(\Sigma)$ is scalarly P-integrable if and only if it is $\langle Px, x^* \rangle$-integrable for all $x \in X$, $x^* \in X^*$.

Since the seminorms ρ_x for $x \in X$ (see (3.1)) generate the strong operator topology τ_s, it follows that the topology $\tau(P)$ is generated by the seminorms

$$\rho_x(P) : f \mapsto \sup\Big\{ \int_\Omega |f|\, d|\langle P, \xi\rangle| : \xi \in U^\circ_{\rho_x} \Big\}, \quad f \in L^1(P); \tag{3.7}$$

see (2.2) with $Y := \mathcal{L}_s(X)$ and ρ_x in place of q and $m := P$. Using the X-valued vector measure Px (see (3.5)) enables us to express $\rho_x(P)$ simply as

$$\rho_x(P)(f) = \sup_{x^* \in \mathbb{B}[X^*]} \int_\Omega |f|\, d|\langle Px, x^* \rangle|, \quad f \in L^1(P). \tag{3.8}$$

Indeed, for each $x \in X$, consider the evaluation map $U_x : T \mapsto Tx$, for $T \in \mathcal{L}(X)$, which is a continuous linear operator from the lcHs $\mathcal{L}_s(X)$ into the Banach space X. With $U_x^* : X^* \to \mathcal{L}_s(X)^*$ denoting the *dual operator* of U_x, it is routine to verify that

$$U_x^*(\mathbb{B}[X^*]) = U^\circ_{\rho_x}; \tag{3.9}$$

adapt [9, Ch. VI, proof of Theorem 1.4], for example. Given $x^* \in \mathbb{B}[X^*]$, since

$$\langle P(E), U_x^*(x^*) \rangle = \langle P(E)x, x^* \rangle = \langle Px(E), x^* \rangle, \quad E \in \Sigma,$$

it follows that $\langle P, U_x^*(x^*) \rangle = \langle Px, x^* \rangle$. This identity, (3.7) and (3.9) yield (3.8).

Given $f, g \in \mathcal{L}^1(P)$, their pointwise product fg is also P-integrable and we have

$$\int_E (fg)\, dP = \Big(\int_\Omega f\, dP \Big) \Big(\int_\Omega g\, dP \Big) P(E), \quad E \in \Sigma,$$

in $\mathcal{L}_s(X)$. Since $\mathcal{N}(P)$ is stable under pointwise multiplication, the pointwise multiplication of individual functions in $\mathcal{L}^1(P)$ induces their P-a.e. multiplication in the quotient space $L^1(P) = \mathcal{L}^1(P)/\mathcal{N}(P)$. These facts appear in [48, Proposition V.3 and p. 80], for example.

The P-null sets are more easily described than those of a general vector measure. Indeed, a set $E \in \Sigma$ is P-null if and only if $P(E) = 0$, which is a consequence of (3.4). Hence, $E \in \Sigma$ satisfies $P(E) = \mathbf{I}$ if and only if $P(\Omega \setminus E) = 0$ if and only if $\Omega \setminus E$ is P-null. This enables us to define a function $f \in \mathcal{L}^0(\Sigma)$ to be P-*essentially bounded* if

$$|f|_P := \inf\Big\{ \sup_{\omega \in E} |f(\omega)| : E \in \Sigma,\, P(E) = \mathbf{I} \Big\} < \infty.$$

In this case there is $E_0 \in \Sigma$ satisfying both $P(E_0) = \mathbf{I}$ and $|f|_P = \sup_{\omega \in E_0} |f(\omega)|$; see [10, pp. 2186–2187] or [48, pp. 72–73]. Clearly, a function $f \in \mathcal{L}^0(\Sigma)$ is P-null if and only if it is P-essentially bounded and $|f|_P = 0$. Accordingly, let $L^\infty(P)$ denote the vector space of all equivalence classes of P-essentially bounded functions modulo $\mathcal{N}(P)$. The norm on $L^\infty(P)$ induced by $|\cdot|_P$ is called the P-*essentially bounded norm*, and is denoted by the same symbol $|\cdot|_P$. We write $(L^\infty(P), |\cdot|_P)$ when we wish to emphasize that $L^\infty(P)$ is equipped with the norm $|\cdot|_P$. We can define the multiplication of equivalence classes in $L^\infty(P)$ via the pointwise multiplication of their representatives, because $\mathcal{N}(P)$ is stable under pointwise multiplication. Then $(L^\infty(P), |\cdot|_P)$ is a commutative Banach algebra whose unit is the equivalence class containing χ_Ω. As in the case of the L^∞-space for a scalar measure, we shall identify each P-essentially bounded function with its equivalence class in $L^\infty(P)$ as long as there is no danger of confusion.

We now collect some known results which will be used in what follows.

LEMMA 3.1. *Let X be a Banach space and $P : \Sigma \to \mathcal{L}_s(X)$ be a spectral measure. Define*

$$\|P(\Sigma)\|_{\mathrm{op}} := \sup_{E\in\Sigma} \|P(E)\|_{\mathrm{op}} < \infty.$$

(i) *The integration operator $I_P : (L^1(P), \tau(P)) \to \mathcal{L}_s(X)$ is a bicontinuous algebra isomorphism onto its range and satisfies*

$$\rho_x(I_P(f)) \le \rho_x(P)(f) \le 4\|P(\Sigma)\|_{\mathrm{op}}\, \rho_x(I_P(f)), \qquad x \in X,\ f \in L^1(P).$$

(ii) *A function $f \in \mathcal{L}^0(\Sigma)$ is P-integrable if and only if it is P-essentially bounded. Consequently, we have the equality*

$$L^1(P) = L^\infty(P) \tag{3.10}$$

of vector spaces.

(iii) *The identity map from $(L^\infty(P), |\cdot|_P)$ onto $(L^1(P), \tau(P))$ is continuous.*

(iv) *The inequalities*

$$|f|_P \le \|I_P(f)\|_{\mathrm{op}} \le 4\|P(\Sigma)\|_{\mathrm{op}} |f|_P, \qquad f \in L^\infty(P) = L^1(P), \tag{3.11}$$

hold and $I_P : (L^\infty(P), |\cdot|_P) \to \mathcal{L}_u(X)$ is a bicontinuous, Banach algebra isomorphism onto an inverse closed Banach subalgebra of $\mathcal{L}_u(X)$.

(v) *The following conditions are equivalent:*

(a) *P is a closed measure.*

(b) *$\mathcal{R}(P) = P(\Sigma)$ is a complete subset of $\mathcal{L}_s(X)$.*

(c) *$\mathcal{R}(P) = P(\Sigma)$ is a closed subset of $\mathcal{L}_s(X)$.*

(d) *$L^1(P)$ is $\tau(P)$-quasi-complete.*

(e) *$L^1(P)$ is $\tau(P)$-complete.*

(f) *$I_P(L^1(P))$ is a τ_s-complete subspace of $\mathcal{L}_s(X)$.*

(g) *$\Sigma(P)$ is a relatively weakly compact subset of $L^1(P)$.*

Proof. (i) See [48, Theorem V.4], for example.

(ii) The first part of (ii) can be derived from [10, Ch. XVIII, Theorem 2.11(c)]. A direct proof is given in [48, Proposition V.4]. The identity (3.10) is a consequence of the first part once we recall that $L^1(P)$ and $L^\infty(P)$ are both the corresponding quotient spaces modulo $\mathcal{N}(P)$.

(iii) Fix $x \in X$. Given $f \in L^\infty(P)$, since $|f(\omega)| \le |f|_P \cdot \chi_\Omega(\omega)$ for P-a.e. $\omega \in \Omega$, it follows from (3.7) or (3.8) that

$$\rho_x(P)(f) \le \rho_x(P)(|f|_P\chi_\Omega) = |f|_P \cdot \rho_x(P)(\chi_\Omega).$$

So, (iii) holds because $\rho_x(P)(\chi_\Omega) < \infty$.

(iv) See [10, Ch. XVII, Theorem 2.10]; the current form is in [48, Theorem V.1].

(v) The equivalences (a)$\Leftrightarrow$(b) and (e)$\Leftrightarrow$(f) follow because I_P is a bicontinuous isomorphism onto its range (via part (i)) and satisfies $I_P(\Sigma(P)) = \mathcal{R}(P)$. As noted earlier in this section, $\mathcal{R}(P)$ is bounded in $\mathcal{L}_s(X)$, which ensures the equivalence (b)⇔(c) because $\mathcal{L}_s(X)$ is quasi-complete. For the equivalences (a)⇔(d)⇔(e), see Remark 2.6(i)–(iii) with $Y := \mathcal{L}_s(X)$ and $m := P$. Since $Y := \mathcal{L}_s(X)$ is quasi-complete, (a)⇔(g) follows from Remark 2.6(vi) with $m := P$. ∎

Part (v) of Lemma 3.1 above provides a simple characterization for a spectral measure to be closed in terms of its range. We refer to [35], [36] and [48] together with the references therein for further features of closed spectral measures.

Since the range of a spectral measure is a B.a., we shall see that certain completeness properties of such a B.a. are related to the closedness of its corresponding spectral measure. Recall, for a Banach space X, that a subset $\mathcal{M} \subseteq \mathcal{L}(X)$ of commuting projections is called a *Boolean algebra of projections* when it is a B.a. relative to the partial order given by range inclusion. The lattice operations are given by

$$Q \wedge R := QR \quad \text{and} \quad Q \vee R := Q + R - QR, \qquad Q, R \in \mathcal{M}.$$

It is always assumed that the zero element of $\mathcal{M}$ is the zero operator, and the unit of $\mathcal{M}$ is $\mathbf{I}$. A B.a. of projections $\mathcal{M}$ is said to be *Bade complete* (resp. *Bade σ-complete*) if it is complete (resp. σ-complete) as an abstract B.a. and if we have

$$\Big(\bigwedge_{\lambda\in\Lambda} Q_\lambda\Big)(X) = \bigcap_{\lambda\in\Lambda} Q_\lambda(X) \quad \text{and} \quad \Big(\bigvee_{\lambda\in\Lambda} Q_\lambda\Big)(X) = \overline{\operatorname{span}}\Big(\bigcup_{\lambda\in\Lambda} Q_\lambda(X)\Big)$$

for each family (resp. each countable family) $\{Q_\lambda\}_{\lambda\in\Lambda}$ in $\mathcal{M}$. The following result is part of [48, Theorem IV.1].

Lemma 3.2. *Let X be a Banach space. A B.a. of projections $\mathcal{M}$ in $\mathcal{L}(X)$ is Bade complete* (*resp. Bade σ-complete*) *if and only if it is the range of a closed spectral measure* (*resp. a spectral measure*).

The B.a. $\Sigma(P) \subseteq L^1(P)$ associated with a spectral measure P was defined in Section 2 (set $m := P$ there). Its connection to the B.a. $\mathcal{R}(P) = P(\Sigma)$ (see Lemma 3.2) is as follows.

Lemma 3.3. *Let X be a Banach space and $P : \Sigma \to \mathcal{L}_s(X)$ be a spectral measure. The restriction of I_P to $\Sigma(P) \subseteq L^1(P)$ is a B.a. isomorphism onto the B.a. of projections $P(\Sigma) = I_P(\Sigma(P))$ in $\mathcal{L}(X)$. Consequently, P is an atomic* (*resp. atomless*) *measure if and only if the B.a. $P(\Sigma)$ is atomic* (*resp. atomless*).

Proof. The first part is a consequence of (3.4). This then ensures the second part once we recall that the B.a. $\Sigma(P)$ is atomic (resp. atomless) if and only if P is an atomic (resp. atomless) measure; see Lemma 2.7(iii) with $Y := \mathcal{L}_s(X)$ and $m := P$. ∎

We now turn our attention to the main topic of this section, namely various completeness properties of $L^1(P)$, for a spectral measure P, with respect to its weak topology $\sigma(L^1(P), L^1(P)^*)$. We begin with a useful description of the dual space $L^1(P)^*$ of $L^1(P)$. Such a description is given in [43, Theorem 1] under the assumption that P is closed; this assumption is actually superfluous.

Lemma 3.4. *Let X be a Banach space and $P : \Sigma \to \mathcal{L}_s(X)$ be a spectral measure. Then*

$$L^1(P)^* = \{(x \otimes x^*) \circ I_P : x \in X,\ x^* \in X^*\}. \tag{3.12}$$

Proof. In view of (3.2), if $x \in X$ and $x^* \in X^*$, then the composition $(x \otimes x^*) \circ I_P$ of $x \otimes x^* \in \mathcal{L}_s(X)^*$ and $I_P \in \mathcal{L}(L^1(P), \mathcal{L}_s(X))$ clearly belongs to $L^1(P)^*$. So, the right-hand side of (3.12) is included in the left-hand side.

Concerning the reverse inclusion, let $\overline{\langle W \rangle}$ denote the τ_s-closed subalgebra generated by a subset W of $\mathcal{L}_s(X)$. Fix $\xi \in L^1(P)^*$. Since $I_P^{-1} : I_P(L^1(P)) \to L^1(P)$ is continuous and linear (see Lemma 3.1(i)), the composition $\xi \circ I_P^{-1} : I_P(L^1(P)) \to \mathbb{C}$ is a continuous linear functional which admits a unique continuous linear extension $\widehat{\xi}$ to the τ_s-closure $\overline{I_P(L^1(P))} \subseteq \mathcal{L}_s(X)$. Since $I_P(L^1(P))$ is a subalgebra of $\mathcal{L}_s(X)$ (see Lemma 3.1(ii)) and since $\operatorname{sim} \Sigma$ is $\tau(P)$-dense in $L^1(P)$ (see Section 2), we have (for $W = P(\Sigma)$) the identity $\overline{\langle P(\Sigma) \rangle} = \overline{I_P(L^1(P))}$. Next, we recall from Lemma 3.2 that $P(\Sigma)$ is a Bade σ-complete B.a. of projections. Its τ_s-closure $\mathcal{M} := \overline{P(\Sigma)}$ in $\mathcal{L}_s(X)$ is then a *Bade complete* B.a. of projections [48, Theorem V.8]. The identity $\overline{\langle P(\Sigma) \rangle} = \overline{\langle \mathcal{M} \rangle}$ clearly holds, and hence $\overline{\langle \mathcal{M} \rangle} = \overline{I_P(L^1(P))}$; i.e., $\overline{I_P(L^1(P))}$ is the τ_s-closed subalgebra of $\mathcal{L}_s(X)$ generated by the Bade complete B.a. of projections $\mathcal{M}$. According to [5, Proposition 3.2] there exist $x \in X$ and $x^* \in X^*$ satisfying $x \otimes x^* = \widehat{\xi}$ on $\overline{I_P(L^1(P))}$, which implies that $x \otimes x^* = \xi \circ I_P^{-1}$ on $I_P(L^1(P))$, and hence that $\xi = (x \otimes x^*) \circ I_P$. As $\xi \in L^1(P)^*$ is arbitrary, the left-hand side of (3.12) is included in the right-hand side. ∎

For a spectral measure P, when is $L^1(P)$ weakly complete? The following result provides an answer: it is so only under very special circumstances. If we write $P : \Sigma \to \mathcal{L}_u(X)$, then P is to be interpreted as a Banach-space-valued measure, typically only *finitely additive*. We say that $P : \Sigma \to \mathcal{L}_u(X)$ has *finite variation* if its total variation, which is defined as in the scalar-valued case [4, Ch. I, Definition 1.4], is a $[0, \infty)$-valued, finitely additive measure on Σ. The set function $P : \Sigma \to \mathcal{L}_u(X)$ is called *strongly additive* if the series $\sum_{n=1}^\infty P(E_n)$ converges in $\mathcal{L}_u(X)$ whenever $\{E_n\}_{n=1}^\infty$ is a sequence of pairwise disjoint sets from Σ [4, Ch. I, Definition 1.14]. Whenever the integration operator I_P is to be regarded as $\mathcal{L}_u(X)$-valued, we write $I_P : (L^1(P), \tau(P)) \to \mathcal{L}_u(X)$.

We point out that a lcHs Y is weakly complete if and only if *every* linear functional on its dual space Y^* is $\sigma(Y^*, Y)$-continuous. This is a consequence of the fact that the completion of $Y_{\sigma(Y,Y^*)}$ equals the algebraic dual of $Y^*_{\sigma(Y^*,Y)}$ [23, §20, 9.(2)]. Thus, if there exists a linear functional on Y^* which is not $\sigma(Y^*, Y)$-continuous, then Y is not weakly complete.

Theorem 3.5. *Let X be a Banach space and $P : \Sigma \to \mathcal{L}_s(X)$ be a spectral measure.*

(I) *If the vector space $L^1(P)$ is infinite-dimensional, then the lcHs $(L^1(P), \tau(P))$ is not weakly complete.*

(II) *The following conditions for P are equivalent:*

- (a) *$(L^1(P), \tau(P))$ is weakly complete.*
- (b) *$L^1(P)$ is finite-dimensional.*
- (c) *$(L^1(P), \tau(P))$ is normable.*
- (d) *$(L^1(P), \tau(P))$ is metrizable.*
- (e) *The linear operator $I_P : (L^1(P), \tau(P)) \to \mathcal{L}_u(X)$ is continuous.*
- (f) *$P : \Sigma \to \mathcal{L}_u(X)$ is σ-additive.*

(g) *There exists a finitely additive measure* $\mu : \Sigma \to [0, \infty)$ *such that* $\|P(E)\|_{\mathrm{op}} \le \mu(E)$ *for every* $E \in \Sigma$.
(h) $P(\Sigma)$ *is a finite subset of* $\mathcal{L}(X)$.
(i) $P : \Sigma \to \mathcal{L}_u(X)$ *has finite variation.*
(j) $P : \Sigma \to \mathcal{L}_u(X)$ *is strongly additive.*
(k) *There exists a finite* Σ*-partition* $\{E_j\}_{j=1}^k$ *of* Ω *(with* $k \in \mathbb{N}$*) such that each projection* $P(E_j)$ *for* $j = 1, \ldots, k$ *is an atom of the B.a. of projections* $P(\Sigma)$*, and*

$$P(F) = \sum_{j=1}^{k} P(F \cap E_j), \qquad F \in \Sigma.$$

(l) $P(\Sigma)$ *is a compact subset of* $\mathcal{L}_u(X)$.
(m) *The Banach space* $(L^\infty(P), |\cdot|_P)$ *is separable.*

Proof. (I) Our proof is an adaptation of that of [30, Proposition 2.5.14]. First observe that $L^1(P)^*$ is also infinite-dimensional (because $\dim L^1(P)^* < \infty$ would imply $\dim L^1(P) < \infty$). So, there exists an infinite, linearly independent subset $V = \{\xi_n : n \in \mathbb{N}\} \subseteq L^1(P)^*$. By Lemma 3.4 above, for each $n \in \mathbb{N}$ there exist (non-zero) vectors $x_n \in X$ and $x_n^* \in X^*$ satisfying $\xi_n = (x_n \otimes x_n^*) \circ I_P$. We may assume that $\|x_n\|_X = 1 = \|x_n^*\|_{X^*}$ for each $n \in \mathbb{N}$ because the infinite set

$$\left\{ \frac{\xi_n}{\|x_n\|_X \cdot \|x_n^*\|_{X^*}} : n \in \mathbb{N} \right\}$$

is linearly independent in $L^1(P)^*$. The set V is then $\sigma(L^1(P)^*, L^1(P))$-bounded because, given $f \in L^1(P)$, we have

$$\begin{aligned}\sup_{n\in\mathbb{N}} |\langle f, \xi_n\rangle| &= \sup_{n\in\mathbb{N}} |\langle f, (x_n \otimes x_n^*) \circ I_P\rangle| = \sup_{n\in\mathbb{N}} |\langle I_P(f)x_n, x_n^*\rangle| \\ &\le \sup_{n\in\mathbb{N}} \|I_P(f)\|_{\mathrm{op}} \cdot \|x_n\|_X \cdot \|x_n^*\|_{X^*} = \|I_P(f)\|_{\mathrm{op}} < \infty.\end{aligned}$$

Now, select any Hamel basis H for the vector space $L^1(P)^*$ such that $V \subseteq H$ [23, §7, 3.(2)]. Then there exists a unique linear functional $u : L^1(P)^* \to \mathbb{C}$ determined by the requirements $u(\xi_n) := n$ for $n \in \mathbb{N}$ and $u(\xi) := 0$ for $\xi \in H \setminus V$ [23, §8.5, p. 63]. Clearly, the image $u(V)$ of the $\sigma(L^1(P)^*, L^1(P))$-bounded subset V is unbounded in $\mathbb{C}$. Thus, u is *not* continuous (as every continuous linear functional on a lcHs maps bounded sets to bounded sets in $\mathbb{C}$). Therefore, $L^1(P)$ is not weakly complete (see the discussion immediately prior to this theorem).

(II) (a)⇒(b). This is the contrapositive of (I).

(b)⇒(a). Clear, as every finite-dimensional lcHs Y is topologically isomorphic to $\mathbb{C}^{\dim Y}$.

(b)⇒(c)⇒(d). Clear.

(d)⇒(e). By (d) the codomain space $L^1(P)$ of the vector measure $[P] : E \mapsto \chi_E$ on Σ is metrizable. So, Lemma 2.3(i) with $Y := L^1(P)$ and $m := [P]$ yields the closedness of $[P]$. Thus, P is also closed (via Remark 2.6(iv) with $Y := \mathcal{L}_s(X)$, and $m := P$), and hence $(L^1(P), \tau(P))$ is complete by Remark 2.6(iii) with $Y := \mathcal{L}_s(X)$ and $m := P$. This together with (d) implies that $(L^1(P), \tau(P))$ is a Fréchet space. The Open Mapping Theorem then shows that the identity operator from the Banach space $(L^\infty(P), |\cdot|_P)$ onto $(L^1(P), \tau(P))$ is a bicontinuous isomorphism (after recalling that this operator is always continuous by Lemma 3.1(iii)). Thus, (e) holds because we already know from Lemma 3.1(iv) that $I_P : (L^\infty(P), |\cdot|_P) \to \mathcal{L}_u(X)$ is continuous.

(e)⇒(f). Let $E(n) \downarrow \emptyset$ in Σ. The σ-additivity of $P : \Sigma \to \mathcal{L}_s(X)$ implies that $I_P(\chi_{E(n)}) = P(E(n)) \to 0$ in $\mathcal{L}_s(X)$. Since $I_P^{-1} : (I_P(L^1(P)), \tau_s) \to (L^1(P), \tau(P))$ is continuous by Lemma 3.1(i), it follows that $\chi_{E(n)} = I_P^{-1}(I_P(\chi_{E(n)})) \to 0$ in $(L^1(P), \tau(P))$ as $n \to \infty$. Now apply (e) to deduce that $P(E(n)) = I_P(\chi_{E(n)}) \to 0$ in $\mathcal{L}_u(X)$, which establishes (f).

(f)⇒(j). This follows immediately from the definitions of σ-additivity and strong additivity.

The equivalence of (g), (h), (i), (j) and (k) is a special case of [47, Proposition 1].

(h)⇒(b). Condition (h) implies $W := \operatorname{span} P(\Sigma) = I_P(\operatorname{sim} \Sigma)$ is *finite-dimensional* and, in particular, is then a τ_s-closed linear subspace of $\mathcal{L}_s(X)$. This together with the $\tau(P)$-denseness of $\operatorname{sim} \Sigma$ in $L^1(P)$ (see Section 2) implies that $W = I_P(L^1(P))$. So, (b) holds because I_P is injective (see Lemma 3.1(i)).

(h)⇒(l). Clear.

(l)⇒(f). Let $E(n) \downarrow \emptyset$ in Σ. Then $P(E(n)) \to 0$ in $\mathcal{L}_s(X)$ as $n \to \infty$. On the other hand, (l) implies that the uniform and strong operator topologies coincide on $P(\Sigma)$. So, $\lim_{n\to\infty} P(E(n)) = 0$ in $\mathcal{L}_u(X)$, which establishes (f).

(b)⇒(m). This is clear because $L^\infty(P) = L^1(P)$ as vector spaces (see Lemma 3.1(ii)) and because every finite-dimensional Banach space is separable.

(m)⇒(h). To prove the corresponding contrapositive statement, assume that $P(\Sigma)$ is an infinite set. According to [47, Lemma 1], there exists a sequence $\{E(n)\}_{n=1}^\infty$ of pairwise disjoint, non-P-null sets in Σ. This enables us to define a linear operator $\Phi : \ell^\infty \to (L^\infty(P), |\cdot|_P)$ by assigning to each $(\alpha_n)_{n=1}^\infty \in \ell^\infty$ the pointwise sum $\sum_{n=1}^\infty \alpha_n \chi_{E(n)} \in L^\infty(P)$. Then Φ is a bicontinuous isomorphism from ℓ^∞ (equipped with the usual uniform norm) onto $\overline{\operatorname{span}}\{\chi_{E(n)} : n \in \mathbb{N}\} \subseteq L^\infty(P)$. Consequently, $(L^\infty(P), |\cdot|)$ is non-separable, which establishes the desired contrapositive statement. ∎

Remark 3.6. There exist Banach spaces X such that *every* $\mathcal{L}_s(X)$-valued spectral measure P has finite range, that is, P satisfies condition (h) in Theorem 3.5(II) above. The class of such Banach spaces contains (at least) two subclasses. One consists of all Grothendieck spaces with the Dunford–Pettis property [42, Proposition 1]. This subclass includes the L^∞-spaces of scalar measures and many others [26]. The second subclass consists of the hereditarily indecomposable Banach spaces; see [45, Proposition 1] and the equivalence (h)⇔(k) in Theorem 3.5(II) together with the fact that the range of every spectral measure is a Bade σ-complete B.a. of projections (via Lemma 3.2). These two subclasses are disjoint because each hereditarily indecomposable space is not a Grothendieck space with the Dunford–Pettis property [45, Proposition 4]. □

The following result characterizes when $L^1(P)$ is weakly quasi-complete (i.e., quasi-complete for $\sigma(L^1(P), L^1(P)^*)$).

Theorem 3.7. *Let X be a Banach space and $P : \Sigma \to \mathcal{L}_s(X)$ be a spectral measure. Then P is a closed measure if and only if the lcHs $(L^1(P), \tau(P))$ is weakly quasi-complete.*

Proof. Suppose that P is closed. Since its codomain space $\mathcal{L}_s(X)$ is quasi-complete, the subset $\Sigma(P) \subseteq L^1(P)$ is relatively weakly compact, that is, its weak closure $\overline{\Sigma(P)}^{\sigma}$ is weakly compact. For this apply Remark 2.6(vi) with $Y := \mathcal{L}_s(X)$ and $m := P$. From Krein's Theorem [23, §24, 5.(4′)], it follows that $\overline{\mathrm{bco}}(\overline{\Sigma(P)}^{\sigma})$ is also weakly compact in $L^1(P)$. Recalling that the weak closure and the $\tau(P)$-closure of any balanced, convex set in $L^1(P)$ coincide, we can conclude that $\overline{\mathrm{bco}}\,\Sigma(P) \supseteq \overline{\Sigma(P)}^{\sigma}$, which implies the identity $\overline{\mathrm{bco}}\,\Sigma(P) = \overline{\mathrm{bco}}(\overline{\Sigma(P)}^{\sigma})$. Thus, $\overline{\mathrm{bco}}\,\Sigma(P)$ is weakly compact, and hence weakly complete.

Select any weakly bounded subset $V \subseteq L^1(P)$, which is then also $\tau(P)$-bounded. The claim is that there exists a constant $C > 0$ such that

$$V \subseteq C \cdot \overline{\mathrm{bco}}\,\Sigma(P); \tag{3.13}$$

To verify this, we first recall that $L^1(P) = L^\infty(P)$ as vector spaces (cf. Lemma 3.1(ii)). Moreover,

$$\mathbb{B}[L^\infty(P)] \subseteq 4 \cdot \overline{\mathrm{bco}}\,\Sigma(P); \tag{3.14}$$

this can be proved by expressing each function in $\mathbb{B}[L^\infty(P)]$ as a uniform limit of appropriate Σ-simple functions. On the other hand, the image $I_P(V) \subseteq \mathcal{L}_s(X)$ is bounded for the strong operator topology τ_s, and hence uniformly bounded by the Banach–Steinhaus Theorem. From this and (3.11), it follows that

$$M := \sup_{f \in V} |f|_P \le \sup_{f \in V} \|I_P(f)\|_{\mathrm{op}} < \infty,$$

which yields

$$V \subseteq M \cdot \mathbb{B}[L^\infty(P)]. \tag{3.15}$$

Therefore, (3.13) holds with $C := 4M$ by (3.14) and (3.15).

Now, since $\overline{\mathrm{bco}}\,\Sigma(P)$ is weakly complete, so is $C \cdot \overline{\mathrm{bco}}\,\Sigma(P)$. In short, each weakly bounded subset V of $L^1(P)$ is contained in some weakly complete subset. In other words, the lcHs $L^1(P)$ is weakly quasi-complete.

Conversely, if $L^1(P)$ is weakly quasi-complete, then it is also $\tau(P)$-quasi-complete [23, §18, 4.(4)]. So, the bounded, closed subset $\Sigma(P) \subseteq L^1(P)$ is also $\tau(P)$-complete. In other words, P is closed. ■

The list of equivalences in Lemma 3.1(v) with the statement of Theorem 3.7 can be extended further. Recalling that $\mathcal{L}_w(X)$ is precisely the lcHs $\mathcal{L}_s(X)$ equipped with its weak topology, let $P_w : \Sigma \to \mathcal{L}_w(X)$ denote the spectral measure $P : \Sigma \to \mathcal{L}_s(X)$ interpreted as being $\mathcal{L}_w(X) = (\mathcal{L}_s(X), \sigma(\mathcal{L}_s(X), \mathcal{L}_s(X)^*))$-valued. The σ-additivity of P_w is guaranteed by the Orlicz–Pettis Theorem. Of course, if $i : \mathcal{L}_s(X) \to \mathcal{L}_w(X)$ is the identity map, necessarily continuous, then $P_w = i \circ P$. Clearly $\mathcal{N}_0(P) = \mathcal{N}_0(P_w)$ and, from the definition of an integrable function for a vector measure (see (I-1) and (I-2) in Section 2), it is clear that $L^1(P) = L^1(P_w)$ as vector spaces.

Proposition 3.8. *Let X be a Banach space and $P : \Sigma \to \mathcal{L}_s(X)$ be a spectral measure. Then P is a closed measure if and only if P_w is a closed measure.*

Proof. For ease of notation set $Y := \mathcal{L}_s(X)$ and $Y_\sigma := (Y, \sigma(Y, Y^*)) = \mathcal{L}_w(X)$, in which case we have $P : \Sigma \to Y$ and $P_w : \Sigma \to Y_\sigma$.

Suppose that P is a closed measure. The seminorms determining the topology of Y_σ are the family $\{q_{y^*} : y^* \in Y^*\}$ given by

$$q_{y^*}(y) := |\langle y, y^*\rangle|, \qquad y \in Y_\sigma, \tag{3.16}$$

and so $\mathcal{P}(Y_\sigma) \subseteq \mathcal{P}(Y)$, from which it is clear via the definition of the topology $\tau(P_w)$ in $L^1(P_w)$ ($= L^1(P)$ as vector spaces) that $\tau(P_w)$ is weaker than the topology $\tau(P)$ in $L^1(P)$. Thus, the identity map from $(L^1(P), \tau(P))$ onto $(L^1(P_w), \tau(P_w))$ is continuous, which implies that $L^1(P_w)^* \subseteq L^1(P)^*$. The claim is that this is an equality. To show this, fix $y^* := x \otimes x^* \in Y_\sigma^* = Y^*$ of the form (3.2), with $x \in X$ and $x^* \in X^*$, and consider the seminorm $q_{y^*} \in \mathcal{P}(Y_\sigma)$ given by (3.16). Since the polar set $\{y^*\}^\circ$ equals $U_{q_{y^*}} = q_{y^*}^{-1}([0,1])$, the Bipolar Theorem ([13, p. 1], [23, §20, 8.(5)]) yields

$$U_{q_{y^*}}^\circ = \overline{\mathrm{bco}}\{y^*\} = \{\lambda y^* : \lambda \in \mathbb{C},\ |\lambda| \le 1\}.$$

Hence, the corresponding $\tau(P_w)$-continuous seminorm $q_{y^*}(P_w)$ for $L^1(P_w)$ satisfies (see (2.2) with q_{y^*} in place of q and P_w in place of m) the equation

$$\begin{aligned} q_{y^*}(P_w)(f) &= \sup_{|\lambda|\le 1} \int_\Omega |f|\, d|\langle P_w, \lambda y^*\rangle| = \int_\Omega |f|\, d|\langle P_w, y^*\rangle| \\ &= \int_\Omega |f|\, d|\langle P, y^*\rangle|, \qquad f \in L^1(P_w). \end{aligned} \tag{3.17}$$

Define now $\xi := y^* \circ I_P = (x \otimes x^*) \circ I_P$. Since the integration operator $I_P : L^1(P) \to Y$ is continuous, it follows that $\xi \in L^1(P)^*$. Moreover, from (3.17) we have

$$\begin{aligned} |\langle f, \xi\rangle| &= \left| \left\langle \int_\Omega f\, dP, y^* \right\rangle \right| = \left| \int_\Omega f\, d\langle P, y^*\rangle \right| \\ &\le \int_\Omega |f|\, d|\langle P, y^*\rangle| = q_{y^*}(P_w)(f), \qquad f \in L^1(P_w) = L^1(P). \end{aligned}$$

Consequently, ξ is $\tau(P_w)$-continuous because $q_{y^*}(P_w) \in \mathcal{P}(L^1(P_w))$. This establishes the reverse inclusion $L^1(P)^* \subseteq L^1(P_w)^*$ because every element of $L^1(P)^*$ is of the form $\xi = (x \otimes x^*) \circ I_P$ for some $x \in X$ and $x^* \in X^*$; see Lemma 3.4.

Hence, we see that $L^1(P)^* = L^1(P_w)^*$, and so the weak topologies $\sigma(L^1(P), L^1(P)^*)$ and $\sigma(L^1(P_w), L^1(P_w)^*)$ coincide on $L^1(P) = L^1(P_w)$. Using Theorem 3.7 we can now conclude that $L^1(P_w)$ is $\sigma(L^1(P_w), L^1(P_w)^*)$-quasi-complete, and so $L^1(P_w)$ is $\tau(P_w)$-quasi-complete by [23, §18, 4.(4)]. In particular, the vector measure $P_w : \Sigma \to \mathcal{L}_w(X)$ is closed as $\Sigma(P_w)$ is a $\tau(P_w)$-closed, bounded subset of $L^1(P_w)$.

The converse implication, namely that the closedness of P follows from that of P_w, is proved (correctly) in [41, proof of Proposition 2] (set $m := P$ there). ■

REMARK 3.9. (i) A comment on [41, Proposition 2] is in order. One direction of the proof given there is based on [21, Corollary 13], which however is yet to be verified because its proof in [21] is incomplete. Because of this problem we have chosen to provide an alternative proof of this direction for Proposition 3.8 above (only for spectral measures).

(ii) Comparing Lemma 3.1(v), Theorem 3.7 and Proposition 3.8 we notice that there is *no claim* that these equivalences are the same as $\mathcal{R}(P_w)$ ($= \mathcal{R}(P)$) being a closed subset of $\mathcal{L}_w(X)$. Indeed, this is in general *false*: see the discussion after Proposition 3.21 below. Actually, since $I_P : (L^1(P))_\sigma \to \mathcal{L}_w(X)$ is a bicontinuous isomorphism of $(L^1(P))_\sigma$ onto

$I_P(L^1(P))$, equipped with the relative topology τ_w from $\mathcal{L}_w(X)$, and satisfies $I_P(\Sigma(P)) = \mathcal{R}(P)$, we see that $\mathcal{R}(P)$ is a closed set for the weak operator topology if and only $\Sigma(P)$ is a weakly closed subset of $(L^1(P), \tau(P))$.

(iii) It was shown in the proof of Proposition 3.8 that whenever P is a *closed* spectral measure, then the lcHs $(L^1(P_w), \tau(P_w))$ is *always quasi-complete*. This is independent of whether or not the codomain space $\mathcal{L}_w(X)$ of P_w is quasi-complete. In fact, $\mathcal{L}_w(X)$ is quasi-complete if and only if the Banach space X is *reflexive*. To see this, recall from Section 1 that X is reflexive if and only if X is weakly quasi-complete (i.e., X_σ is quasi-complete), and hence it suffices to show that $\mathcal{L}_w(X) = (\mathcal{L}_s(X))_\sigma$ is quasi-complete if and only if X_σ is quasi-complete. Suppose first that X_σ is quasi-complete, and let $\{T_\gamma\}$ be any bounded Cauchy net in $\mathcal{L}_w(X)$, that is, $\sup_\gamma |\langle T_\gamma x, x^*\rangle| < \infty$ for all $x \in X$, $x^* \in X^*$. Fix $x \in X$. Then $\{T_\gamma x\}$ is a bounded Cauchy net in X_σ, and so there exists $Tx \in X$ such $T_\gamma x \to Tx$ in X_σ. The map $T : x \mapsto Tx$ for $x \in X$ is linear and, by the Banach–Steinhaus Theorem, also continuous, i.e., $T \in \mathcal{L}(X)$. It is routine to check that $T_\gamma \to T$ in $\mathcal{L}_w(X)$, and so $\mathcal{L}_w(X)$ is quasi-complete. Conversely, suppose that $\mathcal{L}_w(X)$ is quasi-complete. Let $\{x_\gamma\}$ be any bounded Cauchy net in X_σ. Fix $z^* \in X^* \setminus \{0\}$ and define $T_\gamma : X \to X$ by $T_\gamma : x \mapsto \langle x, z^*\rangle x_\gamma$ for $x \in X$ and each γ. For all $x \in X$ and $x^* \in X^*$ we observe (by (3.2) and (3.3)) that

$$\sup_\gamma |\langle T_\gamma, x \otimes x^*\rangle| = |\langle x, z^*\rangle| \sup_\gamma |\langle x_\gamma, x^*\rangle| < \infty$$

because $\{x_\gamma\}$ is bounded in X_σ, i.e., $\{T_\gamma\}$ is a bounded net in $\mathcal{L}_w(X)$. Moreover,

$$|\langle (T_\gamma - T_{\gamma'}), x \otimes x^*\rangle| = |\langle x, z^*\rangle| \cdot |\langle (x_\gamma - x_{\gamma'}), x^*\rangle| \quad \forall \gamma, \gamma',$$

and so $\{T_\gamma\}$ is also Cauchy in $\mathcal{L}_w(X)$. By assumption there exists $T \in \mathcal{L}(X)$ such that $T_\gamma \to T$ in $\mathcal{L}_w(X)$. Choose any $u \in X$ with $\lambda := \langle u, z^*\rangle \neq 0$. Then $\frac{1}{\lambda}T_\gamma \to \frac{1}{\lambda}T$ in $\mathcal{L}_w(X)$, and in particular,

$$\lim_\gamma \langle x_\gamma, x^*\rangle = \lim_\gamma \left\langle \frac{1}{\lambda} T_\gamma, u \otimes x^* \right\rangle = \frac{1}{\lambda}\langle Tu, x^*\rangle, \qquad x^* \in X^*.$$

Hence, $x_\gamma \to \frac{1}{\lambda}Tu$ in X_σ. The quasi-completeness of X_σ is thereby established. □

The following result is an immediate consequence of Lemma 2.3(ii) with $m := P$ together with (3.3) and (3.6).

LEMMA 3.10. *Let X be a Banach space. A spectral measure $P : \Sigma \to \mathcal{L}_s(X)$ is closed whenever there exists a localizable measure $\mu : \Sigma \to [0, \infty]$ such that*

$$\langle Px, x^*\rangle \ll \mu, \qquad x \in X,\ x^* \in X^*. \tag{3.18}$$

EXAMPLE 3.11. Let $\Omega := [0, 1]$ and Σ be the Borel σ-algebra of Ω. Given $1 \le p < \infty$, let $X := L^p([0, 1])$ be the Banach space with its usual L^p-norm for the Lebesgue measure μ on Σ. For each $E \in \Sigma$, multiplication in $L^p([0, 1])$ by χ_E defines an operator $P(E) \in \mathcal{L}(X)$. It is clear that the so-defined set function $P : E \mapsto P(E) \in \mathcal{L}_s(X)$ is a spectral measure. Since P has infinite range, the lcHs $L^1(P)$ is not weakly complete by Theorem 3.5. On the other hand, (3.18) holds for all $x \in X$ and $x^* \in X^* = L^q([0, 1])$, with $1/p + 1/q = 1$, because $\langle Px, x^*\rangle(E) = \int_E x(\omega)x^*(\omega)\, d\mu(\omega)$ for $E \in \Sigma$. Thus, P is closed by Lemma 3.10, and hence Theorem 3.7 shows that $L^1(P)$ is weakly quasi-complete. □

The proof of Theorem 3.7 is somewhat abstract and relies on several general results from functional analysis. Also, the weak quasi-complete lcHs $(L^1(P), \sigma(L^1(P), L^1(P)^*))$ is not identified explicitly. For a large class of spectral measures P an alternative, more direct proof of Theorem 3.7 is available, which is of interest in its own right and has the advantage that it identifies $(L^1(P), \sigma(L^1(P), L^1(P)^*)$ more concretely (as an L^∞-space equipped with its weak-$*$ topology). First we require a definition.

Let X be a Banach space and $P : \Sigma \to \mathcal{L}_s(X)$ be a spectral measure. An element $x_0 \in X$ is called a *separating vector* for P if $P(E) = 0$ whenever $P(E)x_0 = 0$. In other words, $E \in \Sigma$ is P-null if and only if $P(E)x_0 = 0$. For the spectral measure in Example 3.11 the constant function χ_Ω is a separating vector.

Proposition 3.12. *Let X be a Banach space and $P : \Sigma \to \mathcal{L}_s(X)$ be a spectral measure admitting a separating vector $x_0 \in X$. Then there exists a finite, positive measure μ on Σ such that*

$$\big(L^1(P), \sigma(L^1(P), L^1(P)^*)\big) = \big(L^\infty(\mu), \sigma(L^\infty(\mu), L^1(\mu))\big) \tag{3.19}$$

as lcHs'. In particular, P is closed and $L^1(P)$ is weakly quasi-complete.

Proof. Select any *Bade functional* $x_0^* \in X^*$ of P relative to x_0; that is, x_0^* satisfies both

(a) $\mu : E \mapsto \langle P(E)x_0, x_0^*\rangle$, for $E \in \Sigma$, is a finite positive measure on Σ, and
(b) $P(E)x_0 = 0$ whenever $\mu(E) = 0$

(see Lemma 3.2 and [48, Theorem VI.1]). It follows that P and μ have the same null sets because x_0 is a separating vector for P. Consequently,

$$L^1(P) = L^\infty(P) = L^\infty(\mu) \tag{3.20}$$

as vector spaces (see Lemma 3.1(ii) for the first equality), and moreover, (3.18) holds. In particular, P is closed by Lemma 3.10.

Fix $x \in X$ and $x^* \in X^*$. By (3.18) the Radon–Nikodým derivative $\varphi_{x,x^*} \in L^1(\mu)$ of the complex measure $\langle Px, x^*\rangle$ with respect to μ exists. Recalling (3.6) and Lemma 3.4 we deduce, for each $f \in L^1(P) = L^\infty(P) = L^\infty(\mu)$, that

$$\langle f, (x \otimes x^*) \circ I_P\rangle = \int_\Omega f\, d\langle Px, x^*\rangle = \int_\Omega f\varphi_{x,x^*}\, d\mu. \tag{3.21}$$

Let now $g \in L^1(\mu)$. Its corresponding indefinite integral $\mu_g : E \mapsto \int_E g\, d\mu$ for $E \in \Sigma$ satisfies $\mu_g(E) = 0$ whenever $P(E)x_0 = 0$ (see the definition of μ in (a), and use the fact that E is μ-null). According to [16, Theorem 4.2], there exist vectors $x \in \overline{\mathrm{span}}\{P(E)x_0 : E \in \Sigma\} \subseteq X$ and $x^* \in X^*$ such that $\mu_g = \langle Px, x^*\rangle$ on Σ. Thus, (3.21) holds with g in place of φ_{x,x^*}.

Recalling Lemma 3.4, we conclude from the above arguments and (3.20) that $L^1(P)^*$ can be identified with $L^1(\mu)$, and hence (3.21) yields the identity (3.19) between lcHs'. Finally, (3.19) shows that $L^1(P)$ is weakly quasi-complete because $L^\infty(\mu)$ is quasi-complete for the weak-$*$ topology $\sigma(L^\infty(\mu), L^1(\mu))$ [30, Corollary 2.6.19]. ∎

Remark 3.13. (i) If X is a *separable* Banach space, then *every* $\mathcal{L}_s(X)$-valued spectral measure admits a separating vector (see Lemma 3.2 and [48, Proposition VI.3]).

(ii) There exist a non-separable Banach space X and an $\mathcal{L}_s(X)$-valued spectral measure admitting a separating vector [39, Example 1].

(iii) For an alternative proof of the existence of Bade functionals we refer to [10, Ch. XVII, Lemma 3.12]. □

The following example generalizes Example 3.11 and simultaneously illustrates that, in certain settings, the use of Bade functionals can be replaced by an elegant factorizition result of Lozanovskiĭ. Of course, since separating vectors are present, it is covered by Proposition 3.12 above.

EXAMPLE 3.14. Consider a *Banach function space*, briefly B.f.s., $X = X(\mu)$ based on a finite, positive measure space (Ω, Σ, μ), with respect to the μ-a.e. pointwise order. That is, $X(\mu)$ is a complex vector lattice of (equivalence classes of) Σ-measurable functions modulo μ-null functions such that

(a) if $\psi \in \mathcal{L}^0(\Sigma)$ and $\varphi \in X(\mu)$ with $|\psi| \le \varphi$ pointwise μ-a.e., then $\psi \in X(\mu)$,
(b) $\operatorname{sim}\Sigma \subseteq X(\mu)$, and
(c) $X(\mu)$ is equipped with a lattice norm $\|\cdot\|_{X(\mu)}$ for which $X(\mu)$ is complete.

As general references we suggest [28], [37], [53], for example.

We assume further that $X(\mu)$ has *σ-order continuous norm*, i.e., $\lim_{n\to\infty} \|\varphi_n\|_{X(\mu)} = 0$ whenever $\varphi_n \downarrow 0$ in the μ-a.e. pointwise order of $X(\mu)$. For such B.f.s.' $X(\mu)$ we can identify the dual Banach space $X(\mu)^*$ of $X(\mu)$ with its *associate space*

$$X(\mu)' := \{\psi \in \mathcal{L}^0(\Sigma) : \varphi\psi \in L^1(\mu) \text{ for all } \varphi \in X(\mu)\}$$

in the following sense. Every function $\psi \in X(\mu)'$ defines a continuous linear functional $\xi_\psi : \varphi \mapsto \int_\Omega \varphi\psi\, d\mu$ on $X(\mu)$, and conversely, every continuous linear functional on $X(\mu)$ must be equal to ξ_ψ for some $\psi \in X(\mu)'$ [37, Proposition 2.16].

Let $E \in \Sigma$. Via (a), we can define a linear operator $P(E)$ in $X(\mu)$ by $P(E)\varphi := \varphi\chi_E$ for $\varphi \in X(\mu)$. Clearly $P(E)$ is a projection, and moreover is continuous because $\|\cdot\|_{X(\mu)}$ is a lattice norm. The $\mathcal{L}(X(\mu))$-valued set function $P : E \mapsto P(E)$, for $E \in \Sigma$, is clearly finitely additive on Σ and satisfies (3.4). By the σ-order continuity of $\|\cdot\|_{X(\mu)}$, the set function P is actually σ-additive for τ_s. So, $P : \Sigma \to \mathcal{L}_s(X(\mu))$ is a spectral measure, usually referred to as the *canonical spectral measure* in $X(\mu)$.

The constant function $x_0 := \chi_\Omega$, which belongs to $X(\mu)$ by (b), is a separating vector for P. Moreover, $\chi_\Omega \in X(\mu)^* = X(\mu)'$ is a Bade functional relative to x_0 and satisfies $\langle P(E)x_0, \chi_\Omega\rangle = \mu(E)$ for $E \in \Sigma$, i.e., $\langle Px_0, \chi_\Omega\rangle = \mu$ on Σ. In particular, $L^1(P) = L^\infty(P) = L^\infty(\mu)$ as vector spaces.

Let $\varphi \in X(\mu)$ and $\psi \in X(\mu)'$, in which case $\varphi\psi \in L^1(\mu)$ and

$$\langle P\varphi, \psi\rangle(E) = \int_E \varphi\psi\, d\mu, \qquad E \in \Sigma. \tag{3.22}$$

Then $(\varphi \otimes \psi) \circ I_P \in L^1(P)^*$ (see Lemma 3.4) has the form

$$\langle f, (\varphi \otimes \psi) \circ I_P\rangle = \int_\Omega f\, d\langle P\varphi, \psi\rangle = \int_\Omega f(\varphi\psi)\, d\mu$$

for $f \in L^1(P) = L^\infty(P) = L^\infty(\mu)$.

Conversely, let $g \in L^1(\mu)$. Define a bounded, Σ-measurable function g_0 by $g_0(\omega) := g(\omega)/|g(\omega)|$ if $g(\omega) \neq 0$, and by $g_0(\omega) := 0$ otherwise. Then $g = |g|g_0$ pointwise on Ω. By Lozanovskiĭ's Theorem (see [17, Theorem 1], [27, Theorem 6], [40, Proposition 6]) applied to the $\mathbb{R}$-valued function $|g| \in L^1(\mu)$, there exist $\mathbb{R}$-valued functions $\varphi_0 \in X(\mu)$ and $\psi \in X(\mu)^*$ such that $|g| = \varphi_0\psi$ on Ω. So, with $\varphi := g_0\varphi_0 \in X(\mu)$ (recall condition (a)) we see that $g = |g|g_0 = (g_0\varphi_0)\psi = \varphi\psi$. This together with (3.22) implies, for every $f \in L^1(P) = L^\infty(P) = L^\infty(\mu)$, that

$$\int_\Omega fg\,d\mu = \int_\Omega f(\varphi\psi)\,d\mu = \int_\Omega f\,d\langle P\varphi,\psi\rangle = \langle f, (\varphi\otimes\psi)\circ I_P\rangle.$$

We can conclude from the above arguments that the dual space $L^1(P)^*$ of $L^1(P)$ can be identified with $L^1(\mu)$ and that (3.19) holds as lcHs'. □

We now address the weak sequential completeness of $L^1(P)$, beginning with the role played by a certain completeness property of the co-domain space $\mathcal{L}_s(X)$. Even though the next result follows from Theorem 3.7, we provide a simpler and more direct proof.

Proposition 3.15. *Let X be a weakly sequentially complete Banach space. If $P : \Sigma \to \mathcal{L}_s(X)$ is any closed spectral measure, then $L^1(P)$ is also weakly sequentially complete.*

Proof. Since P is closed, the linear subspace $I_P(L^1(P)) \subseteq \mathcal{L}_s(X)$ is complete (see Lemma 3.1(v)), and in particular weakly closed. So, $I_P(L^1(P))$ is weakly sequentially complete because the weak sequential completeness of X implies that of $\mathcal{L}_s(X)$; this was noted earlier in this section. Thus $L^1(P)$, which is topologically isomorphic to $I_P(L^1(P))$ by Lemma 3.1(i), is weakly sequentially complete. ■

In view of Theorem 3.7, the question of the weak sequential completeness of $L^1(P)$ is only relevant for non-closed spectral measures. In this regard, the first point is whether the closedness assumption on P can be omitted in Proposition 3.15. The answer is no, as can be seen from the following

Example 3.16. We refer to [15, 4.2 and 4.3] which exhibit a non-separable Hilbert space X and an $\mathcal{L}_s(X)$-valued spectral measure defined on some (specific) measurable space (Ω,Σ) such that $P(\Sigma)$ is *not* sequentially τ_s-closed; equivalently, $P(\Sigma)$ is not sequentially complete in the quasi-complete lcHs $\mathcal{L}_s(X)$. In particular, $P(\Sigma)$ is not complete in $\mathcal{L}_s(X)$, and hence P is not a closed measure. Moreover, $L^1(P)$ is not $\tau(P)$-sequentially complete (due to the $\tau(P)$-closedness of $\Sigma(P)$ in $L^1(P)$); see [38, Example 6.5(i)] for the detailed arguments. Thus, the lcHs $L^1(P)$ is not weakly sequentially complete [23, §18, 4.(4)], whereas the Hilbert space X is surely weakly sequentially complete. □

Concerning Example 3.16, the point is that the range $P(\Sigma)$ of P is a Bade σ-complete B.a. of projections (see Lemma 3.2), but it is not weakly sequentially complete because it is not sequentially τ_s-complete.

For a Banach-space-valued vector measure m, it was pointed out in Section 1 that $L^1(m)$ is a Banach lattice with order continuous norm, and that $L^1(m)$ is weakly sequentially complete if and only if it does not contain an isomorphic copy of c_0. For analogues of this to the locally convex Riesz space setting we refer to [8], [51]. However, the full analogue is not available. In Example 3.16, the lcHs $L^1(P)$ does not contain a copy of c_0 (see

Proposition 3.17 below), but it *fails* to be weakly sequentially complete. Recall that a lcHs Y is said to *contain an isomorphic copy* of the Banach space c_0 if there exists a Y-valued, bicontinuous isomorphism from c_0 onto its range. A weakly sequentially complete lcHs cannot contain an isomorphic copy of c_0 because c_0 itself is not weakly sequentially complete.

PROPOSITION 3.17. *Let X be a Banach space and $P : \Sigma \to \mathcal{L}_s(X)$ be any spectral measure.*

(i) *Every scalarly P-integrable function is also P-integrable.*
(ii) *The lcHs $(L^1(P), \tau(P))$ cannot contain an isomorphic copy of c_0.*

Proof. (i) Let $f : \Omega \to \mathbb{C}$ be a scalarly P-integrable function. Our proof is motivated by that of [48, Proposition V.4], which provides a direct proof of Lemma 3.1(ii). To show that the subset

$$\{I_P(s) : s \in \operatorname{sim}\Sigma,\ |s| \le |f|\} \subseteq \mathcal{L}(X) \tag{3.23}$$

is uniformly bounded (i.e., bounded in $\mathcal{L}_u(X)$), fix $x \in X$. Given $x^* \in X^*$, the function f is $\langle P, x \otimes x^*\rangle$-integrable by assumption (see (3.3)), and hence $\langle Px, x^*\rangle$-integrable by (3.6). Thus, for each $s \in \operatorname{sim}\Sigma$ satisfying $|s| \le |f|$ pointwise, it follows that

$$\begin{aligned}|\langle I_P(s)x, x^*\rangle| &= \left|\left\langle \int_\Omega s\, dPx, x^* \right\rangle\right| = \left|\int_\Omega s\, d\langle Px, x^*\rangle\right| \\ &\le \int_\Omega |s|\, d|\langle Px, x^*\rangle| \le \int_\Omega |f|\, d|\langle Px, x^*\rangle| < \infty.\end{aligned}$$

Therefore, the subset $\{I_P(s)(x) : s \in \operatorname{sim}\Sigma,\ |s| \le |f|\} \subseteq X$ is weakly bounded, and hence norm bounded. Since $x \in X$ is arbitrary, the Banach–Steinhaus Theorem ensures that the subset (3.23) is uniformly bounded.

Given $k \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$, define the set

$$E(k) := \{\omega \in \Omega : k^2 \le |f(\omega)| < (k+1)^2\} \in \Sigma.$$

Then $0 \le k^2\chi_{E(k)} \le |f|$ pointwise on Ω, so that $k^2P(E(k)) = I_P(k^2\chi_{E(k)})$ belongs to the uniformly bounded subset (3.23). Let

$$M := \sup_{k\in\mathbb{N}_0} \|k^2P(E(k))\|_{\mathrm{op}} < \infty. \tag{3.24}$$

Given $n \in \mathbb{N}$, the set $F(n) := \{\omega \in \Omega : |f(\omega)| \ge n^2\}$ equals the disjoint union $\bigcup_{k=n}^\infty E(k)$. So, for every $x \in X$, it follows from (3.24) that

$$\begin{aligned}\|P(F(n))x\|_X &= \Big\|\sum_{k=n}^\infty P(E(k))x\Big\|_X \le \sum_{k=n}^\infty \|P(E(k))x\|_X \\ &\le \sum_{k=n}^\infty \|P(E(k))\|_{\mathrm{op}} \cdot \|x\|_X \le M\Big(\sum_{k=n}^\infty k^{-2}\Big)\|x\|_X.\end{aligned}$$

Then $\|P(F(n))\|_{\mathrm{op}} \le M\sum_{k=n}^\infty k^{-2}$ for all $n \in \mathbb{N}$. Consequently, $\lim_{n\to\infty} \|P(F(n))\|_{\mathrm{op}} = 0$. But every non-zero projection $Q \in \mathcal{L}(X)$ satisfies $\|Q\|_{\mathrm{op}} \ge 1$. Therefore, we can select $N \in \mathbb{N}$ such that $P(F(N)) = 0$, which means that $F(N)$ is P-null, that is, $|f| < N^2$ pointwise P-a.e. on Ω. Thus, f is P-essentially bounded, and hence P-integrable via Lemma 3.1(ii).

(ii) Since the codomain space $\mathcal{L}_s(X)$ of P is quasi-complete, part (ii) follows from the equivalence (a)$\Leftrightarrow$(c) in [38, Proposition 6.4(i)]. ∎

Part (i) of Proposition 3.17 above is of interest in its own right because, given a general vector measure m, one of the natural questions is whether or not scalarly m-integrable functions are always m-integrable. For arbitrary m the answer is: not always. This topic is extensively explored in [37, Ch. 3], [38], for general vector measures.

According to Example 3.16, there exist non-closed spectral measures whose L^1-space *fails* to be weakly sequentially complete. There also exist non-closed spectral measures whose L^1-space *is* weakly sequentially complete. Actually, as we now show, there is a large class of spectral measures whose L^1-space is *always* weakly sequentially complete, independent of whether the spectral measure is closed or not!

THEOREM 3.18. *Let X be a Banach space and $P : \Sigma \to \mathcal{L}_s(X)$ be any atomic spectral measure. Then the lcHs $L^1(P)$ is weakly sequentially complete. If, in addition, P is a closed measure, then $P(\Sigma)$ is necessarily a compact subset of $\mathcal{L}_s(X)$.*

Proof. Fix a weak Cauchy sequence $\{f_n\}_{n=1}^\infty$ in $L^1(P)$, in which case it is weakly bounded, and hence $\tau(P)$-bounded. The continuous linear operator $I_P : L^1(P) \to \mathcal{L}_s(X)$ maps the $\tau(P)$-bounded sequence $\{f_n\}_{n=1}^\infty$ to the bounded sequence $\{I_P(f_n)\}_{n=1}^\infty$ in $\mathcal{L}_s(X)$, which is then uniformly bounded via the Banach–Steinhaus Theorem. So, we have $K := \sup_{n\in\mathbb{N}} \|I_P(f_n)\|_{\mathrm{op}} < \infty$. This and (3.11) give

$$\sup_{n\in\mathbb{N}} |f_n|_P \le K. \tag{3.25}$$

For each individual function f_n with $n \in \mathbb{N}$, there is $E(n) \in \Sigma$ satisfying both $\Omega \setminus E(n) \in \mathcal{N}_0(P)$ and $|f_n|_P = \sup_{\omega\in E(n)} |f_n(\omega)|$. Consequently, (3.25) implies that

$$|(f_n\chi_{E(n)})(\omega)| \le K, \qquad n \in \mathbb{N}, \omega \in \Omega. \tag{3.26}$$

The claim is both that the set

$$\mathbf{L} := \Big\{\omega \in \Omega : \lim_{n\to\infty} (f_n\chi_{E(n)})(\omega) \text{ exists in } \mathbb{C}\Big\}$$

belongs to Σ, and that $\Omega \setminus \mathbf{L} \in \mathcal{N}_0(P)$. Indeed, with $g_n := f_n\chi_{E(n)}$ for $n \in \mathbb{N}$, let

$$\mathbf{L}_1 := \Big\{\omega \in \Omega : \lim_{n\to\infty} (\mathcal{R}e\, g_n)(\omega) \text{ exists in } \mathbb{R}\Big\},$$
$$\mathbf{L}_2 := \Big\{\omega \in \Omega : \lim_{n\to\infty} (\mathcal{I}m\, g_n)(\omega) \text{ exists in } \mathbb{R}\Big\}.$$

Define extended real-valued functions φ and ψ on Ω by

$$\varphi(\omega) := \limsup_{n\to\infty} (\mathcal{R}e\ g_n)(\omega) \quad \text{and} \quad \psi(\omega) := \liminf_{n\to\infty} (\mathcal{I}m\ g_n)(\omega), \qquad \omega \in \Omega.$$

Since both φ and ψ are Σ-measurable, we have

$$\mathbf{L}_1 = \varphi^{-1}(\mathbb{R}) \cap \psi^{-1}(\mathbb{R}) \cap \{\omega \in \Omega : \varphi(\omega) = \psi(\omega)\} \in \Sigma.$$

Similarly, $\mathbf{L}_2 \in \Sigma$. So, $\mathbf{L} = \mathbf{L}_1 \cap \mathbf{L}_2 \in \Sigma$.

In order to verify that $\Omega \setminus \mathbf{L} \in \mathcal{N}_0(P)$, we shall apply Lemma 2.9 with $Y := \mathcal{L}_s(X)$ and $m := P$. With the notation from there, $\{\chi_{F(\alpha)}\}_{\alpha\in\mathcal{A}(P)}$ is the family of all atoms in $\Sigma(P)$.

First, fix $\alpha \in \mathcal{A}(P)$ and select vectors $x_\alpha \in X$ and $x_\alpha^* \in X^*$ satisfying $\langle P(F(\alpha))x_\alpha, x_\alpha^*\rangle = 1$. Consider the linear functional

$$\xi_\alpha := (x_\alpha \otimes x_\alpha^*) \circ I_P \circ M_{F(\alpha)} \in L^1(P)^*$$

(see Lemma 3.4), where $M_{F(\alpha)} \in \mathcal{L}(L^1(P))$ is the operator of multiplication by $\chi_{F(\alpha)}$ (see (2.7) or (2.8) with $E := F(\alpha)$). By recalling that $g_n = f_n$ pointwise P-a.e. on Ω for $n \in \mathbb{N}$, the sequence $\{g_n\}_{n=1}^\infty$ is the same as $\{f_n\}_{n=1}^\infty$ in the quotient space $L^1(P) = \mathcal{L}^1(P)/\mathcal{N}(P)$, and hence $\{g_n\}_{n=1}^\infty$ is weakly Cauchy in $L^1(P)$. So, $\xi_\alpha \in L^1(P)^*$ maps $\{g_n\}_{n=1}^\infty$ to a Cauchy sequence in $\mathbb{C}$. From (2.11), with $m := P$ and $f := g_n$ for $n \in \mathbb{N}$, we can see that this Cauchy sequence in $\mathbb{C}$ is precisely $\{a(\alpha, g_n)\}_{n=1}^\infty$ because

$$\langle g_n, \xi_\alpha\rangle = \left\langle \left(\int_{F(\alpha)} g_n\, dP\right) x_\alpha, x_\alpha^* \right\rangle = \langle a(\alpha, g_n)P(F(\alpha))x_\alpha, x_\alpha^*\rangle = a(\alpha, g_n)$$

for $n \in \mathbb{N}$. Moreover, it follows from (2.12) with $m := P$ and $f := g_n$ that the set

$$G(\alpha, n) := g_n^{-1}(\{a(\alpha, g_n)\}) \cap F(\alpha) \subseteq F(\alpha), \qquad n \in \mathbb{N},$$

satisfies

$$F(\alpha) \setminus G(\alpha, n) \in \mathcal{N}_0(P). \tag{3.27}$$

With $G(\alpha) := \bigcap_{n=1}^\infty G(\alpha, n) \subseteq F(\alpha)$, it follows that

$$g_n(\omega) = a(\alpha, g_n), \qquad \omega \in G(\alpha),\ n \in \mathbb{N}. \tag{3.28}$$

Since the Cauchy sequence $\{a(\alpha, g_n)\}_{n=1}^\infty$ converges in $\mathbb{C}$, we deduce from (3.28) that $G(\alpha) \subseteq \mathbf{L}$, i.e., $\Omega \setminus \mathbf{L} \subseteq \Omega \setminus G(\alpha)$, and hence (3.27) gives

$$(\Omega \setminus \mathbf{L}) \cap F(\alpha) \subseteq F(\alpha) \setminus G(\alpha) = \bigcup_{n=1}^\infty (F(\alpha) \setminus G(\alpha, n)) \in \mathcal{N}_0(P).$$

Therefore, $(\Omega \setminus \mathbf{L}) \cap F(\alpha) \in \mathcal{N}_0(P)$. As this holds for arbitrary $\alpha \in \mathcal{A}(P)$, Lemma 2.9 with $Y := \mathcal{L}_s(X)$, $m := P$ and $E := \Omega \setminus \mathbf{L}$ shows that $\Omega \setminus \mathbf{L}$ is P-null. So, the claim has been verified.

Now, let f denote the pointwise limit of the individual functions $f_n\chi_{E(n)\cap\mathbf{L}} = g_n\chi_{\mathbf{L}}$ for $n \in \mathbb{N}$. Then $|f| \le K\chi_\Omega$ pointwise by (3.26). Since $K\chi_\Omega$ is P-integrable and the codomain space $\mathcal{L}_s(X)$ of P is quasi-complete (hence, sequentially complete), the Dominated Convergence Theorem applied to P [25, Theorem 2.2(2)] shows that f is the $\tau(P)$-limit of $\{f_n\chi_{E(n)\cap\mathbf{L}}\}_{n=1}^\infty$. So, f is the $\tau(P)$-limit of $\{f_n\}_{n=1}^\infty$ as $f_n = f_n\chi_{E(n)\cap\mathbf{L}}$ pointwise P-a.e. on Ω. In particular, f is also the weak limit of $\{f_n\}_{n=1}^\infty$ in $L^1(P)$, which establishes the weak sequential completeness of $L^1(P)$ because $\{f_n\}_{n=1}^\infty$ was an arbitrary weak Cauchy sequence in $L^1(P)$.

If, in addition, P is closed, then the compactness of $P(\Sigma)$ in $\mathcal{L}_s(X)$ follows from Proposition 2.12(i). ■

We recall that every σ-atomic spectral measure is necessarily closed; see the discussion immediately prior to Lemma 2.8. A general atomic spectral measure may or may not be closed; see Example 3.22 below, for instance.

Corollary 3.19. *Let X be a Banach space and $P : \Sigma \to \mathcal{L}_s(X)$ be any atomic spectral measure.*

(i) *A sequence in* $L^1(P)$ *is* $\tau(P)$*-convergent if and only if it is weakly convergent.*
(ii) *The closed subset* $\Sigma(P) \subseteq L^1(P)$ *is weakly sequentially complete.*

Proof. (i) The 'only if' portion is obvious. On the other hand, the 'if' portion has been established in the proof of Theorem 3.18.

(ii) Since $\Sigma(P)$ is weakly closed in $L^1(P)$ by Proposition 2.10 (with $m := P$), this follows immediately from Theorem 3.18. ∎

REMARK 3.20. Part (ii) of Corollary 3.19 need *not* be valid for spectral measures which are not atomic. Fix any $1 \le p < \infty$ and let $X := L^p([0,1])$. Consider the *atomless*, closed spectral measure $P : \Sigma \to \mathcal{L}_s(X)$ given in Example 3.11; the notation is from there, and so μ is the Lebesgue measure on the Borel σ-algebra Σ of $\Omega := [0,1]$. According to Proposition 3.12 and Example 3.14, we can identify $L^1(P)$ with $L^\infty(\mu)$, as vector spaces, and identify the weak topology on $L^1(P)$ with the weak-$*$ topology $\sigma(L^\infty(\mu), L^1(\mu))$ on $L^\infty(\mu)$. It is shown in [20, Example 4] that there exists a sequence $\{A(n)\}_{n=1}^\infty \subseteq \Sigma$ such that $\chi_{A(n)} \to \frac{1}{2}\chi_\Omega$ in $(L^\infty(\mu), \sigma(L^\infty(\mu), L^1(\mu)))$ as $n \to \infty$. Hence, $\{\chi_{A(n)}\}_{n=1}^\infty$ is a weak Cauchy sequence in $L^1(P)$ whose weak limit $\frac{1}{2}\chi_\Omega$ does not belong to $\Sigma(P)$. So, $\Sigma(P) \subseteq L^1(P)$ is *not* weakly sequentially complete.

This specific spectral measure P also shows that part (i) of Corollary 3.19 may fail for a general spectral measure. Indeed, as already noted, the sequence $\{\chi_{A(n)}\}_{n=1}^\infty$ is weakly convergent to $\frac{1}{2}\chi_\Omega$ in $L^1(P)$. However, it cannot be $\tau(P)$-convergent because the closedness of $\Sigma(P)$ in $L^1(P)$ would then imply that the $\tau(P)$-limit (necessarily $\frac{1}{2}\chi_\Omega$) belongs to $\Sigma(P)$, which is clearly not the case. □

Given a Banach space X, the question arises of whether or not every Bade σ-complete B.a. of projections in $\mathcal{L}_s(X)$ is weakly sequentially complete, i.e., sequentially complete in the space $\mathcal{L}_w(X) = (\mathcal{L}_s(X), \sigma(\mathcal{L}_s(X), \mathcal{L}_s(X)^*))$. The answer is no, in general; see the discussion after Example 3.16. However, if the B.a. happens to be atomic, then the answer is in the affirmative as we now show.

PROPOSITION 3.21. *Let* X *be a Banach space and* $\mathcal{M} \subseteq \mathcal{L}_s(X)$ *be any atomic, Bade* σ*-complete B.a. of projections. Then* $\mathcal{M}$ *is sequentially complete in* $\mathcal{L}_w(X)$.

Proof. According to Lemma 3.2, there exists a measurable space (Ω, Σ) and a spectral measure $P : \Sigma \to \mathcal{L}_s(X)$, necessarily atomic by Lemma 3.3, whose range equals $\mathcal{M}$. By Corollary 3.19(ii), the subset $\Sigma(P) \subseteq L^1(P)$ is weakly sequentially complete, and hence so is its image $I_P(\Sigma(P))$ under the bicontinuous isomorphism I_P from $L^1(P)$ into $\mathcal{L}_s(X)$; use Lemma 3.1(i) and the fact that any continuous linear map between lcHs' Y and Z is also continuous between $(Y, \sigma(Y, Y^*))$ and $(Z, \sigma(Z, Z^*))$ [23, §20, 4.(5)]. But $\mathcal{M} = P(\Sigma) = I_P(\Sigma(P))$, which establishes the proposition. ∎

An immediate consequence of Proposition 3.21 is that such a B.a. of projections $\mathcal{M}$ is necessarily sequentially complete in $\mathcal{L}_s(X)$, which is exactly Theorem 2.2 in [15].

For the interested reader we provide an alternative and more direct proof of Proposition 3.21 in the Appendix. It does not make use of spectral measures, but is based purely on the structure of Bade σ-complete B.a.'s of projections.

The subtlety of Proposition 3.21 can be gleaned from Remark 3.20. From the notation and the example there, since the isomorphism $I_P : L^1(P) \to \mathcal{L}_s(X)$ is bicontinuous, it is also weak-to-weak bicontinuous. But $\Sigma(P)$ is not weakly sequentially complete, and hence $\mathcal{M} := P(\Sigma) = I_P(\Sigma(P))$ is not sequentially complete in $\mathcal{L}_w(X)$. Of course, $\mathcal{M}$ is not atomic, and so there is no contradiction to Proposition 3.21.

The feature exhibited by this example is actually more widespread. Indeed, let X be any infinite-dimensional Hilbert space and $\mathcal{M} \subseteq \mathcal{L}(X)$ be any *atomless*, *Bade complete B.a.* of selfadjoint projections. A classical result of H. Dye [12, Lemma 2.3] asserts that the closure $\overline{\mathcal{M}}^w$ of $\mathcal{M}$ in $\mathcal{L}_w(X)$ never consists entirely of projections. Nevertheless, $\overline{\mathcal{M}}^w$ is always a uniformly bounded, quasi-complete subset of $\mathcal{L}_w(X)$. This follows from the facts that $\mathcal{M} = P(\Sigma)$ for some *closed* spectral measure $P : \Sigma \to \mathcal{L}_s(X)$ (see Lemma 3.2), that $L^1(P)$ is weakly quasi-complete (by Theorem 3.7) and, as noted before, that $I_P : L^1(P) \to \mathcal{L}_s(X)$ being a bicontinuous isomorphism implies it is also a bicontinuous isomorphism of $(L^1(P), \sigma(L^1(P), L^1(P)^*))$ onto $I_P(L^1(P))$ equipped with the relative topology τ_w from $\mathcal{L}_w(X)$. Similar examples can be constructed in certain reflexive Banach spaces [1, p. 354]. Curiously, for *any* Bade complete B.a. of projections $\mathcal{M}$ in *any* Banach space X, it is known that if a net $\{Q_\gamma\}_{\gamma\in\Gamma} \subseteq \mathcal{M}$ converges in $\mathcal{L}_w(X)$ *to a projection* Q, then necessarily $Q \in \mathcal{M}$ [10, Ch. XVII, Lemma 3.6], and $\{Q_\gamma\}_{\gamma\in\Gamma}$ actually converges to Q in $\mathcal{L}_s(X)$ [10, Ch. XVII, Theorem 3.27]. The point is that for certain *atomless* B.a.'s of projections $\mathcal{M}$ in various Banach spaces X, there always exist nets/sequences $\{Q_\gamma\}_{\gamma\in\Gamma} \subseteq \mathcal{M}$ which converge in $\mathcal{L}_w(X)$ to operators which are *not* projections!

We end this section with two relevant examples of atomic spectral measures taken from [15]. To this end, let X be a Banach space and $\mathcal{M} \subseteq \mathcal{L}(X)$ be a B.a. of projections. According to [15, p. 46] (see also [46, p. 366]), we say that $\mathcal{M}$ is *Bade atomic* if it has a family $\{Q_\alpha\}_{\alpha\in\mathcal{A}}$ of atoms such that, for every $Q \in \mathcal{M} \setminus \{0\}$, there is a subset $\mathcal{B} \subseteq \mathcal{A}$ for which the series $\sum_{\alpha\in\mathcal{B}} Q_\alpha$ is unconditionally convergent in the lcHs $\mathcal{L}_s(X)$ with Q as its sum, i.e., $Q = \sum_{\alpha\in\mathcal{B}} Q_\alpha$. If $\mathcal{M}$ is Bade atomic, then clearly it is atomic. The converse holds under the additional assumption that $\mathcal{M}$ is Bade complete [15, Lemma 2.1(ii)].

EXAMPLE 3.22. With $\Omega := [0,1]$, let $\Sigma \subseteq 2^\Omega$ be any σ-algebra such that $\{\omega\} \in \Sigma$ for all $\omega \in \Omega$. We denote by X the non-separable Banach space $\ell^p(\Omega)$ $(1 \le p < \infty)$ or $c_0(\Omega)$. For each $E \in \Sigma$, define a projection $P(E) \in \mathcal{L}(X)$ by $P(E) := \chi_E x$ for $x \in X$. The set function $P : \Sigma \to \mathcal{L}_s(X)$ is an atomic spectral measure whose atoms are all the singleton sets $\{\omega\}$ with $\omega \in \Omega$. The case when $X := \ell^2(\Omega)$ and Σ is the Borel σ-algebra was considered in [15, pp. 45–46]; the arguments given there still apply here. Regarding the range $\mathcal{M} := P(\Sigma)$ of P, it is an atomic, Bade σ-complete B.a. of projections in $\mathcal{L}_s(X)$ (see Lemmas 3.2 and 3.3) and its atoms are the projections $P(\{\omega\})$ for $\omega \in \Omega$. Moreover, $\mathcal{M}$ is also Bade atomic because for every $E \in \Sigma$ the series $\sum_{\omega\in E} P(\{\omega\})$ is unconditionally convergent in $\mathcal{L}_s(X)$ with $P(E) = \sum_{\omega\in E} P(\{\omega\})$.

It turns out that P is closed if and only if $\Sigma = 2^\Omega$. This can be proved by adapting the arguments in Example 2.13. Note also that [36, Example 2.22] gives the 'if' portion, when $X = \ell^2(\Omega)$, by Lemma 2.3(ii) with $Y := \mathcal{L}_s(X)$, $m := P$ and μ being the counting measure. This is possible because μ, defined on 2^Ω, is localizable. In particular, if Σ is the Borel σ-algebra in Ω or the σ-algebra of all countable/co-countable subsets of Ω, then P

is *not* closed. In this case, $L^1(P)$ is not weakly quasi-complete (by Theorem 3.7), but in view of Theorem 3.18, it is weakly sequentially complete. □

EXAMPLE 3.23. The following construction occurs in [15, Example 2.5]. Let $\Omega := [0,1]$ and Σ be the Borel σ-algebra of Ω. For any subset $G \subseteq \Omega$, let $\mathcal{F}(G)$ denote the family of all the finite subsets of G. We denote by X the Banach space of all σ-additive complex measures on Σ, equipped with the total variation norm $\|\mu\|_X := |\mu|(\Omega)$ for $\mu \in X$, where $|\mu| : \Sigma \to [0,\infty)$ is the *total variation measure* of μ [49, §6.1]. Given $\mu \in X$ and $E \in \Sigma$, define $\mu_E \in X$ by $\mu_E(F) := \mu(E \cap F)$ for $F \in \Sigma$. The set function $P : \Sigma \to \mathcal{L}_s(X)$ given by $P(E)\mu := \mu_E$ for $\mu \in X$ and $E \in \Sigma$ is an atomic spectral measure whose atoms are all the singleton sets $\{\omega\} \in \Sigma$ with $\omega \in \Omega$. The range $\mathcal{M} := P(\Sigma)$ of P is an atomic, Bade σ-complete B.a. of projections in $\mathcal{L}_s(X)$; its atoms are precisely the projections $P(\{\omega\})$ with $\omega \in \Omega$ (see Lemmas 3.2 and 3.3). However, $\mathcal{M}$ is *not* Bade atomic. Indeed, take any non-zero, *continuous measure* $\nu \in X$ (i.e., $\nu(\{\omega\}) = 0$ for all $\omega \in \Omega$). Then

$$P(\Omega)\nu = \nu \neq 0 \quad \text{but} \quad \lim_{F \in \mathcal{F}(\Omega)} \sum_{\omega \in F} P(\{\omega\})\nu = 0, \tag{3.29}$$

which implies that $\mathcal{M}$ is not Bade atomic. Consequently, $\mathcal{M}$ is not Bade complete (see the discussion immediately prior to Example 3.22); equivalently, P is not a closed measure (see Lemma 3.2). So $L^1(P)$ is weakly sequentially complete (see Theorem 3.18) but not weakly quasi-complete (see Theorem 3.7). One example of a specific projection belonging to the τ_s-closure of $\mathcal{M}$ but not to $\mathcal{M}$ itself is given in [15, Example 2.5], where it is denoted by P_λ. We now exhibit a further projection of this kind.

Given $\mu \in X$, define $\Omega(\mu) := \{\omega \in \Omega : \mu(\{\omega\}) \neq 0\}$. Then $\Omega(\mu)$ is a countable set. Indeed, given $n \in \mathbb{N}$, the subset $\Omega(|\mu|, n) := \{\omega \in \Omega : |\mu|(\{\omega\}) > 1/n\}$ is finite because $|\mu|(\Omega) < \infty$. So, $\Omega(|\mu|) = \bigcup_{n=1}^{\infty} \Omega(|\mu|, n)$ is countable. Thus, $\Omega(\mu)$ is also countable because $\Omega(\mu) = \Omega(|\mu|)$. In particular, $\Omega(\mu) \in \Sigma$. Observe that $\mu(F \setminus \Omega(\mu)) = 0$ for every countable set $F \subseteq \Omega$.

Define a map $Q : X \to X$ by $Q\mu := \mu_{\Omega(\mu)}$ for $\mu \in X$. To show that Q is additive, first observe the general fact that

$$\eta_{\Omega(\eta) \cup F} = \eta_{\Omega(\eta)}, \qquad \eta \in X, \tag{3.30}$$

for each *countable* subset $F \subseteq \Omega$. This is a consequence of the σ-additivity of $\eta \in X$:

$$\begin{aligned}\eta_{\Omega(\eta)\cup F}(E) &:= \eta(E \cap (\Omega(\eta) \cup F)) \\ &= \eta(E \cap \Omega(\eta)) + \eta(E \cap (F \setminus \Omega(\eta))) = \eta(E \cap \Omega(\eta)) = \eta_{\Omega(\eta)}(E)\end{aligned}$$

for each $E \in \Sigma$. Second, fix $\mu, \nu \in X$. Since

$$|(\mu+\nu)(\{\omega\})| \leq |\mu(\{\omega\})| + |\nu(\{\omega\})|, \qquad \omega \in \Omega,$$

we have $\Omega(\mu+\nu) \subseteq \Omega(\mu) \cup \Omega(\nu)$. This yields

$$(\mu+\nu)_{\Omega(\mu)\cup\Omega(\nu)} = (\mu+\nu)_{\Omega(\mu+\nu)} \tag{3.31}$$

by (3.30) with $\eta := (\mu+\nu)$ and $F := (\Omega(\mu) \cup \Omega(\nu)) \setminus \Omega(\mu+\nu)$. Next, (3.30) with $\eta := \mu$ and $F := \Omega(\nu)$ gives

$$\mu_{\Omega(\mu)\cup\Omega(\nu)} = \mu_{\Omega(\mu)}. \tag{3.32}$$

Similarly we have

$$\nu_{\Omega(\nu)\cup\Omega(\mu)} = \nu_{\Omega(\nu)}. \tag{3.33}$$

Since $(\mu+\nu)_{\Omega(\mu)\cup\Omega(\nu)} = \mu_{\Omega(\mu)\cup\Omega(\nu)} + \nu_{\Omega(\mu)\cup\Omega(\nu)}$, it follows from (3.31)–(3.33) that

$$Q(\mu+\nu) := (\mu+\nu)_{\Omega(\mu+\nu)} = (\mu+\nu)_{\Omega(\mu)\cup\Omega(\nu)} = \mu_{\Omega(\mu)} + \nu_{\Omega(\nu)} = Q(\mu)+Q(\nu),$$

which verifies the additivity of Q as $\mu,\nu \in X$ are arbitrary.

Next, let $a \in \mathbb{C}$. To establish the identity $Q(a\mu) = aQ(\mu)$, we may assume that $a \neq 0$ because this identity clearly holds when $a = 0$. But for $a \neq 0$ we have $\Omega(a\mu) = \Omega(\mu)$, which implies that

$$Q(a\mu) := (a\mu)_{\Omega(a\mu)} = (a\mu)_{\Omega(\mu)} = a\cdot\mu_{\Omega(\mu)} = aQ(\mu).$$

So, it has been verified that Q is *linear.*

The linear operator Q is also continuous because, for $\mu \in X$, we have

$$\|Q\mu\|_X = \|\mu_{\Omega(\mu)}\|_X = |\mu_{\Omega(\mu)}|(\Omega) \le |\mu|(\Omega) = \|\mu\|_X.$$

Moreover, Q is a *projection.* Indeed, for $\mu \in X$ it follows that

$$Q^2\mu = Q(\mu_{\Omega(\mu)}) = (\mu_{\Omega(\mu)})_{\Omega(\mu)} = \mu_{\Omega(\mu)} = Q\mu.$$

The claim is that the series $\sum_{\omega\in\Omega} P(\{\omega\})$ is unconditionally convergent in $\mathcal{L}_s(X)$ and that its sum equals Q. In fact, fix $\mu \in X$ and $\varepsilon > 0$. Since $\Omega(\mu)$ is countable and $\sum_{\omega\in\Omega(\mu)} |\mu|(\{\omega\}) = |\mu|(\Omega(\mu))$, we can select $F_0 \in \mathcal{F}(\Omega(\mu))$ such that $|\mu|(\Omega(\mu)\setminus F_0) < \varepsilon$. Whenever $F \in \mathcal{F}(\Omega)$ satisfies $F \supseteq F_0$, we have

$$\begin{aligned}\Big\|Q\mu - \sum_{\omega\in F} P(\{\omega\})\mu\Big\|_X &= \|\mu_{\Omega(\mu)} - \mu_{\Omega(\mu)\cap F}\|_X = \|\mu_{\Omega(\mu)\setminus F}\|_X = |\mu|(\Omega(\mu)\setminus F)\\ &\le |\mu|(\Omega(\mu)\setminus F_0) < \varepsilon.\end{aligned}$$

This establishes the claim as $\mu \in X$ and $\varepsilon > 0$ are arbitrary.

Finally, to see that $Q \notin \mathcal{M}$ assume that, on the contrary, $Q = P(E_0)$ for some $E_0 \in \Sigma$. Then

$$\delta_\omega = Q(\delta_\omega) = P(E_0)\delta_\omega = (\delta_\omega)_{E_0}, \quad \omega \in \Omega,$$

with δ_ω denoting the Dirac measure at $\omega \in \Omega$. This implies that $\omega \in E_0$ for all $\omega \in \Omega$. In other words, $E_0 = \Omega$, and hence $P(\Omega) = Q$, so that $\mathbf{I} = P(\Omega) = \sum_{\omega\in\Omega} P(\{\omega\})$. This contradicts (3.29) and so $Q \notin \mathcal{M}$. □

4. Appendix

In this final section we provide the proofs of those results in Section 2 which are yet to be verified, together with the alternative proof of Proposition 3.21.

Proof of Lemma 2.7. (i) Since the 'only if' portion is obvious, let us verify the 'if' portion. Suppose then that $m(G) = 0$. To prove that G is m-null assume, on the contrary, that G is *not* m-null. Then there exists $H \in \Sigma\cap G$ satisfying $m(H) \neq 0$. Since $m(H) \in m(\Sigma\cap G) \subseteq m(\Sigma\cap E) = \{0, m(E)\}$ (see (2.10)), we must have $m(H) = m(E)$, and hence

$$m(G\setminus H) = m(G) - m(H) = -m(H) = -m(E) \neq 0. \tag{4.1}$$

It then follows that $m(G\setminus H) = m(E)$ because $m(G\setminus H) \neq 0$ by (4.1) and because

$m(G \setminus H) \in m(\Sigma \cap E) = \{0, m(E)\}$. On the other hand, $m(G \setminus H) = -m(E)$ again by (4.1). So, $m(E) = m(G \setminus H) = -m(E)$, that is, $m(E) = 0$, which is impossible as E is an m-atom. Thus, G must be m-null.

(ii) (a)$\Rightarrow$(b). By assumption $m(E) \neq 0$. Let $F \in \Sigma$. Since E is an m-atom, either $m(E \cap F) = 0$ or $m(E \setminus F) = 0$. From part (i) we have either $E \cap F \in \mathcal{N}_0(m)$ or $E \setminus F \in \mathcal{N}_0(m)$, which establishes (b).

(b)$\Rightarrow$(a). First we show that $m(E) \neq 0$. So, select $G \in \Sigma \cap E$ with $m(G) \neq 0$, which is possible because $E \notin \mathcal{N}_0(m)$. Then $E \cap G = G \notin \mathcal{N}_0(m)$. This and (b) yield $E \setminus G \in \mathcal{N}_0(m)$, and hence we have

$$m(E) = m(G) + m(E \setminus G) = m(G) \neq 0.$$

Now, let $F \in \Sigma \cap E$. Then either $E \cap F \in \mathcal{N}_0(m)$ or $E \setminus F \in \mathcal{N}_0(m)$ by assumption, which implies that either $m(E \cap F) = 0$ or $m(E \setminus F) = 0$. Thus, (a) holds.

(b)$\Leftrightarrow$(c). This equivalence is a consequence of the fact that a set $G \in \Sigma$ satisfies $G \in \mathcal{N}_0(m)$ if and only if $[m](G) := \chi_G = 0$ in $L^1(m) = \mathcal{L}^1(m)/\mathcal{N}(m)$.

(c)$\Rightarrow$(d). Observe first that the $[m]$-atom E must satisfy $\chi_E = [m](E) \neq 0$ in $\Sigma(m) \subseteq L^1(m)$. Next, fix a set $G \in \Sigma$ such that $\chi_G \leq \chi_E$ in the order of $\Sigma(m)$. We may assume that $\chi_G \leq \chi_E$ pointwise everywhere on Ω. Again, E being an $[m]$-atom implies that either $\chi_{E \cap G} = [m](E \cap G) = 0$ or $\chi_{E \setminus G} = [m](E \setminus G) = 0$, that is,

$$\chi_G = \chi_{E \cap G} = 0 \quad \text{or} \quad \chi_G = \chi_{E \setminus (E \setminus G)} = \chi_E$$

in the B.a. $\Sigma(m)$. This establishes (d).

(d)$\Rightarrow$(b). The atom χ_E in the B.a. $\Sigma(m)$ is by definition a non-zero element, and hence $E \notin \mathcal{N}_0(m)$. Next let $F \in \Sigma$. The element $\chi_{E \cap F} \in \Sigma(m)$, which is dominated by the atom χ_E in the B.a. $\Sigma(m)$, must satisfy either $\chi_{E \cap F} = 0$ or $\chi_{E \cap F} = \chi_E$ as elements of $\Sigma(m)$. Now, the identity $\chi_{E \cap F} = 0$ in $\Sigma(m)$ means exactly that $E \cap F \in \mathcal{N}_0(m)$. On the other hand, if $\chi_{E \cap F} = \chi_E$ in $\Sigma(m)$, then $\chi_{E \setminus F} = \chi_{E \setminus (E \cap F)} = \chi_E - \chi_{E \cap F} = 0$ in $\Sigma(m)$, that is, $E \setminus F \in \mathcal{N}_0(m)$. So, we have established (b).

(iii) This follows immediately from the equivalence of (a), (c) and (d) in part (ii). ∎

Proof of Lemma 2.8. Fix $\alpha \in \mathcal{A}(m)$, with corresponding atom $\chi_{F(\alpha)}$ in $\Sigma(m)$, and $f \in L^1(m)$. Observe first that

$$\int_{F(\alpha)} s\,dm \in \operatorname{span} m(\Sigma \cap F(\alpha)), \qquad s \in \operatorname{sim} \Sigma. \tag{4.2}$$

Next, it follows from [25, Theorem 2.4] that there is a sequence $\{s_n\}_{n=1}^{\infty}$ in $\operatorname{sim}\Sigma$ which is $\tau(m)$-convergent to f. The multiplication operator $M_{F(\alpha)} \in \mathcal{L}(L^1(m))$ and the integration operator $I_m \in \mathcal{L}(L^1(m), Y)$ were defined in Section 2. It follows from (4.2) with $s := s_n$ for each $n \in \mathbb{N}$ that

$$\begin{aligned}\int_{F(\alpha)} f\,dm &= (I_m \circ M_{F(\alpha)})(f) = \lim_{n\to\infty} (I_m \circ M_{F(\alpha)})(s_n) \\ &= \lim_{n\to\infty} \int_{F(\alpha)} s_n\,dm \in \overline{\operatorname{span}}\, m(\Sigma \cap F(\alpha)) \end{aligned}\tag{4.3}$$

in Y. On the other hand, $F(\alpha)$ is an m-atom by Lemma 2.7(ii) with $E := F(\alpha)$, so that $m(\Sigma \cap F(\alpha)) = \{0, m(F(\alpha))\}$ (see (2.10) with $E := F(\alpha)$). Consequently, $\overline{\operatorname{span}}\, m(\Sigma \cap F(\alpha))$

equals the one-dimensional subspace span$\{m(F(\alpha)\}$. Hence, (4.3) guarantees that there is a unique complex number $a(\alpha, f)$ satisfying (2.11).

Clearly, to verify (2.12) is equivalent to showing that

$$\int_F (f\chi_{F(\alpha)} - a(\alpha, f)\chi_{F(\alpha)})\, dm = 0, \qquad F \in \Sigma \cap F(\alpha). \tag{4.4}$$

To prove (4.4), fix $F \in \Sigma \cap F(\alpha)$. Then either $F = F(\alpha) \cap F \in \mathcal{N}_0(m)$ or $F(\alpha) \setminus F \in \mathcal{N}_0(m)$ as $F(\alpha)$ is an m-atom (see Lemma 2.7(ii) with $F(\alpha)$ in place of E). If $F \in \mathcal{N}_0(m)$, then (4.4) clearly holds. So, assume that $F(\alpha) \setminus F \in \mathcal{N}_0(m)$. Then

$$\int_{F(\alpha)\setminus F} (f\chi_{F(\alpha)} - a(\alpha, f)\chi_{F(\alpha)})\, dm = 0,$$

which means that

$$\int_{F(\alpha)} (f\chi_{F(\alpha)} - a(\alpha, f)\chi_{F(\alpha)})\, dm = \int_F (f\chi_{F(\alpha)} - a(\alpha, f)\chi_{F(\alpha)})\, dm. \tag{4.5}$$

Since the left-hand side of (4.5) equals 0 by (2.11), so does the right-hand side. Hence, (4.4) holds for arbitrary $F \in \Sigma \cap F(\alpha)$, and thus (2.12) is established. ∎

Proof of Lemma 2.9. For each $E \in \Sigma$, define

$$\mathcal{A}_E(m) := \{\alpha \in \mathcal{A}(m) : \chi_{F(\alpha)} \le \chi_E \text{ in the B.a. } \Sigma(m)\}.$$

Then $\chi_E = \bigvee_{\alpha \in \mathcal{A}_E(m)} \chi_{F(\alpha)}$ in the B.a. $\Sigma(m)$ (see, for example, [18, §16, Lemma 1]). Fix now $E \in \Sigma$. Then E is m-null if and only if $\chi_E = 0$ in $\Sigma(m)$ if and only if $\mathcal{A}_E(m) = \emptyset$. This establishes the lemma because $\mathcal{A}_E(m) = \emptyset$ is equivalent to the condition that $E \cap F(\alpha)$ is in $\mathcal{N}_0(m)$ for all $\alpha \in \mathcal{A}(m)$. ∎

Proof of Proposition 2.10. Let $f \in L^1(m)$ be the weak limit in $L^1(m)$ of a net $\{\chi_{E(\gamma)}\}_{\gamma\in\Gamma} \subseteq \Sigma(m)$. Define members of Σ by

$$G(0) := f^{-1}(\{0\}) \quad \text{and} \quad G(1) := f^{-1}(\{1\})$$

for the individual function f. It suffices to show that

$$f = \chi_{G(1)} \quad \text{pointwise } m\text{-a.e. on } \Omega, \tag{4.6}$$

as then $f = \chi_{G(1)}$ in $\Sigma(m)$. To this end, fixing $\alpha \in \mathcal{A}(m)$, we claim that

$$\big(\Omega \setminus (G(0) \cup G(1))\big) \cap F(\alpha) \in \mathcal{N}_0(m). \tag{4.7}$$

Indeed, via the Hahn–Banach Theorem, choose $y^*_\alpha \in Y^*$ (not unique) satisfying

$$\langle m(F(\alpha)), y^*_\alpha \rangle = 1. \tag{4.8}$$

The $\tau(m)$-continuous linear functional $y^*_\alpha \circ I_m \circ M_{F(\alpha)} : L^1(m) \to \mathbb{C}$ is also continuous for the weak topology $\sigma(L^1(m), L^1(m)^*)$, which implies that

$$\langle f, y^*_\alpha \circ I_m \circ M_{F(\alpha)} \rangle = \lim_{\gamma\in\Gamma} \langle \chi_{E(\gamma)}, y^*_\alpha \circ I_m \circ M_{F(\alpha)} \rangle. \tag{4.9}$$

On the other hand, Lemma 2.8 yields that

$$\langle f, y^*_\alpha \circ I_m \circ M_{F(\alpha)} \rangle = \left\langle \int_{F(\alpha)} f\, dm, y^*_\alpha \right\rangle = \langle a(\alpha, f) m(F(\alpha)), y^*_\alpha \rangle = a(\alpha, f), \tag{4.10}$$

via (4.8), and that

$$f^{-1}(\mathbb{C} \setminus \{a(\alpha, f)\}) \cap F(\alpha) \in \mathcal{N}_0(m). \tag{4.11}$$

Given $\gamma \in \Gamma$, by Lemma 2.8 with $\chi_{E(\gamma)}$ in place of f, we find (by arguing similarly to (4.10)) that

$$\langle \chi_{E(\gamma)}, y_\alpha^* \circ I_m \circ M_{F(\alpha)} \rangle = a(\alpha, \chi_{E(\gamma)}) \tag{4.12}$$

and that $\chi_{E(\gamma)\cap F(\alpha)} = \chi_{E(\gamma)}\chi_{F(\alpha)}$ is m-a.e. pointwise equal to $a(\alpha, \chi_{E(\gamma)})\chi_{F(\alpha)}$. The latter means that

$$a(\alpha, \chi_{E(\gamma)}) = \begin{cases} 0 & \text{if } E(\gamma) \cap F(\alpha) \in \mathcal{N}_0(m), \\ 1 & \text{if } (F(\alpha) \setminus E(\gamma)) \in \mathcal{N}_0(m) \end{cases} \tag{4.13}$$

because $F(\alpha)$ is an m-atom. It now follows from (4.9), (4.10) and (4.12) that $a(\alpha, f) = \lim_{\gamma\in\Gamma} a(\alpha, \chi_{E(\gamma)})$. This and (4.13) yield $a(\alpha, f) \in \{0, 1\}$, which ensures that

$$\big(\Omega \setminus (G(0) \cup G(1))\big) \cap F(\alpha) = (f^{-1}(\mathbb{C} \setminus \{0, 1\})) \cap F(\alpha) \subseteq \big(f^{-1}(\mathbb{C} \setminus \{a(\alpha, f)\})\big) \cap F(\alpha).$$

Now an appeal to (4.11) verifies (4.7).

Since (4.7) holds for every $\alpha \in \mathcal{A}(m)$, Lemma 2.9 with the set $\Omega \setminus (G(0) \cup G(1))$ in place of E yields $\Omega \setminus (G(0) \cup G(1)) \in \mathcal{N}_0(m)$. Thus, (4.6) holds because

$$f = (\chi_{G(0)} + \chi_{G(1)} + \chi_{\Omega\setminus(G(0)\cup G(1))})f = \chi_{G(1)} + f\chi_{\Omega\setminus(G(0)\cup G(1))},$$

with $\Omega \setminus (G(0) \cup G(1)) \in \mathcal{N}_0(m)$. ∎

Proof of Proposition 2.12. (i) Let $\{F(\alpha)\}_{\alpha\in\mathcal{A}(m)}$ be the family of all m-atoms. Apply Lemma 2.7(iii) to see that the corresponding B.a. $\Sigma(m)$ is also atomic. The set $\{0, 1\}^{\mathcal{A}(m)}$ of all functions from $\mathcal{A}(m)$ into $\{0, 1\}$ is compact with respect to the product topology. We proceed to construct a Y-valued, continuous function Φ defined on the compact space $\{0, 1\}^{\mathcal{A}(m)}$ whose range coincides with $\mathcal{R}(m)$ in Y.

Fix $\varepsilon \in \{0, 1\}^{\mathcal{A}(m)}$. Let $\mathcal{A}_\varepsilon := \varepsilon^{-1}(\{1\})$, in which case ε equals the characteristic function of the subset $\mathcal{A}_\varepsilon \subseteq \mathcal{A}(m)$. The family of all finite subsets of $\mathcal{A}_\varepsilon$ is denoted by $\mathcal{F}_\varepsilon$ and is directed by inclusion. We claim that the series $\sum_{\alpha\in\mathcal{A}(m)} \varepsilon(\alpha)m(F(\alpha))$ is unconditionally summable in Y, and that its sum lies within $\mathcal{R}(m)$; in other words,

$$\sum_{\alpha\in\mathcal{A}(m)} \varepsilon(\alpha)m(F(\alpha)) = m(E) \in \mathcal{R}(m) \tag{4.14}$$

for some $E \in \Sigma$. To verify this, observe that $\sum_{\alpha\in\mathcal{A}(m)} \varepsilon(\alpha)m(F(\alpha))$ and $\sum_{\alpha\in\mathcal{A}_\varepsilon} m(F(\alpha))$ are the same series. Since $\Sigma(m)$ is *complete* as an abstract B.a. (see Lemma 2.2), there exists $E \in \Sigma$ such that $\chi_E = \bigvee_{\alpha\in\mathcal{A}_\varepsilon} \chi_{F(\alpha)}$ in $\Sigma(m)$. For each $A \in \mathcal{F}_\varepsilon$ define (the finite disjoint union) $F(A) := \bigcup_{\alpha\in A} F(\alpha) \in \Sigma$ so that $m(F(A)) = \sum_{\alpha\in A} m(F(\alpha))$. Then the net $\{\chi_{E\setminus F(A)}\}_{A\in\mathcal{F}_\varepsilon}$ is downwards filtering to 0 in $\Sigma(m)$. So, by Lemma 2.2 it follows that $\lim_{A\in\mathcal{F}_\varepsilon} m(E \setminus F(A)) = 0$ in Y, which gives

$$m(E) = \lim_{A\in\mathcal{F}_\varepsilon} m(F(A)) = \lim_{A\in\mathcal{F}_\varepsilon} \sum_{\alpha\in A} m(F(\alpha)).$$

In other words, $\sum_{\alpha\in\mathcal{A}_\varepsilon} m(F(\alpha))$ is an unconditionally convergent series in Y with sum $m(E)$. So, (4.14) holds and the claim is established.

The above claim enables us to define a function $\Phi : \{0, 1\}^{\mathcal{A}(m)} \to Y$ by

$$\Phi(\varepsilon) := \sum_{\alpha\in\mathcal{A}(m)} \varepsilon(\alpha)m(F(\alpha)), \quad \varepsilon \in \{0, 1\}^{\mathcal{A}(m)}.$$

It also gives the inclusion $\Phi(\{0,1\}^{\mathcal{A}(m)}) \subseteq \mathcal{R}(m)$ (see (4.14)). To obtain the reverse inclusion, fix any non-m-null set $E \in \Sigma$. Define $\varepsilon_{_E} : \mathcal{A}(m) \to \{0,1\}$ to be the characteristic function of the subset

$$\{\alpha \in \mathcal{A}(m) : \chi_{F(\alpha)} \le \chi_E\} \quad \text{in the B.a. } \Sigma(m), \tag{4.15}$$

in which case $\mathcal{A}_{\varepsilon_{_E}} = \varepsilon_{_E}^{-1}(\{1\})$. In the complete, atomic B.a. $\Sigma(m)$, the equality $\chi_E = \bigvee_{\alpha\in\mathcal{A}_{\varepsilon_{_E}}} \chi_{F(\alpha)}$ holds [18, §16, Lemma 1]. By recalling the above process of defining $\Phi(\varepsilon)$, now for this particular $\varepsilon_{_E}$, we see that $m(E) = \Phi(\varepsilon_{_E})$, which establishes the reverse inclusion $\mathcal{R}(m) \subseteq \Phi(\{0,1\}^{\mathcal{A}(m)})$ as $E \in \Sigma$ is an arbitrary non-m-null set. Thus, $\Phi(\{0,1\}^{\mathcal{A}(m)}) = \mathcal{R}(m)$.

To prove the continuity of Φ, let $q \in \mathcal{P}(Y)$ and $\delta > 0$. As the series $\sum_{\alpha\in\mathcal{A}(m)} m(F(\alpha))$ is unconditionally convergent in Y (choose $\varepsilon := \chi_{\mathcal{A}(m)}$ above), there exists a finite subset $A(q,\delta) \subseteq \mathcal{A}(m)$ satisfying

$$q\Big(\sum_{\alpha\in A} m(F(\alpha)) - \sum_{\alpha\in A'} m(F(\alpha))\Big) < \delta/2 \tag{4.16}$$

whenever A and A' are finite subsets of $\mathcal{A}(m)$ such that $A \cap A' \supseteq A(q,\delta)$. Now take any two functions $\varepsilon, \varepsilon' \in \{0,1\}^{\mathcal{A}(m)}$ satisfying

$$\varepsilon(\alpha) = \varepsilon'(\alpha), \qquad \alpha \in A(q,\delta), \tag{4.17}$$

and define $\mathcal{B}_1 := \mathcal{A}_\varepsilon \setminus \mathcal{A}_{\varepsilon'}$ and $\mathcal{B}_2 := \mathcal{A}_{\varepsilon'} \setminus \mathcal{A}_\varepsilon$. Given $j = 1,2$, for the special choice $\varepsilon := \chi_{\mathcal{B}_j}$ in the second paragraph of this proof, the corresponding set $\mathcal{A}_\varepsilon$ is $\mathcal{B}_j$, and so the series $\sum_{\alpha\in\mathcal{B}_j} m(F(\alpha))$ is unconditionally convergent in Y. Moreover, for any finite subset $B \subseteq \mathcal{B}_j$, it follows from (4.17) that

$$A(q,\delta) \cap B \subseteq A(q,\delta) \cap \mathcal{B}_j = \emptyset, \qquad j = 1,2.$$

This and (4.16), with $A(q,\delta) \cup B$ in place of A and $A(q,\delta)$ in place of A', yield

$$q\Big(\sum_{\alpha\in B} m(F(\alpha))\Big) = q\Big(\sum_{\alpha\in A(q,\delta)\cup B} m(F(\alpha)) - \sum_{\alpha\in A(q,\delta)} m(F(\alpha))\Big) < \delta/2,$$

and hence $q(\sum_{\alpha\in\mathcal{B}_j} m(F(\alpha))) \le \delta/2$ for $j = 1,2$. It follows that

$$\begin{aligned}
q(\Phi(\varepsilon) - \Phi(\varepsilon')) &= q\Big(\sum_{\alpha\in\mathcal{A}_\varepsilon} m(F(\alpha)) - \sum_{\alpha\in\mathcal{A}_{\varepsilon'}} m(F(\alpha))\Big) \\
&= q\Big(\sum_{\alpha\in\mathcal{A}_\varepsilon\cap\mathcal{A}_{\varepsilon'}} m(F(\alpha)) + \sum_{\alpha\in\mathcal{B}_1} m(F(\alpha)) - \sum_{\alpha\in\mathcal{A}_\varepsilon\cap\mathcal{A}_{\varepsilon'}} m(F(\alpha)) - \sum_{\alpha\in\mathcal{B}_2} m(F(\alpha))\Big) \\
&= q\Big(\sum_{\alpha\in\mathcal{B}_1} m(F(\alpha)) - \sum_{\alpha\in\mathcal{B}_2} m(F(\alpha))\Big) \\
&\le q\Big(\sum_{\alpha\in\mathcal{B}_1} m(F(\alpha))\Big) + q\Big(\sum_{\alpha\in\mathcal{B}_2} m(F(\alpha))\Big) \le \delta.
\end{aligned}$$

So, Φ is continuous as $q \in \mathcal{P}(Y)$ and $\delta > 0$ are arbitrary. We conclude that $\mathcal{R}(m)$ is a compact subset of Y because it equals the range of the continuous function Φ defined on the compact space $\{0,1\}^{\mathcal{A}(m)}$.

(ii) Fix $E \in \Sigma$. Let $\varepsilon \in \{0,1\}^{\mathcal{A}(m)}$ denote the characteristic function of the subset (4.15) of $\mathcal{A}(m)$. Then, as demonstrated in the proof of (i), the vector $m(E) \in Y$ is the sum of the unconditionally convergent series $\sum_{\alpha\in\mathcal{A}(m)} \varepsilon(\alpha)m(F(\alpha))$. This proves (ii) because $\alpha \in \mathcal{A}(m)$ satisfies $\varepsilon(\alpha) = 1$ if and only if $m(F(\alpha)) = m(E \cap F(\alpha))$. ∎

Proof of Proposition 2.16. (a)⇒(b). Because of (a) the associated $L^1(m)$-valued vector measure $[m] : E \mapsto \chi_E$ on Σ is also closed (see Remark 2.6(iv)). Moreover, $[m]$ is atomic (see Lemma 2.7(iii)). So, by Proposition 2.12(i) applied to $[m]$, the range $\mathcal{R}([m])$ is compact in $L^1(m)$, and hence (b) holds because $\Sigma(m) = \mathcal{R}([m])$.

(b)⇒(c)⇒(d). These implications are clear.

(d)⇒(a). This is exactly Proposition 2.4. ∎

Note that we can also prove (d)⇒(a) via (c) in Proposition 2.16. In fact, (d) implies (c) because $\Sigma(m)$ is weakly closed in $L^1(m)$ by Proposition 2.10. Now, (c) is equivalent to the weak completeness of $\Sigma(m)$ [13, 0.6, p. 3]. Finally, the weak completeness of $\Sigma(m)$ implies its $\tau(m)$-completeness [23, §18, 4.(4)], that is, (a) holds.

We end this section with the promised alternative

Proof of Proposition 3.21. The *bidual* of the Banach space X is denoted by X^{**}. Let $J : X \to X^{**}$ denote the natural embedding, which is a linear isometry.

Take an arbitrary sequence $\{H_n\}_{n=1}^{\infty} \subseteq \mathcal{M}$ which is Cauchy in $\mathcal{L}_w(X)$. We need to show that it has a limit in $\mathcal{L}_w(X)$ and that this limit belongs to $\mathcal{M}$. This will require several steps.

Step 1. *There exists $H \in \mathcal{L}(X, X^{**})$ such that*

$$\lim_{n\to\infty} \langle H_n x, x^*\rangle = \lim_{n\to\infty} \langle x^*, (JH_n)x\rangle = \langle x^*, Hx\rangle, \qquad x \in X,\ x^* \in X^*. \tag{4.18}$$

To verify this, all we need is to obtain the second equality in (4.18) because the first equality is obvious from the definition of J. By the Banach–Steinhaus Theorem, $C := \sup_{n\in\mathbb{N}} \|H_n\|_{\mathrm{op}} < \infty$. Fix $x \in X$. Then the sequence $\{H_n x\}_{n=1}^{\infty}$ is weakly Cauchy in X. Since $J \in \mathcal{L}(X_{\sigma(X,X^*)}, X^{**}_{\sigma(X^{**},X^*)})$ [30, Proposition 2.6.24], it follows that $\{(JH_n)x\}_{n=1}^{\infty}$ is a $\sigma(X^{**}, X^*)$-Cauchy sequence, and hence has a $\sigma(X^{**}, X^*)$-limit $\xi_x \in X^{**}$, i.e.,

$$\lim_{n\to\infty} \langle x^*, (JH_n)x\rangle = \langle x^*, \xi_x\rangle, \qquad x^* \in X^* \tag{4.19}$$

(apply [30, Corollary 2.6.21] to the Banach space X^*). Next, observe that $\|\xi_x\|_{X^{**}} \le C\|x\|_X$ by (4.19) because

$$\begin{aligned}\|\xi_x\|_{X^{**}} &= \sup_{\|x^*\|_{X^*}\le 1} |\langle x^*, \xi_x\rangle| = \sup_{\|x^*\|_{X^*}\le 1} \lim_{n\to\infty} |\langle x^*, (JH_n)x\rangle| \\ &\le \sup_{\|x^*\|_{X^*}\le 1} \limsup_{n\to\infty} \|x^*\|_{X^*} \|H_n x\|_X \le C\|x\|_X.\end{aligned}$$

Since $x \in X$ is arbitrary, this enables us to define $H \in \mathcal{L}(X, X^{**})$ by $Hx := \xi_x$. Thus, the second equality in (4.18) holds in view of (4.19), thereby establishing Step 1.

Since $\mathcal{M}$ is σ-complete as an abstract B.a., the projections

$$Q := \bigvee_{n=1}^{\infty} \bigwedge_{k=n}^{\infty} H_k \quad \text{and} \quad R := \bigwedge_{n=1}^{\infty} \bigvee_{k=n}^{\infty} H_k$$

both belong to $\mathcal{M}$.

STEP 2. *The equality $JQ = HQ$ holds in $\mathcal{L}(X, X^{**})$.*

Indeed, set $\widetilde{H}_n := \bigwedge_{k=n}^{\infty} H_k \in \mathcal{M}$ for $n \in \mathbb{N}$. Since $\widetilde{H}_n \uparrow Q$ in the order of $\mathcal{M}$, it follows from [10, Ch. XVII, Lemma 3.4] that $\lim_{n\to\infty} \widetilde{H}_n = Q$ in $\mathcal{L}_s(X)$. In other words, relative to the norm in X, we have

$$\lim_{n\to\infty} \widetilde{H}_n x = Qx, \qquad x \in X. \tag{4.20}$$

Fix $n \in \mathbb{N}$ and $x \in X$ for the moment. Given any $k \in \mathbb{N}$ with $k \geq n$, observe that $\widetilde{H}_n \leq H_k$ by definition, i.e., $H_k\widetilde{H}_n = \widetilde{H}_n$. This identity and (4.18), with $\widetilde{H}_n x$ in place of x, give for every $x^* \in X^*$ the equality

$$\langle x^*, (H\widetilde{H}_n)x\rangle = \lim_{k\to\infty} \langle (H_k\widetilde{H}_n)x, x^*\rangle = \langle \widetilde{H}_n x, x^*\rangle = \langle x^*, (J\widetilde{H}_n)x\rangle. \tag{4.21}$$

So, $(H\widetilde{H}_n)x = (J\widetilde{H}_n)x$ in $X^{**} = \mathcal{L}(X^*, \mathbb{C})$: to see this, apply [30, Corollary 2.2.22] to the lcHs $(X^{**}, \sigma(X^{**}, X^*))$. Using this identity and (4.20) yields

$$(HQ)x = \lim_{n\to\infty} H(\widetilde{H}_n x) = \lim_{n\to\infty} (J\widetilde{H}_n)x = (JQ)x$$

in $(X^{**}, \|\cdot\|_{X^{**}})$ for each $x \in X$. Thus we have $HQ = JQ \in \mathcal{L}(X, X^{**})$.

STEP 3. *The identity $HR = H$ holds in $\mathcal{L}(X, X^{**})$.*

We shall prove this similarly to Step 2. Define $H_n^\sharp := \bigvee_{k=n}^{\infty} H_k \in \mathcal{M}$ for $n \in \mathbb{N}$. Then $H_n^\sharp \downarrow R$ in the order of $\mathcal{M}$, and so $\lim_{n\to\infty} H_n^\sharp = R$ in $\mathcal{L}_s(X)$, again by [10, Ch. XVII, Lemma 3.4]. Fix $x \in X$ and $n \in \mathbb{N}$. If $k \in \mathbb{N}$ satisfies $k \geq n$, then $H_k \leq H_n^\sharp$ or equivalently $H_k H_n^\sharp = H_k$. From this identity and (4.18), with $H_n^\sharp x$ in place of x, it follows, for every $x^* \in X^*$, that

$$\langle x^* H(H_n^\sharp x)\rangle = \lim_{k\to\infty} \langle H_k(H_n^\sharp x), x^*\rangle = \lim_{k\to\infty} \langle H_k x, x^*\rangle.$$

Again by (4.18) we also have

$$\lim_{k\to\infty} \big\langle H_k x, x^*\big\rangle = \langle x^*, Hx\rangle, \qquad x^* \in X^*,$$

and so $\langle x^*, HH_n^\sharp x\rangle = \langle x^*, Hx\rangle$. Hence, $H(H_n^\sharp x) = Hx$ in X^{**}. Since $\lim_{n\to\infty} H_n^\sharp x = Rx$ in X, we have

$$(HR)x = H\Big(\lim_{n\to\infty} H_n^\sharp x\Big) = \lim_{n\to\infty} H(H_n^\sharp x) = Hx$$

in $(X^{**}, \|\cdot\|_{X^{**}})$. As this holds of all $x \in X$, we conclude that $HR = H$ in $\mathcal{L}(X, X^{**})$.

STEP 4. *We have the identity $Q = R$.*

First fix an arbitrary *atom* $K \in \mathcal{M}$. The distributive laws in $\mathcal{M}$ [18, §7, Lemma 4] give

$$QK = \Big(\bigvee_{n=1}^{\infty} \bigwedge_{k=n}^{\infty} H_k\Big) \wedge K = \bigvee_{n=1}^{\infty} \bigwedge_{k=n}^{\infty} H_k K, \tag{4.22}$$

$$RK = \Big(\bigwedge_{n=1}^{\infty} \bigvee_{k=n}^{\infty} H_k\Big) \wedge K = \bigwedge_{n=1}^{\infty} \bigvee_{k=n}^{\infty} H_k K. \tag{4.23}$$

Observe that $H_n K \in \{0, K\}$ for all $n \in \mathbb{N}$ because K is an atom.

We claim that there exists $N \in \mathbb{N}$ satisfying either

$$H_n K = 0, \qquad n \geq N, \tag{4.24}$$

or

$$H_n K = K, \qquad n \geq N. \tag{4.25}$$

Assume, on the contrary, that there exist increasing sequences $\{n(k)\}_{k=1}^{\infty}$ and $\{r(j)\}_{j=1}^{\infty}$ such that $H_{n(k)}K = 0$ for all $k \in \mathbb{N}$ and $H_{r(j)}K = K$ for all $j \in \mathbb{N}$. Fix $x \in X$ and $x^* \in X^*$. Then $\{\langle (H_{n(k)}K)x, x^*\rangle\}_{k=1}^{\infty}$ and $\{\langle (H_{r(j)}K)x, x^*\rangle\}_{j=1}^{\infty}$ are both subsequences of the convergent sequence $\{\langle (H_n K)x, x^*\rangle\}_{n=1}^{\infty}$ in $\mathbb{C}$ (see (4.18) with Kx in place of x). Therefore

$$\lim_{k\to\infty} \langle (H_{n(k)}K)x, x^*\rangle = \lim_{j\to\infty} \langle (H_{r(j)}K)x, x^*\rangle.$$

But $\lim_{k\to\infty}\langle (H_{n(k)}K)x, x^*\rangle = 0$ and $\lim_{j\to\infty}\langle (H_{r(j)}K)x, x^*\rangle = \langle Kx, x^*\rangle$, which implies that $K = 0$ because $x \in X$ and $x^* \in X^*$ are arbitrary. However, this contradicts K being an atom. Thus, there must exist $N \in \mathbb{N}$ satisfying either (4.24) or (4.25).

We consider each of the possibilities (4.24) and (4.25) separately.

CASE 1: Assume that (4.24) holds. Then $\bigwedge_{k=n}^{\infty} H_k K = 0$ for all $n \in \mathbb{N}$, and so (4.22) implies that $QK = 0$. On the other hand, as (4.24) yields $\bigvee_{k=n}^{\infty} H_k K = 0$ for all $n \geq N$, we have $RK = 0$ from (4.23). Thus, we conclude that $QK = RK = 0$.

CASE 2: Assume that (4.25) holds. Then $\bigwedge_{k=n}^{\infty} H_k K = K$ for all $n \geq N$. Since $\bigwedge_{k=n}^{\infty} H_k K \leq K$ for $1 \leq k \leq N$, it follows by (4.22) that $QK = K$. To obtain the identity $RK = K$, observe first from (4.25) that $\bigvee_{k=n}^{\infty} H_k K = K$ for all $n \geq N$, and hence $\bigvee_{k=n}^{\infty} H_k K \downarrow K$ in the order of $\mathcal{M}$. This together with (4.23) gives $RK = K$. Thus we have shown that $QK = RK = K$.

Cases 1 and 2 show that $QK = RK$ whenever K is an atom of $\mathcal{M}$.

Let $\mathcal{A}_{\mathcal{M}}$ denote the set of all atoms in $\mathcal{M}$. Since $\mathbf{I} = \bigvee_{K\in\mathcal{A}_{\mathcal{M}}} K$ in the order of the B.a. $\mathcal{M}$ (see, for example, [18, §16, Lemma 1]), Step 4 now follows by [18, §7, Lemma 4] which yields

$$Q = Q\Big(\bigvee_{K\in\mathcal{A}_{\mathcal{M}}} K\Big) = \bigvee_{K\in\mathcal{A}_{\mathcal{M}}} QK = \bigvee_{K\in\mathcal{A}_{\mathcal{M}}} RK = R\Big(\bigvee_{K\in\mathcal{A}_{\mathcal{M}}} K\Big) = R.$$

STEP 5. *The identity $JQ = H$ holds in $\mathcal{L}(X, X^*)$.*

Indeed, $JQ = HQ$ by Step 2, and $HQ = HR$ by Step 4. On the other hand, $HR = H$ by Step 3. Thus, we have $JQ = H$.

STEP 6. *The sequence $\{H_n\}_{n=1}^{\infty}$ converges to $Q \in \mathcal{M}$ in $\mathcal{L}_w(X)$.*

In fact, Step 6 follows readily from Steps 1 and 5 together with

$$\lim_{n\to\infty} \big\langle H_n x, x^*\big\rangle = \langle x^*, Hx\rangle = \langle x^*, (JQ)x\rangle = \langle Qx, x^*\rangle$$

for all $x \in X$ and $x^* \in X^*$.

Finally, we conclude from Step 6 that $\mathcal{M}$ is sequentially complete in $\mathcal{L}_w(X)$ because $\{H_n\}_{n=1}^{\infty}$ is an arbitrary τ_w-Cauchy sequence in $\mathcal{M}$. ∎

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